Acceleration methods: Reducing variance

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Acceleration methods

Clever methods to reduce the variance of a Monte Carlo estimate of an integral.

- Antithetic variables
- Control variates

Antithetic variables

This is another variance reduction method. The idea is that it may be helpful to generate samples of correlated variables in estimating integrals

Recall, covariance of X and Y is

$$Cov(X,Y) = E((X-EX)(Y-EY)) = E(XY) - (EX)(EY)$$

Suppose X_1 and X_2 are identically distributed then

$$Var(\frac{X_1+X_2}{2}) = \frac{1}{4}(Var(X_1) + Var(X_2) + 2Cov(X_1, X_2))$$

So, variance is reduced if X_1 and X_2 are negatively correlated.

Antithetic Variables approach

Given two samples $X_1, X_2, \ldots, X_m \sim f$ and $Y_1, Y_2, \ldots, Y_m \sim f$ for estimating

$$I = \int h(x)f(x)dx$$

where X_i and Y_i are **negatively correlated** and h is a monotone(increasing or decreasing) function.

Define an *antithetic pair*

$$\frac{h(X_i) + h(Y_i)}{2}$$

then the *antithetic estimator*

$$\hat{I} = \frac{1}{m}\sum_{i=1}^{m} \frac{h(X_i) + h(Y_i)}{2}$$

is more efficient than the estimate based only on X's and Y's.

Question:

How to get negatively correlated X_i and Y_i

- Using inversion
  - $U \sim \mathrm{Unif}(0,1)$
  - U and $1-U$ are negatively correlated
  - So, $X = F^{-1}(U)$ and $Y = F^{-1}(1-U)$ are negatively correlated.
- Use symmetry
  - $X \sim f(x)$ which is symmetric about μ
  - $X - \mu$ has the same distribution as $\mu - X$
  - So X has the same distribution as $2\mu - X$

Example: Want to compute

$$\int_0^\infty x^2 e^{-x} dx$$

$g(x) = x^2$ is monotone on 0 to ∞, so could use antithetic approach.

- Be careful, $g(x) = x^2$ is not monotone on $(-\infty, \infty)$.

1. Boring way
    - Generate $X_1, X_2, \ldots, X_m \sim \mathrm{Exp}(1)$

$$\hat{I} = \frac{1}{m} \sum_{i=1}^{m} X_i^2$$

2. Better way
    - Generate $U_1, U_2, \ldots, U_m \sim \mathrm{Unif}(0, 1)$
    - Let $X_i = -\ln(U_i)$ and $Y_i = -\ln(1 - U_i)$

$$\hat{I} = \frac{1}{m} \sum_{i=1}^{m} \frac{X_i^2 + Y_i^2}{2}$$

Activity:

For the above example, compare antithetic variable approach with simple monte-carlo. What is the reduction in variance?

Control Variates

Another variance reduction technique.

Problem: We are looking to estimate EY.

The idea here is that you have a bi-variate random variable (C, Y) and a simulated sample of size m:

$$(c_1, y_1), (c_2, y_2), \cdots, (c_m, y_m)$$

Suppose we know EC. We can use this to speed up convergence of EY.

When C and Y are strongly positively correlated then variance is improved over the naive estimator $\bar{y}$. C is called the control variable.

Control Variates (continued)

Problem: We are looking to estimate EY.

Boring way to estimate EY is

$$\bar{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$$

Notation: $EC = \mu_c$ is known in advance. σ_C^2 is the variance of C, σ_Y^2 is the variance of Y and $\rho_{C,Y}$ is the correlation between C and Y

Consider,

$$\bar{Y}' = \bar{Y} - b(\bar{C} - \mu_C) = \frac{1}{m}\sum_{i=1}^{m} Y_i - C_i + \mu_C$$

Then, $EY = EY'$ and $Var(Y') = Var(Y) + b^2 Var(C) - 2b\rho_{C,Y}$.

If $b > 0$, C, Y are positively correlated, then $Var(Y') < Var(Y)$ but other possibilities exist.

Finance examples use these methods (Geometric Brownian Motion, Asian call options)