Bakytzhan Urazbekov

In this work, the following light weakly bound atomic nuclei are studied: He, Li, and Be. A three-body model + 2N for He and Li, and 2 + N model for Be are applied. The wave function of the three-body system is obtained within the framework of the Stochastic Variational Model based on the Gaussian basis. The interaction potentials for three-body systems have a Pauli projector, which excludes forbidden states. An analytical expression is obtained for the density distribution function of nuclear matter using the three-body wave function. The root-mean-square matter radii of He, Li, Be nuclei are calculated and given comparisons with other sources.

The interactions of He, Li, and Be with the simplest particles, such as , are studied in detail. On the basis of the calculated density distribution functions, the folding interaction potentials are obtained. The resulting folding interaction potential is applied to calculate the differential cross sections of elastic scattering in the framework of the optical model. Using the Coupled Channels approach for the Be induced reactions the role of different reaction channels is studied, and deformation parameters are derived. The comparisons of theoretical cross sections with experimental data in inelastic channels for nuclear reaction + Be + Be are given.

Within the framework of the Coupled Reaction Channels method, the cross sections for nuclear reactions of single-nucleon and cluster transfers are calculated taking into account the internal structure of He, Li, and Be nuclei. The contributions of the -cluster and nucleon involved transfer channels for He, Li, and Be are demonstrated. The obtained results based on the proposed theoretical approach reflect the cluster structure of the exploring nuclei.

Doctoral Thesis

*Author:*

*Supervisor:*  
 *Co-Supervisor:*  
[Prof. Andrey Denikin](https://www.researchgate.net/profile/A_Denikin)

image

I, , declare that this thesis titled, “” and the work presented in it are my own. I confirm that:

* This work was done wholly while in candidature for a research degree at .
* Where I have consulted the published work of others, this is always clearly attributed.
* Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
* I have acknowledged all main sources of help.

   
Signed:

Date:

“ *By the love of God, the nations are created,*  
*And you love them like yourself.*  
*Love people as brothers, like freedom,*  
*Then truth and life are for you.* ”

Abay Qunanbayev  
*Kazakh poet and philosopher*

I believe that if it were not for these people, then the thesis in the form, in which you are reading it, would not have taken place.

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ll

NCSM & No Core Shell Model  
RGM & Resonating Group Method  
GCM & Generator Coordinate Method  
SVM & Stochastic Variational Method  
DWBA & distorted wave Born approximation  
CRC & coupled reaction channels  
OM & optical model  
DF & double folding  
CC & coupled channel  
SA & spectroscopic amplitude  
cm & center of mass  
rms & root-mean-square

lr@l

Elementary electric charge & & 1.602 C  
Reduced Planck constant & & 6.582 eVs

lll

Symbol & Name & Unit

& distance & fm  
 & wavenumber & fm  
 & energy & MeV  
 & density function & fm  
 & atomic mass & u  
 & cross section & mb  
 & differential cross section & ${\rm{mb}}\setminus{\rm{sr}}$

# Introduction

Humanity has always been interested in the structure of the Universe - how it works, what it consists of. The understanding and naming of modern objects such as molecules and atoms began in antiquity. The meaning of the word atom, in Greek , is indivisible, or the uncut particle of matter. Modern science has gone much deeper and determined that the atom is actually a complex particle. According to the standard model, the fundamental particles are now six quarks, six leptons and their corresponding anti-particles (for more details, see ). This has been achieved mainly through theoretical predictions and the development of integrated complex technology, detection systems designed to detect even the smallest particle of fundamental interactions. Particularly, in the modern nuclear physics the experimental techniques have also been successfully developed in the field of the production of secondary beams. Secondary beams, being rare and unstable nuclei, allow studying the properties and characteristics of the dripline nuclei, challenging the nuclear physics.

In recent years, the study of light weakly bound nuclei has not become less interesting due to the successful development of experimental facilities . It is known that in nuclei the nucleons tend to group into clusters. A well known manifestation of the cluster structure in heavy nuclei is the -decay of U, first experimentally discovered by E. Rutherford . The -decay means that -cluster can be formed in the uranium nucleus as an individual subsystem with the following emission. The first quantitative theory of -decay was developed by G. Gamov (see ). In addition to -decay, there is also a cluster decay, where an emitted particle can be a heavy nucleus, for example – C .

There are many theoretical approaches devoted to studying the structure of exotic nuclei. In particular, the nuclear excitation function for light exotic nuclei is well reproduced by the No Core Shell Model (NCSM) . The method uses one-particle basis function of the harmonic oscillator, realistic NN, NNN interactions. This method is reduced to solving the -nucleon Schrödinger equation on the basis containing all possible configurations of -nucleon oscillator functions. However, the size of the basis grows rapidly with the number of nucleons, the reliability of the NCSM model calculations decreases in the case of heavy nuclei. At present, the capabilities of modern computational machines make it possible to calculate with sufficient accuracy for nuclei with masses . For example, in Ref. within the framework of this model, the excitation spectrum of the one-particle basis for the 9Be nucleus is in good agreement with the experimental data .

The microscopic description, that is, taking into account the NN effective potential, of the break-up reaction is presented in the Resonating Group Method (RGM) . Generalization of the RGM consists in constructing overall mathematically equivalent methods for the simultaneous calculation in the nuclear structures and nuclear reactions. The practical application of this method is also limited in the region of the lightest nuclei due to the lack of computational power in the antisymmetrization process. In the work the algebraic RGM was successfully applied in describing the internal structure 6He, 6Li, and He nuclei, their rms matter and proton and neutron radii, as well as nuclear matter distribution density were calculated. Also the fusion reactions at low energies with 6He involved were presented.

Similar to the RGM , is the Generator Coordinate Method (GCM), if one takes the full model space of the basis Bloch-Brink wave functions . In the framework of the GCM it is possible to carry out calculations for the nuclei with medium masses taking into account a few clusters. For scattering problems, the RGM can be applied rather straightforwardly because the inter-cluster wave function is explicitly treated. On the other hand, in the application of the GCM to scattering problems, it is necessary to connect the basis wave functions in the internal region with continuum states in the asymptotic region at a chosen channel radius . Microscopic three-body calculations for two valence neutrons around a core nucleus have been achieved by many groups to investigate the neutron halo and two-neutron correlation in drip-line nuclei such as 6He and Be . The Be using the GCM has been studied . In this work a negative-parity band was found, electromagnetic transition probabilities and partial widths of molecular states were calculated.

In addition to the above theoretical approaches, there is the Stochastic Variational Method (SVM) . The SVM uses the Gaussian function as a basis function, which is convenient for deriving analytically matrix elements of the Hamiltonian. The corresponding Hamiltonian contains a Pauli projector , that removes forbidden states. The variational method describes well the internal structures of 6He, 6Li and 9Be. In particular, the electromagnetic properties, geometric parameters and configuration weights are in good agreement with the methods, where they are obtained by exact solutions of three-body problems.

To describe the elastic scattering in the field of a complex potential, one usually uses the Optical Model (OM), first presented in Ref. . Within the framework of this model, the nucleus is considered as a transparent substance that has the ability to absorb. In practice, this method describes the experimental data well providing the direct information on the nucleus-nucleus interaction potential. However, the ambiguity of the optical potential parameters reduces the merits of optical model. To study the inelastic channels, transfer reactions and their use to describe experimental data the Distorted Wave Born Approximation (DWBA) method has been successfully applied . The DWBA method proved to be appropriate and successful in describing the cross section of nuclear reactions involving the ground and low-lying excited nuclear states of single-particle as well as collective nature. The method becomes inadequate, since the reaction leads to the broad or strongly overlapped final states. In this case the couples channel formalism has to be applied.

Schematic representation of the cluster structure of some nuclei, and their half-life, if unstable. Alpha particle, proton and neutron are marked in yellow, red and orange, respectively. The allocation of 6He, 6Li and 9Be nuclei means the selected as objects of study. Read the text for more details..

In light nuclei, the cluster structure can often clearly manifest itself in the dripline region of the nuclear map. These series include such nuclei as 6He, Li, Be and other exotic nuclei. However, there are also stable nuclei with apparent cluster structures. These include such nuclei as 6Li, 9Be, B and others. Numerous experimental data allow us to treat such nuclei as multi-cluster systems, including tightly bound -clusters and valence nucleons. A visual representation of how these nuclei can be arranged in the cluster model is illustrated in Fig [1](#fig:exotic_nuclei). For these nuclei the relative motion of internal subsystems determines the property and mechanisms of nuclear reactions. In the figure, we see that the simplest nuclei with a cluster structure are 6He, 6Li, and 9Be. Their structures are ideally suited to consideration within the framework of the three-body model. Moreover, it should be noted that the interaction of pairs inside these nuclei is well known, which can be used in constructing a solution to the Schrödinger equation. An important factor is also the number of scattering experiments conducted to study the structures of these nuclei. Therefore, the objects for investigation in this work have been taken 6He, 6Li, and 9Be nuclei.

