9. Named Discrete distributions

Principles of Data Science with R

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PSTAT 10

we did:

- Random Variables: Discrete or Continuous
- Discrete Random variables (By hand and using R)
  - P.m.f
  - E(X)
  - V(X)
  - C.d.f

Next we will see...

Certain distributions are so common that they get their own name!

- Named distributions
  - Discrete Uniform Distributon
  - Binomial Distribution
  - Continuous Uniform Distribution
  - (Continuous) Normal Distribution

Discrete uniform distribution

All outcomes are equally likely/possible.

k	1	2	3	4	5	6
$P(X = k)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$X \sim \mathrm{DUnif}(\{1, 2, \ldots, 6\})$

Read as X follows *discrete uniform distribution on 1, 2, . . . , 6*.

More generally, $X \sim \mathrm{DUnif}(\{1, 2, \ldots, n\})$ then $P(X = k) = 1/n$

Binomial distribution

Only two outcomes.

Binomial Experimental setup:

1. There are a fixed number of trials (denoted by n)
2. These n trials are independent.
3. Each trial has two possible outcomes, 0 or 1. We call an outcome of 1 a *success*.
4. The probability of success for each trial is the same and is denoted by p. Correspondingly, the probability of a failure is denoted by $1 - p$.

If $X =$ the total number of successes in the n trials of a binomial experiment then

$$X \sim \mathrm{Binom}(n, p).$$

n and p are the parameters of the Binomial distrubiton.

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- totally different experiments, yet both are Binomial RV albeit with different parameters

Recall

For discrete r.v.s, $P(X = k)$ is the *probability mass function* or **pmf** of X.

It is a function of the values k that the random variable takes.

Probabilities for Binomial RVs

$X \sim \mathrm{Binom}(n, p)$

The p.m.f of X is given by

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k = 0, 1, \ldots, n$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{1.2.3\cdots n}{1.2.3\cdots k(1.2.3\cdots n-k)}$ gives the number of ways to choose k of the n outcomes to be successes.

We'll use the `choose(n,k)` function in R to calculate $\binom{n}{k}$

- **Mean** $\mu = np$
- **variance** $\sigma^2 = np(1-p)$; **SD** $\sigma = \sqrt{np(1-p)}$

Let $X \sim \text{Binom}(10, 1/4)$.

Determine the probability of getting 4 successes ie $P(X = 4)$.

```
n <- 10
p <- 1/4
k <- 4
choose(n, k) * p^k * (1-p)^(n-k)
```

```
## [1] 0.145998
```

Use R-built in function `dbinom` for binomial probability

```
dbinom(4, size = 10, prob = 1/4)
```

```
## [1] 0.145998
```

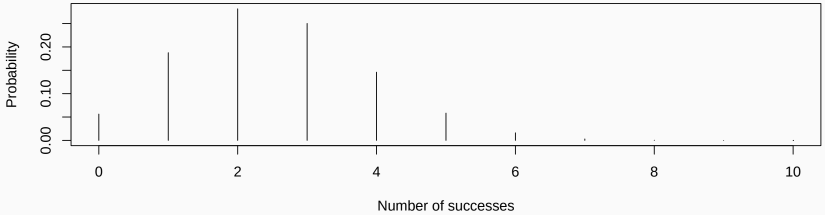
The syntax for the function is,

```
dbinom(x, size, prob, log = FALSE)
```

Let $X \sim \text{Binom}(10, 1/4)$, then the pmf of X is

```
##    0    1    2    3    4    5    6    7    8    9   10
## 0.06 0.19 0.28 0.25 0.15 0.06 0.02 0.00 0.00 0.00 0.00
```

```r
plot(0:10, dbinom(0:10, 10, 1/4),
     xlab="Number of successes",
     ylab="Probability", type="h")
```



$E(X) = np = 10 * 0.25 = 2.5$,
$Var(X) = np(1 - p) = 10 * 0.25 * 0.75 = 1.875$, SD $= 1.37$,

Typical values lie between 2.5 -1.37 = 0.625 to 2.5+1.3 = 4.375

Let $X \sim \mathrm{Binom}(10, 1/4)$.

