Useful Mathematical Tools

Summation Notation: $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$

Product Notation: $\prod_{i=1}^{n} b_i = b_1 b_2 b_3 \cdots b_n$

Power Properties:

$$a^b a^c = a^{(b+c)} \quad \Rightarrow \quad \prod_{i=1}^{n} (a^{x_i}) = a^{x_1} a^{x_2} \cdots a^{x_n} = a^{x_1+x_2+\cdots+x_n} = a^{\sum_{i=1}^{n} x_i}$$

$$b^a c^a = (bc)^a \quad \Rightarrow \quad \prod_{i=1}^{n} (x_i^a) = x_1^a x_2^a \cdots x_n^a = (x_1 x_2 \cdots x_n)^a = \left(\prod_{i=1}^{n} x_i \right)^a$$

Indicator Function:

Let Ω be a space with elements ω. Let A be a subset of Ω. Then the indicator function of A is given by

$$\mathbb{1}_A(\omega) = 1_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A \end{cases}$$

Properties of Indicator Function:

$$\mathbb{1}_A(\omega) = 1 - \mathbb{1}_{A^c}(\omega)$$

$$\mathbb{1}_{A_1 \cup A_2 \cup \ldots \cup A_n}(\omega) = \max\{\mathbb{1}_{A_1}(\omega), \mathbb{1}_{A_2}(\omega), \ldots, \mathbb{1}_{A_n}(\omega)\} = \begin{cases} 1, & \text{if } \omega \in A_i \text{ for some } i \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{1}_{A_1 \cap A_2 \cap \ldots \cap A_n}(\omega) = \min\{\mathbb{1}_{A_1}(\omega), \mathbb{1}_{A_2}(\omega), \ldots, \mathbb{1}_{A_n}(\omega)\} = \begin{cases} 1, & \text{if } \omega \in A_i \text{ for all } i \\ 0, & \text{o.w.} \end{cases}$$

Special Case of Indicator Function: $\mathbb{1}_{(a,b)}(\omega) = \begin{cases} 1, & \text{if } \omega \in (a,b) \\ 0, & \text{if } \omega \notin (a,b) \end{cases}$

Arithmetic Series: $1 + 2 + \ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Geometric Series:

Finite Sum: $\quad a + a \cdot r + a \cdot r^2 \ldots + a \cdot r^n = \sum_{i=0}^{n} ar^i = \frac{a\left(1 - r^{n+1}\right)}{1 - r}$

Infinite Sum: $\quad$ if $|r| < 1$, then $\sum_{i=o}^{\infty} ar^i = \frac{a}{1-r}$

Binomial Theorem:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \quad \text{for } a, b \in \mathbb{R},\ n \in \mathbb{Z}^+$$

Expansion of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for any } x$$

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \quad \text{for any } x$$

Integration by Parts:

Suppose that f, g are continuously differentiable function on the interval $[a, b]$, then

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx.$$

Monotone Functions:

A monotone function is the one which is always either increasing or decreasing.
A sufficient condition to check monotonicity:

- if $g'(x) > 0$ for all x, then $g(x)$ is a monotone increasing function.
- if $g'(x) < 0$ for all x, then $g(x)$ is a monotone decreasing function.

Factorial: $n! = n(n-1)(n-2)\cdots(3)(2)(1)$

Gamma Function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \quad \text{for any } \alpha > 0$$

Facts about the gamma function:

- For $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
- If $n \in \mathbb{Z}^+$, then $\Gamma(n) = (n-1)!$
- $\Gamma(1) = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$