The experimental data of elastic scattering of the 6He+ and 6Li+ data is available at laboratory energies 151 MeV and 166 MeV . This experimental data can be used to show the elastic transfer phenomenon through two-neutron transfer between the two core. The main motivation of these studies was to probe the relative importance of the "di-neutron" and "cigar" configurations of the valence neutrons in the 6He ground state. The data show the backward angle rise in cross section. Usually it is considered as an indication of characteristic of elastic transfer. The DWBA analysis using the 6He and 6Li wave functions built on hyper-spherical harmonics , showed good agreement with the backward angle data. Consequently, it is concluded that the di-neutron configuration dominates the 6He ground state wave function. The more detailed study of the 6He+ scattering data has carried out in Ref. in the framework of the Coupled Reaction Channels (CRC) method. In this work the spectroscopic information of the di-neutron was obtained, a sequential transfer of the di-neutron was implemented in CRC calculations. It was concluded that the two valence neutrons are transferred mainly simultaneously, the rising of cross section at backward angles comes due to the elastic transfer of the system.

Based on the Borromean structure of Be, special attention was paid to the breakup processes resulting from the Be(Li, Li)Be nuclear reaction . The excited nucleus Be can decay either directly into the three-body system or through one of the unstable nuclei, such as He and Be. These relatively recent experimental studies explicitly confirm the cluster structure of Be. The calculated branching ratios show that the low-lying excited states, at E 4.0 MeV, are mostly populated with the Be+n configuration. In other conditions, the He+ configuration plays a significant role. Another aspect of finding the cluster structure is its effect on the nuclear reaction mechanisms. The investigations of works show that the multi-particle-multi-hole structures are expected at rather low excitation energies in nuclei. In such a case, it can be understood that the correlated nucleons are transferred as a whole strongly correlated cluster, which has the internal quantum numbers of a free particle.

In the present work, the three-body structure of nuclei and their role in the mechanisms of nuclear reactions are investigated. Usually, in the descriptions of the cross sections for nuclear reactions, the intrinsic structure of colliding nuclei in the interaction potentials is neglected, taking them in a phenomenological way, for example, in Gaussian form, or in the form of the Woods-Saxon potential . In such cases, one should take the potential corresponding to the realistic shapes of colliding nuclei. To describe direct nuclear reactions, we proceed from the realistic shape of colliding nuclei within the three-body model. Therefore, the motivation for this work is to minimize the number of free parameters in the calculations of nuclear reactions as much as possible, and to use the density distribution functions of nuclear matter based on the three-body model. And also the main goal of this work is to understand how the mechanisms of interaction of nuclear reactions with the participation of weakly bound nuclei 6He, 6Li, and 9Be occur.

This work is organized as follows. The first chapter is devoted to an introduction to the three-body model based on the SVM method. This chapter describes the advantage of using the Gaussian basis function, the Hamiltonian, which includes the Pauli principle, the transformation of the basis function, and provides formulae for calculating the density function of nuclear matter within the three-body model. The next chapter presents widely used theoretical approaches to describe the scattering wave function. Methods for calculating differential cross sections are presented: for elastic scattering – Optical model, inelastic scattering – Coupled channel, and nuclear transfer reactions – Coupled Reactions Channels. For a complete understanding of how one-particle spectroscopic amplitudes (SA) are calculated within the framework of the shell model used in CRC calculations, a separate section is devoted. The details of building the potential of interaction in the framework of DF model are presented. The last three chapters present the results of the calculations. The comparative analysis of experimental data and theoretical calculations has been performed. Finally, the main findings are presented in the conclusion.

Some parts of this work have been published as follows:

* Urazbekov, B. A., Denikin, A. S., Lukyanov, S. M., Itaco, N., Janseitov, D. M., Mendibayev, K., ... and Harakeh, M. N. (2019). Journal of Physics G: Nuclear and Particle Physics, 46(10), 105110.
* Urazbekov, B. A., Denikin, A. S. (2018). Eurasian Journal of Physics and Functional Materials, 2(1), 17-22.
* Urazbekov, B.A., Denikin A.S., Itaco, N., Janseitov, D., (2021) Physics of Atomic Nuclei, to be published.
* Janseitov, D. M., Urazbekov, B.A., Lukyanov, S. M., Mendibayev, K., Penionzhkevich, Y. E., Skobelev, N. K., Sobolev, Y. G., ... and Khlebnikov, S. V. (2018). International Journal of Modern Physics E, 27(10), 1850089.
* Lukyanov, S. M., Urazbekov, B.A., Denikin, A. S., Naumenko, M. A., Burjan, V., Trzaska, W. H., Harakeh, M., ... and Glagolev, V. (2017). Exotic Nuclei, pp. 74-80 (2019).
* Urazbekov, B. A., Denikin, A. S., Itaco, N., and Tursunbayev, N. T. (2020). Eurasian Journal of Physics and Functional Materials, 4(3), 201-212.
* Burtebayev, N., Janseitov, D. M., Urazbekov, B. A., Kerimkulov, Z., Alimov, D., Nassurlla, M., Valiolda, D. S., ... and Danilov, A. N. (2020). Journal of Physics: Conference Series (Vol. 1555, p. 012032).

The materials of this work have been presented in the international conferences:

* International conference “Nucleus – 2018” “Fundamental problems of nuclear physics, nuclear energy industry, and nuclear technology”, 2-6 July 2018, Voronezh, Russia;
* The LXIX International Conference "Nucleus-2019" on Nuclear Spectroscopy and Nuclear Structure "Fundamental Problems of Nuclear Physics, Nuclei at Borders of Nucleon Stability, High Technologies", 1-5 July 2019, Dubna, Russia;
* The LXX International conference "NUCLEUS – 2020. Nuclear physics and elementary particle physics. Nuclear physics technologies", 11-17 October 2020, online;
* The XXIV International Scientific Conference of Young Scientists and Specialists (AYSS-2020), 9 - 13 November 2020, online;

# The three body model

The wave function for the three-body system in the current work was calculated within the model proposed in Ref. . The work based on the three body model provides great progress in understanding the structure and properties of the light weakly bound nuclei, like He, Li, Be nuclei. In addition, a lot of research has been done in different directions of physics using this wave function. Below we list some of them:

* electromagnetic properties;
* low lying excitation spectra;
* root-mean-square charge radii , magnetic , quadrupole and octupole moments;
* processes of Li interaction with hadrons, including quasi-elastic scattering (, 2), (, ) and (, );
* high energy proton and the lightest nuclei scattering on Li
* scattering of -mesons and -capture by Li
* parameters of -decay of He;
* the thermonuclear reactions in the D-He-Be plasma;
* properties of the six-quark dibaryons in nuclei with =6 and *etc*.

Since this wave function was widely used, we chose it as the wave function to describe the three-body system in this work.

## Three body wave function

Consider a three body system containing and particles. A total wave function of this system with total spin and spin projection can be represented as a sum of components

$$\Psi^{JM}\left( {\bf x}\_k, {\bf y}\_k \right)=\sum\_\gamma\Psi^{ J M}\_\gamma\left( {\bf x}\_k, {\bf y}\_k \right).
\label{totwf}$$

The vector ${\bf x}\_k$ is a vector of the relative distance between the pair of particles and , and ${\bf y}\_k$ is the vector of the relative distance between the center of mass of the pair and the particle (see Fig. [1.1](#fig:jacobiSet)). The designation is equivalent to the set of quantum numbers . The momenta , are the orbital momenta conjugated to the coordinates ${\bf x} \_k$ and ${\bf y} \_k$ respectively, is the total orbital momentum of the system and is the total spin of three body system. The each component defines spatial and spin parts as follow

$$\begin{aligned}
\Psi^{ J M}\_\gamma\left( {\bf x}\_k, {\bf y}\_k \right) = & \left[
\Phi^{(\lambda,l)}\_L \left({\bf x}\_k, {\bf y}\_k \right) \otimes
\chi\_S(k, pq)
\right]\_{JM} = \\
=&
\sum\_{M\_L M\_S} (LM\_LSM\_S | JM) ~
\Phi^{(\lambda,l)}\_{LM\_L}\left( {\bf x}\_k, {\bf y}\_k \right)~
\chi\_{SM\_S}(k, pq) \nonumber\end{aligned}$$

The spin function of three body system having two fermions is given by

In the case of system, which has only fermion , the total spin function becomes

The spatial part of the wave function is chosen to be multidimensional Gaussian functions of the form

$$\begin{aligned}
\Phi^{(\lambda,l)}\_{LM\_L} \left({\bf x}\_k, {\bf y}\_k \right)=&
x^\lambda y^l
\sum\_i C\_i
\exp\left( - \alpha\_i^{(k)} x^2\_k - \beta\_i^{(k)} y^2\_k \right)~
\times \nonumber \\
& \times \left[
Y\_\lambda \left(\hat{x}\_k \right) \otimes Y\_l \left(\hat{y}\_k \right)
\right]\_{LM\_L}.
\label{spatial\_part}\end{aligned}$$

Here, the coefficients are the parameters of the wave function expansion and are found as a result of solving the generalized eigenvalue problem. The linear parameters , make the wave function more complete, if they are represented in the following form

The schemes of Jacobi coordinate sets for the three body system.