What's the probability of at most 4 successes? $P(X \leq 4) = ?$.

$$P(X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$

```r
# `dbinom` gives the pmf
sum(dbinom(0:4, size = 10, prob = 1/4))
```

```
## [1] 0.9218731
```

```r
pbinom(4, size = 10, prob = 1/4)
```

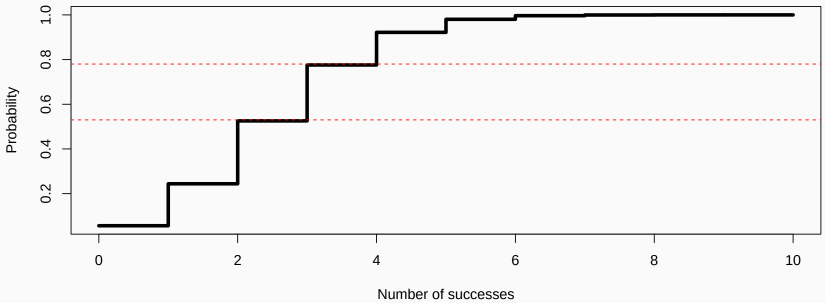
```
## [1] 0.9218731
```

`pbinom` gives the *cumulative probabilities*, the *cdf*.

Let $X \sim \text{Binom}(10, 1/4)$, then the cdf of X is

```
##    0    1    2    3    4    5    6    7    8    9   10
## 0.06 0.24 0.53 0.78 0.92 0.98 1.00 1.00 1.00 1.00 1.00
```

```r
plot(0:10, pbinom(0:10, 10, 1/4),
     xlab="Number of successes",
     ylab="Probability", type="s", lwd=4)
abline(0.53, 0, lty = 2, col = "red")
abline(0.78, 0, lty = 2, col = "red")
```



Let $X \sim \mathrm{Binom}(10, 1/4)$.

What's the probability of getting between 4 and 8 successes?

$P(4 \leq X \leq 8) = ?$

$$\underbrace{\underbrace{\underline{0}\ \ \underline{1}\ \ \underline{2}\ \ \underline{3}}_{X\leq 3}\ \ \underbrace{\underline{4}\ \ \underline{5}\ \ \underline{6}\ \ \underline{7}\ \ \underline{8}}_{4\leq X\leq 8}}_{X\leq 8}\ \ \underbrace{\underline{9}\ \ \underline{10}}_{9\leq X\leq 10}$$

$P(4 \leq X \leq 8) = P(X \leq 8) - P(X \leq 3)$

```r
pbinom(8, 10, 1/4) - pbinom(3, 10 , 1/4)
```

```
## [1] 0.2240953
```

```r
sum(dbinom(4:8, 10,1/4))
```

```
## [1] 0.2240953
```

Let $X \sim \mathrm{Binom}(10, 1/4)$.

What's the probability of getting more than 4 successes?

$P(X > 4) = ?$

$P(X > 4) = 1 - P(X \leq 4)$

```
1 - pbinom(4, 10, 1/4)
```

```
## [1] 0.07812691
```

Generating binomial observations

If $X \sim Bino(10, 0.8)$ then X is total number of successes in 10 trials of a binomial experiment with $p = 0.8$

```
sum(sample(0:1, 10, replace = T, prob = c(0.2, 0.8)))
```

```
## [1] 10
```

Using built in binomial generator:

```
rbinom(1, size = 10, prob = 0.8)
```

```
## [1] 8
```

Generate a sample of size 5 from this binomial distribution.

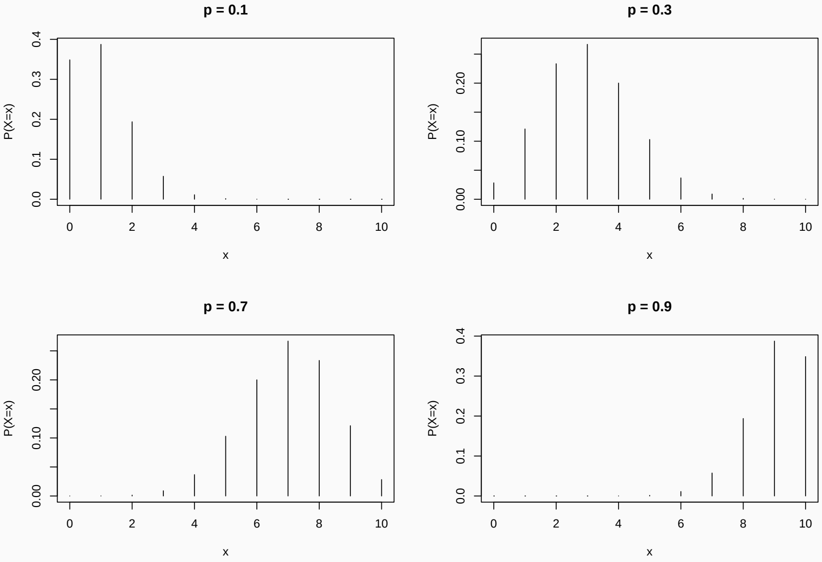
```
rbinom(5, size = 10, prob = 0.8)
```

```
## [1]  9  8 10  8  8
```

Binomial distributions for $n = 10$ and different p

```
par(mfrow = c(2,2))
plot(0:10,dbinom(0:10, 10, 0.1),
     type="h", xlab = "x", ylab = "P(X=x)",
     main = "p = 0.1")
plot(0:10,dbinom(0:10, 10, 0.3),
     type="h", xlab = "x", ylab = "P(X=x)",
     main = "p = 0.3")
plot(0:10,dbinom(0:10, 10, 0.7),
     type="h",xlab = "x", ylab = "P(X=x)",
     main = "p = 0.7")
plot(0:10,dbinom(0:10, 10, 0.9),
     type="h", xlab = "x", ylab = "P(X=x)" ,
     main = "p = 0.9")
```

Binomial distributions for $n = 10$ and different p



Distributions in R

R comes with many named distributions built in.

The functions below illustrate a pattern in R:

- `dbinom(x, size, prob)`
- `pbinom(q, size, prob)`
- `rbinom(n, size, prob)`

We will soon see

- `dunif`, `punif`, `runif`
- `dnorm`, `pnorm`, `rnorm`

Where does $\binom{n}{k}$ come from?

A Gallup survey suggests that 25% of Americans are obese. Suppose we randomly select four individuals to participate in this experiment. What is the probability that exactly 1 of them will be obese?

Where does $\binom{n}{k}$ come from?

Let's call these people Alex (A), Brian (B), Carol (C), and Dalia (D). Only four scenarios will satisfy the condition of "exactly 1 of them is obese":

A	B	C	D	Probability
$\underbrace{NO}_{0.75}$	$\underbrace{NO}_{0.75}$	$\underbrace{NO}_{0.75}$	$\underbrace{O}_{0.25}$	$0.75^3 0.25 = 0.1055$
NO	NO	O	NO	$0.75^3 0.25 = 0.1055$
NO	O	NO	NO	$0.75^3 0.25 = 0.1055$
O	NO	NO	NO	$0.75^3 0.25 = 0.1055$

The probability of exactly one 1 of 4 people being obese is the sum of all of these probabilities.

$$0.1055 + 0.1055 + 0.1055 + 0.1055 = 4 \times 0.1055 = 0.422$$

$$4 \times 0.25 \times 0.75^3 = 0.422$$

number of ways of choosing 1 slot for obese out of 4 slots $\times$ $P(O) \times P(NO)^3$

$$\binom{4}{1} \times p \times (1-p)^4 = 0.422$$

$$\binom{4}{1} = \frac{4!}{1!(4-1)!} = \frac{4.3.2.1}{1(3.2.1)} = 4$$

More generally, k successes in n trials

- Step 1: an outcome with k success and $(n-k)$ failures has probability $p^k(1-p)^{n-k}$.

$$\underbrace{\underline{S}\quad\underline{S}\quad\cdots\quad\underline{S}}_{k\text{ trials}}\quad\underbrace{\underline{F}\quad\cdots\quad\underline{F}}_{n-k\text{ trials}}$$

- Step 2: there are $\binom{n}{k}$ possible outcomes with k success and $(n-k)$ failure.
- Step 3:

$$\begin{aligned}P(\#\text{success}=k) &= (\#\text{outcomes with } k \text{ successes}) \times P(\text{each outcome})\\ &= \binom{n}{k} \times p^k(1-p)^{n-k}.\end{aligned}$$

Summary:

- Named discrete distributions
  - Discrete uniform
  - Binomial Distribution

Next:

- Named continuous distributions
  - Uniform Distribution
  - Normal Distribution