A pair of the three body system may have forbidden states according to the principle Pauli. For instance, lets say, the pair has the state must be excluded. This forbidden state is projected out by implementing the pseudo-potential :

where is a regular interaction potential between and particles, – a constant usually equal to . By varying this parameter the binding energy of three body system remains stable. The operator is given by:

So the three body Hamiltonian including kinetic energy and pseudo potentials looks as

The Pauli principle plays a huge role in the structure of the nucleus, particularly it does not allow overlapping of two constituent particles, while maintaining the Paili principle. This approach, so-called method of orthogonalizing pseudo-potentials, was previously developed by the group of V. Kukulin . The method widely used not only in the construction of the wave function of the bound state, but also in the scattering theory .

## Transformation of the basis function

The main advantage of the Gaussian basis functions, is in simplicity of transformation into an alternative set of Jacobi coordinates . The transformation from the sets to may be accomplished as follows:

$$\begin{pmatrix}
{\bf x}\_k \\
{\bf y}\_k
\end{pmatrix} = {\bf T}^{(kp)}
\begin{pmatrix}
{\bf x}\_p \\
{\bf y}\_p
\end{pmatrix},$$

where, the ${\bf T}^{(kp)}$ matrix elements is given by

$${\bf T}^{(kp)} =
\begin{pmatrix}
c\_{kp}&s\_{kp} \\
-s\_{kp}&c\_{kp} \\
\end{pmatrix},$$

where the coefficients and are given by

$$\begin{aligned}
c\_{kp}=-\left( \frac{m\_k m\_p}{(m\_k+m\_q)(m\_p+m\_q)} \right)^{1/2} \\
s\_{kp}=(-1)^{k-p}~{\rm sign} (k-p) (1-c^2\_{kp})^{1/2}.\end{aligned}$$

Consequently, these coefficients are depended only on the masses of particles. Moreover the coefficients follow rules as

The transformation of the spatial part ([[spatial\_part]](#spatial_part)) of the wave function can be expressed in the following way

$$\label{transform\_spatial\_part}
\Phi^{(\lambda,l)}\_{LM\_L}\left( {\bf x}\_k, {\bf y}\_k \right) = \sum\_{\tilde{\lambda}, \tilde{l}} A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf T}^{(kq)} }~
\Phi\_{LM\_L}^{(\tilde{\lambda},\tilde{l}) } \left( {\bf x}\_q, {\bf y}\_q \right) ,$$

where the summation is proceeded over new quantum numbers in condition , and the new spacial part is given by

$$\begin{aligned}
\Phi\_{LM\_L}^{(\tilde{\lambda},\tilde{l}) } \left( {\bf x}\_q, {\bf y}\_q \right) = &
x^{\tilde{\lambda}}\_q y^{\tilde{l}}\_q
\sum\_{i=1}^{N\_\gamma} C\_i
\exp\left( - \tfrac{1}{2} \alpha\_i^{(q)} x^2\_q -\tfrac{1}{2} \beta\_i^{(q)} y^2\_q
- \rho\_i^{(q)} {\bf x}\_q \cdot {\bf y}\_q
\right)~
\times \nonumber \\
& \times \left[
Y\_{\tilde{\lambda}} \left(\hat{x}\_k \right) \times Y\_{\tilde{l}} \left(\hat{y}\_k \right)
\right]\_{LM\_L}.
\label{new\_spacial\_part}\end{aligned}$$

where new linear parameters can be expressed as

$$\begin{aligned}
\begin{pmatrix}
\alpha\_i^{(q)} & \rho\_i^{(q)} \\
\rho\_i^{(q)} & \beta\_i^{(q)}
\end{pmatrix} = \left( {\bf T}^{(kq)} \right)^{T} \times ~
\begin{pmatrix}
\alpha\_i^{(k)} & \rho\_i^{(k)} \\
\rho\_i^{(k)} & \beta\_i^{(k)}
\end{pmatrix} ~ \times
{\bf T}^{(kq)}.\end{aligned}$$

Note, that for the initial coordinate set the radial wave function ([[spatial\_part]](#spatial_part)) does not include the scalar product ${\bf x}\_k \cdot {\bf y}\_k$, which means .

The coupling coefficient $A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf T}^{(kq)} }$ given in  ([[transform\_spatial\_part]](#transform_spatial_part)) is defined as follows

$$\begin{aligned}
A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf T}^{(kq)} } = & \sum\_{\lambda\_1 \lambda\_2 l\_1 l\_2}
\left({\rm T}\_{11}^{(kq)} \right)^{\lambda\_1}
\left({\rm T}\_{12}^{(kq)} \right)^{\lambda\_2}
\left({\rm T}\_{21}^{(kq)} \right)^{l\_1}
\left({\rm T}\_{22}^{(kq)} \right)^{l\_2}
\times \nonumber
\\
& \times {E}^{\lambda\_1 \lambda\_2 \lambda l\_1 l\_2 l L}\_{\tilde{\lambda} \tilde{l} } \mathcal{D}(\lambda,\lambda\_1,\lambda\_2) \mathcal{D}(l,l\_1,l\_2).
\label{trans\_coef}\end{aligned}$$

Here, the summation is satisfied to the conditions , , , the coefficients and come from re-coupling of the solid spherical harmonics , which are given in Appendix [7](#AppendixA). Switching into the Jacobi set is carried out in analogous way.

The spin part of the wave function transforms as

As for the one fermionic system the spin part remains unchanged.

## Overlap matrix elements

For the convenience of further expressions for the matrix elements we define a basis function for the three body system in form

Here, the coordinate arguments, ${\bf x}\_k$ and ${\bf y}\_k$, are down for convenience, and the radial part of the basis function has form

The overlap matrix elements for the basis function ([[basis\_function\_1]](#basis_function_1)) is expressed as follows

$$\begin{aligned}
\langle \phi^{ J M}\_{\gamma i}\left(k, pq \right) \vert
\phi^{ J M}\_{\gamma^{\prime}j}\left(k, pq \right) \rangle=&
\int \int d{\bf x}\_k d{\bf y}\_k
~\phi^{ J M}\_{\gamma i}\left(k, pq \right)
\left( \phi^{ J M}\_{\gamma^{\prime} j}\left(k, pq \right) \right)^{\*}
= \label{overlap\_in\_basic\_set}
\\
=\int\_0^\infty \int\_0^\infty dx\_k dy\_k~ & x\_k^{2\lambda+2}y\_k^{2 l +2} {\rm exp}\left( - \tfrac{1}{2}\alpha^{(k)}\_{ij} x^2\_k - \tfrac{1}{2} \beta^{(k)}\_{ij} y^2\_k \right) \delta\_{\gamma \gamma'} =
\nonumber \\
=&~
C\_i C\_j
\mathcal{I} \left( 2\lambda+2,\alpha\_{ij}^{(k)} \right)~ \mathcal{I} \left(2l+2,\beta\_{ij}^{(k)} \right) ,\nonumber\end{aligned}$$

where and , the table integral is given in Eq. ([[table\_integral\_1]](#table_integral_1)).

It is also useful to give an expression of overlap matrix elements for alternative Jacobi coordinate set. For instance, the matrix elements for the three body system may be given as follows

$$\langle \phi^{ J M}\_{\gamma i}\left(k,pq \right) \vert
\phi^{ J M}\_{\gamma^{\prime}j}\left(k,pq \right) \rangle
=
\sum\_{\tilde{\gamma} \tilde{\gamma}'}
A^{\tilde{\lambda}' \tilde{l}' \lambda' l'}\_{{\bf T}^{(kq)} }
A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf T}^{(kq)} }
\langle \phi^{ J M}\_{\tilde{\gamma}i}\left(q,kp \right) \vert
\phi^{ J M}\_{\tilde{\gamma}^{\prime}j}\left(q,kp \right) \rangle
\label{overlap\_alternative}$$

The Jacobian matrix ${\bf J}^{(kq)}$ for transformation from the ${\bf x}\_k, {\bf y}\_k$ coordinates to the ${\bf x}\_q, {\bf y}\_q$ coordinates gives the ${\bf T}^{(kq)}$ matrix

$${\bf J}^{(kq)} =
\begin{pmatrix}
\frac{\partial {\bf x}\_k \left({\bf x}\_q,{\bf y}\_q \right)}{ \partial {\bf x}\_q }
& \frac{\partial {\bf x}\_k \left({\bf x}\_q,{\bf y}\_q \right)}{ \partial {\bf y}\_q } \\
\frac{\partial {\bf y}\_k \left({\bf x}\_q,{\bf y}\_q \right)}{ \partial {\bf x}\_q }
& \frac{\partial {\bf y}\_k \left({\bf x}\_q,{\bf y}\_q \right)}{ \partial {\bf y}\_q }
\end{pmatrix} = {\bf T}^{(kq)}.$$

Accordingly, the determinant $\vert {\bf J}^{(kq)} \vert$ is a determinant of the ${\bf T}^{(kq)}$ matrix, which equals to :

$$\vert {\bf J}^{(kq)} \vert = \vert {\bf T}^{(kq)} \vert =1.$$

Therefore, the integration variables in Eq. ([[overlap\_alternative]](#overlap_alternative)) can be changed without any factorization.

The spacial part ([[new\_spacial\_part]](#new_spacial_part)) in the alternative coordinate set has the scalar product, $\exp \left(- \alpha\_i^{(q)} \beta\_i^{(q)} {\bf x}\_q \cdot {\bf y}\_q \right)$. With this form the overlap matrix elements can be calculated directly, using the expansion of the exponential function . However it is possible to project the scalar product into ${\bf T}$ matrix as follows:

$${\bf Q}\_i^{(kq)}= {\bf T}^{(kq)} \times
\begin{pmatrix}
1 & 0 \\
-\tfrac{\rho^{(q)}\_{i}}{\alpha^{(q)}\_i} & 1
\end{pmatrix} .
\label{q\_rotation\_matrix}$$

Consequently, the radial part ([[new\_spacial\_part]](#new_spacial_part)) of the wave function can be rewritten without the scalar product term as

$$\begin{aligned}
\Phi\_{LM\_L}^{(\tilde{\lambda},\tilde{l}) } \left( {\bf x}\_q, {\bf y}\_q \right) = &
x^{\tilde{\lambda}}\_q y^{\tilde{l}}\_q
\sum\_{i=1}^{N\_\gamma} C\_i
\exp\left( - \tfrac{1}{2}
{\alpha'}^{(q)}\_{i}
x^2\_q -\tfrac{1}{2} \beta\_i^{(q)} y^2\_q \right)
\times \nonumber \\
& \times \left[
Y\_{\tilde{\lambda}} \left(\hat{x}\_k \right) \times Y\_{\tilde{l}} \left(\hat{y}\_k \right)
\right]\_{LM\_L}.
\label{new\_spacial\_part}\end{aligned}$$

with

One can get easily then an expression for the overlap matrix element in alternative Jacobi coordinate set as follows

Taking into account Eq.([[matrix\_element\_alter\_set]](#matrix_element_alter_set)) the matrix element ([[overlap\_alternative]](#overlap_alternative)) can be then rewritten as follows

$$\begin{aligned}
\langle \phi^{ J M}\_{\gamma i}\left(k,pq \right) \vert
\phi^{ J M}\_{\gamma^{\prime}j}\left(k,pq \right) \rangle
= \label{overlap\_to\_q}
\sum\_{\tilde{\gamma} \tilde{\gamma}'}
A^{\tilde{\lambda}' \tilde{l}' \lambda' l'}\_{{\bf T}^{(kq)} }
A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf T}^{(kq)} }
\langle \phi^{ J M}\_{\tilde{\gamma} i}\left(q,kp \right) \vert
\phi^{ J M}\_{\tilde{\gamma}^{\prime}j}\left(q,kp \right) \rangle=
\nonumber \\
=
\sum\_{\substack{\tilde{\gamma} \tilde{\gamma}'\\i=1..N\_\gamma \\j=1..N\_{\gamma'}}}
A^{\tilde{\lambda}' \tilde{l}' \lambda' l'}\_{{\bf Q}^{(kq)}\_{ij} }
A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q}^{(kq)}\_{ij} }
C\_i C\_j
\mathcal{I} \left( 2 \tilde{\lambda}+2,~
\alpha'^{(q)}\_{ij}
\right)
\mathcal{I} \left( 2 \tilde{l}+2, ~
\beta\_{ij}^{(q)} \right)
\delta\_{\tilde{\gamma}\tilde{\gamma}^{\prime}}. \nonumber \\
{}\end{aligned}$$

Notably, the coefficient $A^{\tilde{\lambda} \tilde{l} \lambda l}\_{ \tiny {\bf Q}^{(kq)}\_{ij} }$ should now become depended on the rotation matrix ${\bf Q}^{(kq)}\_{ij}$. For simplicity the indexes of the rotation matrix are down in the next sections, consequently, the recoupling coefficient turns $A^{\tilde{\lambda} \tilde{l} \lambda l}\_{ {\bf Q} }$.

## Normalization and correlation density function

By means of the overlap matrix elements ([[overlap\_in\_basic\_set]](#overlap_in_basic_set)) the weight of the component may be represented as follows

This, in turn, gives the the normalization of the total three-body wave function as follows

A correlation density function of the total wave function ([[totwf]](#totwf)) can be expressed in the following way

with

Here, the summation goes over and .

## Density distribution functions

The density distribution function of nuclear matter within the three body model can be expressed as follows

$$\rho({\bf R}) = \sum\_{\iota=\{kpq\}} \rho^{(\iota)} ({\bf R})$$

where . The density function of a cluster is given by

$$\rho^{(\iota)}({\bf R})=
\langle~ \Psi\_{tot}^{JM} ~\vert~ \hat{\rho}\_i~ \vert~ \Psi\_{tot}^{JM} ~\rangle,$$

here, the operator of density is defined as

$$\hat{\rho}\_i\equiv
\begin{cases}
\delta \left( {\bf y}\_i - y\_0^{(i)} {\bf R} \right)
& ~{\rm for}~ i{\rm \text -th~nucleons} , \\
\rho\_{\alpha} \left( {\bf y}\_i - y\_0^{(i)} {\bf R} \right) &
~{\rm for}~ i{\rm \text -th~\alpha\text -clusters},
\end{cases}$$

where $\delta({\bf z})$ – delta function, $\rho\_\alpha({\bf r})$ – internal density distribution function of -cluster:

$$\rho\_{\alpha}({\bf r}) = \rho\_0 \exp \left( - \gamma\_0 {\bf r}^2 \right)$$

The -particle density function is normalized to unity with the following parameters

Here, the square root of – rms matter radius of -particle, which equals to 1.461 .

Write a matrix element of the nucleon operator $\delta \left( {\bf y}\_k-y^{(k)}\_0 {\bf R} \right)$ via the basis functions:

$$\begin{aligned}
&\langle \phi^{JM}\_{\gamma i} (k,pq) \vert
\delta \left( {\bf y}\_k-y^{(k)}\_0 {\bf R} \right)
\vert \phi^{JM}\_{\gamma' j} (k,pq) \rangle =
\label{me\_delta\_k}\\
& =C\_i C\_j \int \int d{ x}\_k d{ y}\_k ~x\_k^{2} ~y^{2}\_k
~\varphi^{(\lambda,l)}\_i(x\_k,y\_k)
~\varphi^{(\lambda,l)}\_j (x\_k,y\_k)
\frac{\delta \left( { y}\_k-y^{(k)}\_0 { R} \right)}{y\_k^2} ~
\delta\_{\gamma \gamma'}~ =
\nonumber \\
&=
C\_iC\_j
\left( y\_0^{(k)} \right)^{2l}
R^{2l}
\exp \left( -\tfrac{1}{2} {y\_0^{(k)}}^2 \beta^{(k)}\_{ij} R^2 \right)
\mathcal{I}\left( 2\lambda+2, \tfrac{1}{2} \alpha\_{ij}^{(k)} \right) \delta\_{\gamma \gamma'}.~
\nonumber
\end{aligned}$$

Based on this expression the density function of the nucleon has form

where

If the cluster is -particle, a relevant matrix element has form

$$\begin{aligned}
&\langle \phi^{JM}\_{\gamma i} (k,pq) \vert
\rho\_\alpha \left( {\bf y}\_k-y^{(k)}\_0 {\bf R} \right)
\vert \phi^{JM}\_{\gamma' j} (k,pq) \rangle =
\label{me\_rho\_alpha}
\\
\nonumber
& = \rho\_0 C\_i C\_j \exp\left(-\gamma\_0 R^2 \right)
\int \int d{ x}\_k d{ y}\_k ~x\_k^{2} ~y^{2}\_k
~\varphi^{(\lambda,l)}\_i(x\_k,y\_k)
~\varphi^{(\lambda,l)}\_j (x\_k,y\_k) \times \\
& \times \exp \left(-\gamma\_0 {y\_0^{(k)}}^2 y^2\_k \right)
i\_0(2\gamma\_0 y\_0^{(k)} y\_k R) \delta\_{\gamma \gamma'},
\nonumber\end{aligned}$$

where, is the modified spherical Bessel function of the first kind ([[mod\_sph\_bessel]](#mod_sph_bessel)). The density function of the -particle is built as follows

where

Here, is given in Eq. ([[table\_integral\_2]](#table_integral_2)).

Let’s turn to the density functions of particles. In order to calculate the matrix elements for these particle, one must switch basis functions into Jacobi coordinate set, as in the case with overlap matrix element ([[overlap\_alternative]](#overlap_alternative)). In particular, the matrix element for the nucleon may be represented in the form

$$\begin{aligned}
&\langle \phi^{JM}\_{\gamma i} (k,pq) \vert
\delta \left( {\bf y}\_q-y^{(q)}\_0 {\bf R} \right)
\vert \phi^{JM}\_{\gamma' j} (k,pq) \rangle =
\\
&\sum\_{\tilde{\gamma} \tilde{\gamma}'}
A^{\tilde{\lambda}' \tilde{l}' \lambda' l'}\_{{\bf Q} }
A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q} }
\langle \phi^{ J M}\_{\tilde{\gamma} i}\left(q,kp \right) \vert
\delta \left( {\bf y}\_q-y^{(q)}\_0 {\bf R} \right) \vert
\phi^{ J M}\_{\tilde{\gamma}^{\prime} j}\left(q,kp \right) \rangle.
\nonumber\end{aligned}$$

Analogously to the Eq. ([[me\_delta\_k]](#me_delta_k)) the density function of the nucleon may be represented as follows

where the component is

$$\begin{aligned}
&\rho\_\gamma^{(N\_q)}=
\sum\_{\tilde{\gamma}ij}
{A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q} }}^2
C\_iC\_j
\left( y\_0^{(q)} \right)^{2\tilde{l}}
R^{2\tilde{l}}
\exp \left( -\tfrac{1}{2} {y\_0^{(q)}}^2 \beta^{(q)}\_{ij} R^2 \right) \times
\\ \nonumber
& \times
\mathcal{I}\left( 2\tilde{\lambda}+2, \tfrac{1}{2} {\alpha'}\_{ij}^{(q)} \right).\end{aligned}$$

Here, it should be reminded that the rotation matrix ${\bf Q}$ also depends on indexes (see. Eq. ([[q\_rotation\_matrix]](#q_rotation_matrix)) and ([[overlap\_to\_q]](#overlap_to_q))). And, the matrix element of the density operator for -cluster may be introduced in the following expression

$$\begin{aligned}
&\langle \phi^{JM}\_{\gamma i} (k,pq) \vert
\rho\_\alpha \left( {\bf y}\_q-y^{(q)}\_0 {\bf R} \right)
\vert \phi^{JM}\_{\gamma' j} (k,pq) \rangle =
\\
&\sum\_{\tilde{\gamma} \tilde{\gamma}'}
A^{\tilde{\lambda}' \tilde{l}' \lambda' l'}\_{{\bf Q} }
A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q} }
\langle \phi^{ J M}\_{\tilde{\gamma} i}\left(q,kp \right) \vert
\rho\_\alpha \left( {\bf y}\_q-y^{(q)}\_0 {\bf R} \right) \vert
\phi^{ J M}\_{\tilde{\gamma}^{\prime} j}\left(q,kp \right) \rangle.
\nonumber\end{aligned}$$

Following simple derivations one may write the density function of the -cluster of three body system in the scheme as

where the function is

$$\begin{aligned}
&\rho\_\gamma^{(\alpha\_q)}(R)=
4 \pi \rho\_0 \exp\left(-\gamma\_0 R^2 \right)
\sum\_{\tilde{\gamma}ij}
{A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q} }}^2
C\_i C\_j
\mathcal{I}\left( 2\tilde{\lambda}+2, \tfrac{1}{2} {\alpha'}^{(q)}\_{ij}\right) \times \\
& \times \mathcal{I}\left( \tilde{l},~0,~\beta\_{ij}^{(q)}+2~{y^{(q)}\_0}^2 \gamma\_0, ~2 \gamma\_0 y^{(q)}\_0 R \right).
\nonumber\end{aligned}$$

Density functions for the nucleons and -clusters are obtained in the similar way, as with density functions for the nucleon and -cluster .

## Root mean square radii

The above deduced density functions are useful for deriving the root mean square (rms) radii. Moreover, the explicit form of some integrals allows to express the rms radius in a convenient form. This would enable the exact value than the value obtained numerically.

The matter rms radius of in the framework of three body system in this work are defined as follows

while the charge radius is

where – charge of the three body system, the rms radii in general way are determined as follows

In particular, using the Eq. ([[rho\_nk]](#rho_nk)) the rms radius for the nucleon is defined as follows

In the case of the -cluster , by means of the Eq. ([[rho\_alphak]](#rho_alphak)) its rms radii is calculated as follows

where the integral depending on six arguments is expressed in Appendix [7](#AppendixA) (see Eq. ([[table\_integral3]](#table_integral3))).

Likewise, the rms radii for the particle in the three body system may be expressed through the density functions . For instance, putting the Eq. ([[rho\_nq]](#rho_nq)) into the Eq. ([[rms\_general]](#rms_general)) we can get the rms radius for the nucleon :

$$\begin{aligned}
\langle r\_{N\_q}^2 \rangle = &
\sum\_{\gamma \tilde{\gamma}} \sum\_{ij}
{A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q} }}^2
C\_iC\_j
\left( y\_0^{(q)} \right)^{2\tilde{l}+3}
\mathcal{I}(2\tilde{l}+4, ~\tfrac{1}{2} {y\_0^{(q)}}^2 \beta^{(q)}\_{ij})~
\times
\\ \nonumber
& \times
\mathcal{I}\left( 2\tilde{\lambda}+2, \tfrac{1}{2} {\alpha'}\_{ij}^{(q)} \right).\end{aligned}$$

Inserting Eq. ([[rho\_alpha\_q]](#rho_alpha_q)) into Eq. ([[rms\_general]](#rms_general)) we obtain the rms radius for the -cluster the following expression

$$\begin{aligned}
\langle r\_{\alpha\_q}^2 \rangle=&
4 \pi \rho\_0
\sum\_{\gamma \tilde{\gamma}} \sum\_{ij}
{A^{\tilde{\lambda} \tilde{l} \lambda l}\_{{\bf Q} }}^2
C\_i C\_j
\mathcal{I}\left( 2\tilde{\lambda}+2, \tfrac{1}{2} {\alpha'}^{(q)}\_{ij}\right) \times \\
& \times \mathcal{I}\left(1, ~\tilde{l},~0,~2\gamma\_0,~\beta\_{ij}^{(q)}+2{y^{(q)}\_0}^2 \gamma\_0,~2\gamma\_0 y^{(q)}\_0\right)
\nonumber\end{aligned}$$

As regards for the particles in the three body system the rms radii may be extracted in analogous way as in the case with the particles .

For calculating charge radii one must use the different parametrizations and . Provided the -particle has Gaussian density distribution, the parameters may be obtained by using the charge rms radius in the Eq. ([[rho\_alpha\_parameters]](#rho_alpha_parameters)).

# Theoretical models describing nuclear reactions

## Description of elastic scattering

In this chapter the description of scattering of two nuclei is given by a complex potential . The potential may depend on the spins of the two nuclei but not on their internal coordinates. Thus, it cannot excite the nuclei internally or cause the transfer between them. It can only change their relative motion and, perhaps, reorient spins to each other or to the orbital motion .

In the case of colliding + particles without spin the potential is central, depending only on the magnitude of the channel coordinate ${\bf r}$. The corresponding Schrödinger equation may be written explicitly

$$\left( E + \frac{\hbar^2}{2 \mu} {\bf \nabla}^2 - U \left( r \right) \right)
\chi \left( {\bf r} \right) =0
\label{es\_she1}$$

where is the reduced mass of the + system, is the energy in the centre-of-mass system. The $\chi \left( {\bf r} \right)$ wave function is known as distorted waves describing elastic scattering. The expression "distorted wave" is meant to denote distortion away from the plane wave form due to the presence of the distorting potential (see Fig. [2.1](#fig:scattering_scheme)). Asymptotically, it has the form of an incident plane wave plus outgoing (scattered) spherical waves

$$\chi^{(+)} \left( {\bf k}, {\bf r} \right) \rightarrow
e^{i {\bf k} \cdot {\bf r}} + f \left( \theta \right) \frac{1}{{\bf r}}
e^{i k r}, ~~ {\bf r} \rightarrow \infty
\label{es\_chi\_assymp\_with\_f}$$

where is the scattering amplitude. The superscript stands for outgoing plane wave, while incoming spherical waves is the time-reverse of the

$$\chi^{(-)} \left( {\bf k}, {\bf r} \right) =
\left( \chi^{(+)} \left(- {\bf k}, {\bf r} \right) \right)^{\*}$$

An incoming plane wave scattering off a body making a distorted wave

Using the partial wave expansion

$$\chi^{(+)} \left( {\bf k}, {\bf r} \right) =
\frac{4 \pi}{kr} \sum\_{LM} i^{L} \chi\_L \left( k,r \right)
Y\_{LM}\left( \hat{{\bf r}}\right) \left( Y\_{LM}( \hat{{\bf k}})\right)^{\*},$$

A solution for Eq. ([[es\_she1]](#es_she1)) in the absence of any interaction potential, , is given by

$$\chi^{(+)} \left( {\bf k}, {\bf r} \right) \rightarrow
e^{i {\bf k} \cdot {\bf r}}, ~~ {\bf r} \rightarrow \infty,$$

for radial part only is written as

where is the usual spherical Bessel function. The radial function satisfies the radial Schrödinger equation

where .

In the case of the interaction potential has a more sophisticated form, including both Coulomb and nuclear short-range potentials, the radial Schrödinger equation ([[es\_she\_radial]](#es_she_radial)) may be rewritten as

where is the usual Sommerfeld parameter with the charge numbers

At the radius , where the nuclear potential is negligible, Eq. ([[es\_she2]](#es_she2)) has the solution, which can be expressed in terms of the outgoing and incoming Coulomb functions, as follow

where is the Coulomb phase shift, and it is given with the Gamma function as follow

In practice, the radial Eq. ([[es\_she2]](#es_she2)) is solved by numerical integration from , then, matched the value and the slope of the result onto the form ([[es\_assymp\_chi]](#es_assymp_chi)) ar . This procedure then gives a value for the scattering matrix elements. Having the matrix elements, the amplitude of the elastic scattering, analogous from Eq. ([[es\_chi\_assymp\_with\_f]](#es_chi_assymp_with_f)), for the wave function in the Eq. ([[es\_she2]](#es_she2)) is given by

where is the Legendre polynomials, which are solutions to the Legendre differential equation. By means of the Rutherford scattering amplitude

the differential cross section of the elastic scattering has the form

The expression ([[es\_diff\_cross\_section]](#es_diff_cross_section)) allows thus to provide comparison with the obtained experimental data.

## The coupled channels method for inelastic scattering

The elastic scattering of the projectile by the nucleus has been denoted by . Let be an inelastic channel , in which only the nucleus has an extra excitation. In the framework of the coupled channel (CC) approach the total wave function for the system may be written as

$$\Psi = \phi\_\alpha \left( { x} \right) \chi\_\alpha \left( {\bf r}\_\alpha \right)+
\phi\_{\alpha^{\prime}} \left( { x} \right) \chi\_{\alpha^{\prime}} \left( {\bf r}\_{\alpha^{\prime}} \right)
\label{cc\_tot\_wf}$$

where ${\bf r}$ is the channel coordinate for the partitions or , represents the corresponding internal coordinates. The total wave function can be part of the Schrödinger equation kind of

with the Hamiltonian appropriate particular for the partition

where is the internal Hamiltonian for the nuclei and , is the kinetic energy operator of relative motion and is the interaction potential operator. The wave functions of ground and excited state , being eigenfunctions of the internal Hamiltonian

have an orthonormality property of the form

Using the expression of total wave function ([[cc\_tot\_wf]](#cc_tot_wf)), multiplying Eq. ([[cc\_she1]](#cc_she1)) from the left by the function, then, by the function, the two coupled equations can be defined in the following form

$$\begin{aligned}
\left( E -\varepsilon\_\alpha - K\_\alpha - \langle \alpha \vert V\_\alpha \vert \alpha \rangle \right) \chi\_\alpha \left( {\bf r} \right) = & \langle \alpha \vert V\_{\alpha} \vert \alpha^{\prime} \rangle \chi\_{\alpha^{\prime}} \left( {\bf r} \right)
\nonumber \\
\left( E -\varepsilon\_\alpha - K\_\alpha - \langle \alpha^{\prime} \vert V\_\alpha \vert \alpha^{\prime} \rangle \right) \chi\_{\alpha^{\prime}} \left( {\bf r} \right) =& \langle \alpha^{\prime} \vert V\_{\alpha} \vert \alpha \rangle \chi\_{\alpha} \left( {\bf r} \right)
\end{aligned}$$

where , or , is the matrix element of . In particular, the matrix element is given by

$$\begin{aligned}
\langle \alpha^{\prime} \vert V\_\alpha \vert \alpha \rangle = & \int d x \phi\_\alpha^{\prime} \left( { x} \right) V\_\alpha \left( x, {\bf r} \right) \phi\_\alpha \left( { x} \right) =\nonumber \\
= & V\_{\alpha^{\prime} \alpha} \left( {\bf r} \right).\end{aligned}$$

An expansion of the coupling potential $V\_{\alpha^{\prime} \alpha} \left( {\bf r} \right)$ into the multipoles can be given as follow

$$V\_{\alpha^{\prime} \alpha} \left( {\bf r} \right) = \sum\_{\lambda \mu} V^{\lambda \mu}\_{\alpha^{\prime} \alpha} (r) Y\_{\lambda \mu} \left( \hat{r} \right)$$

If the potential shape has a deformation, the nuclear potential can be constructed as

$$V\_\alpha \left( \bf{x}, {\bf r}\right) = U\left( r -\Delta\left( \hat{r}^{\prime} \right) \right)
\label{cc\_coup\_pot\_def}$$

where denotes the angular part of $\bf{r}$ referred to the intrinsic coordinates $\bf{x}$. The shift function is normally expanded in multipoles

In the collective model, the ground and excited states are characterized by their angular momenta and with projections and , respectively. For these state the relevant matrix element of the operator is described using the Wigner-Eckart theorem by

Using the definitions, ([[cc\_coup\_pot\_def]](#cc_coup_pot_def)) and ([[cc\_delta\_def]](#cc_delta_def)), the reduced matrix element of radial multipoles from Eq. ([[cc\_we]](#cc_we)) can be rewritten as follow

where

The matrix element is the expectation value of the operator in the internal state of the deformed nucleus. In the framework of the rotational model it can be given with the deformation length as follow

where is an average radius of the interaction potential.

The general asymptotic behaviour of the elastic channel can be taken from Eq. ([[es\_chi\_assymp\_with\_f]](#es_chi_assymp_with_f)), while the inelastic channel may have

$$\chi\_{\alpha^{\prime}} \left( {\bf r}\_{\alpha^{\prime}} \right) \rightarrow
f\_{\alpha^{\prime}} \left( \theta \right)
\frac{e^{i k r\_{\alpha^{\prime}}}}{r},
~~ {\bf r}\_{\alpha^{\prime}} \rightarrow \infty$$

Note, that the plane wave expression doesn’t not present in this equation - only outgoing wave presents. A relevant differential cross section for the inelastic channel is obtained from the coefficient of the outgoing wave as follow

The wave numbers and follow from the energy conservation

## The coupled-reaction-channels method for the transfer reactions

Consider a model for the nuclear reaction, in which entrance and exit channels are denoted as and respectively. The total wave function for this model may be given as

$$\Psi = \chi\_{\alpha} \left( {\bf r}\_\alpha \right) \phi\_\alpha \left( x\_\alpha \right) +
\chi\_{\beta} \left( {\bf r}\_\beta \right) \phi\_\beta \left( x\_\beta \right).$$

with a model Hamiltonian such that . From the projections of this equation onto the two channels,

with the two equivalent forms of ,

one can get a pair of coupled equations for and :

$$\begin{aligned}
\left[ ~\left(E-\varepsilon\_\alpha \right) -K\_\alpha -
\langle \alpha \vert V\_\alpha \vert \alpha \rangle ~\right]
\chi\_\alpha ({\bf r}\_\alpha) =
\langle \alpha \vert H-E \vert \beta \rangle \chi\_\beta \nonumber \\
\left[ ~\left(E-\varepsilon\_\beta \right) -K\_\beta -
\langle \beta \vert V\_\beta \vert \beta \rangle ~\right]
\chi\_\beta ({\bf r}\_\beta) =
\langle \beta \vert H-E \vert \alpha \rangle \chi\_\alpha .
\label{crc\_couple\_eq1}\end{aligned}$$

These are *the coupled-reaction-channels* (CRC) equations. They are integro-differential equations, as may be seen more explicitly in the form

$$\begin{aligned}
\left[ ~\left(E-\varepsilon\_\alpha \right) -K\_\alpha -
\langle \alpha \vert V\_\alpha \vert \alpha \rangle ~\right]
\chi\_\alpha ({\bf r}\_\alpha) =
\int \text{d} {\bf r}\_\beta K\_{\alpha \beta} \left( {\bf r}\_\alpha, {\bf r}\_\beta \right) \chi\_\beta \left( {\bf r}\_\beta \right)
\nonumber \\
\left[ ~\left(E-\varepsilon\_\beta \right) -K\_\beta -
\langle \beta \vert V\_\beta \vert \beta \rangle ~\right]
\chi\_\beta ({\bf r}\_\beta) =
\int \text{d} {\bf r}\_\alpha K\_{\beta \alpha} \left( {\bf r}\_\beta, {\bf r}\_\alpha \right) \chi\_\alpha \left( {\bf r}\_\alpha \right)\end{aligned}$$

where the kernels are

$$\begin{aligned}
K\_{\alpha \beta} \left( {\bf r}\_\alpha, {\bf r}\_\beta \right) =
J\_{\alpha \beta}
\int \text{d} \zeta\_\alpha \phi\_\alpha^\* \left( x\_\alpha \right)
(H-E) \phi\_\beta \left( x\_\beta \right) \nonumber \\
K\_{\beta \alpha} \left( {\bf r}\_\beta, {\bf r}\_\alpha \right) =
J\_{\beta \alpha}
\int \text{d} \zeta\_\beta \phi\_\beta^\* \left( x\_\beta \right)
(H-E) \phi\_\alpha \left( x\_\alpha \right).\end{aligned}$$

Here the internal coordinates have been transformed from the set to the set $\left( \zeta\_\alpha, {\bf r}\_\beta \right)$, where the are independent of ${\bf r}\_\beta$. Also, is the Jacobian of this transformation. Then is the Jacobian for the analogous transformation form to $\left( \zeta\_\beta,{\bf r}\_\alpha \right)$.

Since the off-diagonal matrix elements of are small, they have little effect on the elastic scattering and may be neglected on the right side. This implies that the elastic scattering in the entrance channel is described well by the potential , and the potential describes well the elastic scattering in the channel. By doing this approximation, Eq. ([[crc\_couple\_eq1]](#crc_couple_eq1)) may be rewritten as

$$\begin{aligned}
\left[ ~\left(E-\varepsilon\_\alpha \right) -K\_\alpha -
\langle \alpha \vert V\_\alpha \vert \alpha \rangle ~\right]
\chi\_\alpha ({\bf r}\_\alpha) \approx & 0
\nonumber \\
\left[ ~\left(E-\varepsilon\_\beta \right) -K\_\beta -
\langle \beta \vert V\_\beta \vert \beta \rangle ~\right]
\chi\_\beta ({\bf r}\_\beta) \approx &
\langle \beta \vert H-E \vert \alpha \rangle \chi\_\alpha
\nonumber \\
\approx &
\langle \beta \vert V\_\alpha \vert \alpha \rangle \chi\_\alpha + \langle \beta \vert \alpha \rangle (H\_\alpha - E\_\alpha) \chi\_\alpha
\nonumber \\
\approx & \langle \beta \vert V\_\alpha \vert \alpha \rangle \chi\_\alpha
\label{crc\_couple\_eq2}\end{aligned}$$

Here the prior interaction, that is the interaction of the channel, is used. The non-orthogonal term vanishes because of the is on-shell. The usual Green function techniques may then be used to solve these equations and give *the distorted wave Born approximation transition* (DWBA) amplitude. A detailed representation of the amplitude will be done in the next section.

## Distorted Wave Born Approximation

Let consider the nuclear reaction of transfer particle from a nucleus into a nucleus:

In the case of weak coupling between intermediate channels, it is reasonable to evaluate the transition amplitude in Born Approximation.

In the rearrangement reactions there are few ways to describe the interaction between the different fragments, one for each partition. For example,if we choose to describe the scattering in terms of the nuclei of the entrance partition, the projectile-target interaction will be written as

Here, – a binding potential of the valence particle with the core, and it is real potential, – a complex optical potential describing the scattering state of the system. In this representation, known as the prior form, the transition amplitude for the transfer process is given by

$$\begin{aligned}
T\_{prior} =& \langle \chi\_\beta^{(-)} \phi\_a \phi\_B \vert
V\_{\nu b}+U\_{ab}-U\_{\alpha} \vert
\chi\_\alpha^{(+)} \phi\_A \phi\_b \rangle =
\nonumber \\
= & \int \int d {\bf r}\_\alpha d {\bf r}\_\beta
\chi^{-}\_\beta \left( {\bf r}\_\beta \right)^\*
I\_{\beta \alpha} \left( \left( {\bf r}\_\beta , {\bf r}\_\alpha \right) \right)
\chi^{+}\_\alpha \left( {\bf r}\_\alpha \right),
\label{dwba\_amplitude}
\end{aligned}$$

where ${\bf r}\_\alpha$ and ${\bf r}\_\beta$ – the radius vectors, illustrated in Fig.[2.2](#fig:dwba_scheme), describing relative distance of the and systems respectively, – the optical potential describing elastic scattering of -channel, and is the kernel function having form

$$I\_{\beta \alpha} \left( \left( {\bf r}\_\beta , {\bf r}\_\alpha \right) \right) =
\left( \phi\_a \phi\_B \vert
V\_{\nu b}+U\_{ab}-U\_{\alpha} \vert
\phi\_A \phi\_b \right).$$

The arrangement of radial vectors in the DWBA on example of the nuclear reaction. Here and .

Analogously, for the exit channel with the transition amplitude turns to

Here, stands for the optical potential corresponding to elastic channel.

The differential cross section for the rearrangement nuclear reaction within the DWBA, in particular for prior form, may be written as

$$\frac{d \sigma}{d \Omega}\_{DWBA} = \frac{\mu\_\beta \mu\_\alpha}{2 \pi \hbar^2}
\left( \frac{k\_\beta}{k\_\alpha}\right)
\vert T\_{prior} \left( {\bf k}\_\beta, {\bf k}\_\alpha \right) \vert^2$$

For the post form the differential cross section is written analogously replacing the corresponding transition amplitude.

In accordance with the DWBA calculations the main ingredients are internal wave functions for both initial and final nuclei. In this scheme, the valence particle is bound to the core to give the composite . In the simplest picture, the valence particle can be considered a pure single-particle state. This means that within this model there is only one possible configuration of the core and valence particle to give the nucleus . Thus, the wave function for the nucleus can be written as

$$\phi^{JM}\_B \left( \xi,~{\bf r}' \right) =
\left[ \phi\_b^I (\xi) \times \phi\_{lsj} ({\bf r}') \right]\_{JM},$$

where is the intrinsic coordinate of the nucleus. In a more sophisticated model, however, the state of the composite contains components of many single particle states coupled to all possible core states. Therefore, the wave function $\phi^{JM}\_B \left( \xi,~{\bf r} \right)$ may be built as a superposition of the form

$$\phi^{JM}\_B \left( \xi,~{\bf r}' \right) =
\frac{1}{\sqrt{n\_B}}
\sum\_{Ilj} \mathcal{A}\_{lsj}^{IJ}
\left[ \phi\_b^I (\xi) \times \phi\_{lsj} ({\bf r}') \right]\_{JM}
\label{eq:composite\_expansion}$$

where the coefficient is the coefficient of fractional parentage or spectroscopic amplitudes. More information about the SA will be introduced in the next section. The squared value of the amplitude is

which is called spectroscopic factors. The spectroscopic factors can be regarded as the probability of finding the valence particle in a single particle state coupled to the core with spin . The quantity is the number of nucleons or clusters in the composite system.

Note that the kernel function $I\_{\beta \alpha} \left( {\bf r}\_\beta , {\bf r}\_\alpha \right)$ involve the overlap between the composite and core wave functions. Using the Eq.[[eq:composite\_expansion]](#eq:composite_expansion) the internal variable in the integral can be explicitly performed giving just unity by normalization

$$\left( \phi^{JM}\_B \left( \xi,~{\bf r}' \right), \phi\_b^I ( \xi ) \right)
\equiv
\int d \xi \phi^{JM}\_B \left( \xi,~{\bf r}' \right) \phi\_b^I ( \xi ) =
\frac{1}{\sqrt{n\_B}}
\sum\_{Ilj} \mathcal{A}\_{lsj}^{IJ} \phi\_{lsj} ({\bf r}')$$

The bound wave function $\phi\_{lsj} ({\bf r}')$ obey the Schrödinger equation

$$\left( T + V\_{\nu b} ({\bf r}') + \epsilon - E \right) \phi\_{lsj} ({\bf r}) = 0$$

where is binding energy of the valence particle .

The spectroscopic amplitude is an important ingredient of the DWBA method. Since the further calculations take into account the transfer reactions of one particle, it is possible to obtain the spectroscopic amplitude using the Shell model calculations. This allows to reduce the number of free parameters in the CRC calculations. The next section is devoted to defining how spectroscopic amplitudes can be obtained with in the Shell model.

## Spectroscopic Amplitudes within the Shell Model

Spectroscopic amplitudes for nuclear reactions can be obtained by calculations within the shell model. However, this approach is limited in calculating the amount of transferred particles. The calculation of matrix elements for the creation operator is suitable only in the case of the one-particle transfers, or at best two particle transfer. In nuclear reactions with the cluster transfer, one should use other theoretical approaches, or achieve certain values by fitting for the best reproduction of the observables .

Observables for the removal or addition of a nucleon from a specific initial state to a specific final state are related to the matrix elements of the creation and destruction operators. The creation operator is a tensor of rank since it creates the single-particle state . Here, stands for the set of single-particle quantum numbers . The destruction operator is not a tensor of rank , however,

is a tensor of rank . And its inverse is . Corresponding matrix elements of these operators can be written as

where – the harmonic oscillator parameter chosen to reproduce the observed mean-square charge radius, – total spin of the -particle state. All many-body matrix elements of can be reduced to these involving a single -state.

Wave-function expansion relations and sum-rules for the states in the particle system can be obtained by operating with the number operator

on the configuration and then inserting a complete set of states with particles

The matrix element of can be reduced with the Wigner-Eckhart theorem

By Eq. [[matrix\_elements\_of\_sa]](#matrix_elements_of_sa) the reduced matrix element of can be converted in a reduced matrix element of to obtain the final result:

Denoting the -particle wave function by

where is the linear combination of operator which create the antisymmetric -particle state, we can thus expand the wave function in terms of those in the basis:

A phase factor arises from commuting with the particles in the state

A sum rule for the matrix elements of can be obtained by multiplying both sides of Eq. [[knmo\_basis]](#knmo_basis) by to obtain

Then, one finds the sum-rule:

Thus:

The matrix elements in which the sum over final states is normalized to unity are historically called coefficients of fractional parentage (CFP) defined by:

The reduced matrix elements of the creation and destruction operators are used to define the SA associated with nuclear reactions. They are also building blocks for the more complicated operators associated with one-body (such as electromagnetic and -decay) and two-body transition amplitudes.

## Double folding potential

The numerical calculations of elastic scattering can be performed in the framework of the OM with the optical potential given by:

where and are real volume, imaginary volume, spin-orbit and Coulomb potentials, respectively. The volume potentials of colliding two spherical nuclei may be represented as parametrized function. For example, in practice the Woods-Saxon potential is often used, and it has the form

where is depth of the potential, is average distance and is diffusion parameter.

The spin-orbit term of the OM potential has standard form

The Coulomb term has been taken as the interaction of a point-charge with a uniformly charged sphere

The situation changes, when the colliding nuclei have sophisticated shape. Assume, that there are nuclear matter distribution of the projectile and target , depending on internal their own radii. Folding the nucleon-nucleon interaction potential over the density distributions of nuclear matter, in the framework of *the Double folding model* an interaction potential may be constructed as follow

$$\begin{aligned}
V^V\left( r \right) = & N\_f V^{DF} \left( r \right) \nonumber \\
V^{DF} \left( {\bf r} \right) = & \int \int \text{d} {\bf r}\_a\text{d} {\bf r}\_A
\rho\_a \left( {\bf r}\_a \right) V\_{nn} \left( r\_{aA}\right) \rho\_A \left( {\bf r}\_A \right).
\label{df\_potential}\end{aligned}$$

where $r\_{aA}=\vert {\bf r} + {\bf r}\_A - {\bf r}\_a \vert$ (see Fig. [2.3](#fig:df_scheme)). The normalization parameter is usually fitted in accordance with the dynamics of nuclear reaction of elastic scattering.

Radius vectors for the double folding model.

The form of DF potential ([[df\_potential]](#df_potential)) used in this work is the Direct folding potential. In addition, if one takes into account the effects of nucleon exchange, the Exchange term of the DF potential must be added (for more, see ). The DF potential may be calculated using the effective M3Y-Paris nucleon-nucleon potential and the nuclear-matter-densities of projectile and target nuclei.

The effective nucleon-nucleon interactions usually are taken as a sum of the three Yukawa potentials, i.e. M3Y-potentials. For both isoscalar (T=0) and isovector (T=1) the M3Y-potentials have a form as

The parameters , of the M3Y-Paris effective NN-potential , often used in the double folding calculations, are given in Tab.[2.1](#m3y_potpar).

The dependence of the folding potential on the collision energy and an additional dependence on the nuclear matter density in the region of their overlap have been phenomenologically established. The folding potential is usually taken as the real part of the optical potential, adjusting its depth to better reproduce the experimental data. The imaginary part of the optical potential is selected in the form of the Woods-Saxon potential, or the same folding potential, while varying its depth.

The potential depending on the collision energy of the projectile, undergoes a change by multiplying a factor

An additional dependence of the potential on the total density of colliding nuclei in the region of their overlap may given as

$$F(\rho)=C \left( 1+ \alpha \text{exp} \left( - \beta \rho \right) - \gamma \rho({\bf r})\right).
\label{F\_dependence}$$

with the parameters

For the particles , He, , which have simple structure, the density distribution of nuclear matter can be represented as parametrized Gaussian function as follow

where the parameters and are defined in condition to reproduce the rms matter radii

here – atomic number of projectile.

The density distributions in ([[df\_potential]](#df_potential)) are normalized so that

$$\begin{aligned}
\int \rho\_a ({\bf r}) \text{d} {\bf r} =a
\end{aligned}$$

where is atomic mass of projectile. For the target nucleus normalization has the equivalent form.

Parameters of the M3Y-Paris NN-potential with the Direct and Exchange terms.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | , MeV |  |  |  | , |
|  | Direct T=0 | Exchange T=0 | Direct T=1 | Exchange T=1 |  |
| 1 | 11061.60 | -1524.25 | 313.62 | -4118.9 | 4.0000 |
| 2 | -2537.5 | -518.75 | 223.5 | 1054.75 | 2.5000 |
| 3 | 0.0 | -7.84 | 0.0 | 2.62 | 0.7072 |

As regards target nuclei, having cluster structure, the density distribution of nuclear matter is calculated by means of the three body wave function, and its details are set out in the Section [1.5](#section_density_function) .

# The three-body wave functions and their properties

## The three-body wave functions

The three-body wave functions are calculated within the SVM. The obtained parameters for ground state of the He nucleus are listed in Application [8](#AppendixB). The calculations were done using the RSC potential for nucleon-nucleon interaction, the SBB potential for - interaction and the BFW potential for - interaction. In Tab. [3.1](#tab:variational_data) the relative weights of the ground state wave function components and eigenvalues the three body systems are listed[[1]](#footnote-40).

The ground state energies of the three-body systems and the weights of the wave function components found by means of the SVM for the 6He, 6Li and 9Be nuclei. The set of quantum numbers , the dimension and the component weights ([[wf\_normalizations]](#wf_normalizations)) are listed.

| He | | | Li | | | Be | | |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | | |  | | |  | | |
|  | | |  | | |  | | |
|  |  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |  |
| 1 1 1 1 |  |  |  |  |  |  |  |  |
| 2 2 0 0 |  |  |  |  |  |  |  |  |
| 3 3 1 1 |  |  |  |  |  |  |  |  |
| 81 |  |  |  |  |  |  |  |  |
| 25 |  |  |  |  |  |  |  |  |
| 25 |  |  |  |  |  |  |  |  |
| 0.102 |  |  |  |  |  |  |  |  |
| 0.010 |  |  |  |  |  |  |  |  |
| 0.004 |  |  |  |  |  |  |  |  |
| 2 0 2 1 |  |  |  |  |  |  |  |  |
| 1 1 1 0 |  |  |  |  |  |  |  |  |
| 2 2 0 1 |  |  |  |  |  |  |  |  |
| 2 2 1 1 |  |  |  |  |  |  |  |  |
| 49 |  |  |  |  |  |  |  |  |
| 49 |  |  |  |  |  |  |  |  |
| 25 |  |  |  |  |  |  |  |  |
| 25 |  |  |  |  |  |  |  |  |
| 0.073 |  |  |  |  |  |  |  |  |
| 0.021 |  |  |  |  |  |  |  |  |
| 0.005 |  |  |  |  |  |  |  |  |
| 0.002 |  |  |  |  |  |  |  |  |
| 2 1 1 1/2 |  |  |  |  |  |  |  |  |
| 2 1 2 1/2 |  |  |  |  |  |  |  |  |
| 2 3 1 1/2 |  |  |  |  |  |  |  |  |
| 2 3 2 1/2 |  |  |  |  |  |  |  |  |
| 4 3 1 1/2 |  |  |  |  |  |  |  |  |
| 4 3 2 1/2 |  |  |  |  |  |  |  |  |
| 49 |  |  |  |  |  |  |  |  |
| 49 |  |  |  |  |  |  |  |  |
| 36 |  |  |  |  |  |  |  |  |
| 25 |  |  |  |  |  |  |  |  |
| 16 |  |  |  |  |  |  |  |  |
| 16 |  |  |  |  |  |  |  |  |
| 0.328 |  |  |  |  |  |  |  |  |
| 0.235 |  |  |  |  |  |  |  |  |
| 0.017 |  |  |  |  |  |  |  |  |
| 0.004 |  |  |  |  |  |  |  |  |
| 0.009 |  |  |  |  |  |  |  |  |
| 0.003 |  |  |  |  |  |  |  |  |

The data in Tab. [3.1](#tab:variational_data) show good agreement between calculated and measured values of the ground state energy for the 9Be nucleus. However, the corresponding values given for the 6He and 6Li nuclei underestimate the experimental data approximately by  0.5–0.7 MeV. The discrepancy, probably, due to the non-completeness of the basis function. In this work it has no symmetrization, and doesn’t enfold an alternative coordinate space. Other reasons might be in the presence of other cluster components, which has not been included in the variational calculations, and the -body forces. For example, 6He can contain two cluster such as H + H, the 6Li nucleus can have He + H, and the 9Be nucleus can be represented as the He + He system. These kind of structures cannot be explicitly included in the current model. In particular, for 6He and 6Li within the SVM the authors Kakenov managed to move closer to experimental values of binding energies by implementing the -force.

image image

In Figures [[fig:he0\_the3D]](#fig:he0_the3D), [[fig:li0\_the3D]](#fig:li0_the3D) and [[fig:be0\_the3D]](#fig:be0_the3D) the correlation densities ([[wf\_cor\_density]](#wf_cor_density)) of the three-body wave functions are illustrated in the 3D-Plot form for the He, Li and Be nuclei. For the purpose of proper visualization the contour-plot are given along.

image image

In Figures [[fig:he0\_the3D]](#fig:he0_the3D) and [[fig:li0\_the3D]](#fig:li0_the3D), two main geometric configurations can be distinguished, cigar-like component with maximum at the coordinates , and di-nucleon component around the coordinates , both for He and Li nuclei. The striking highlight of such geometrical configurations is clear evidence of the cluster structure. In particular, we see that in He, the effect of pairing two nucleons stands out against the background of comparison with Li. Similar results were described in works , and were obtained in particularly within the hyperspherical harmonics method .

image image

The Fig. [[fig:be0\_the3D]](#fig:be0_the3D) shows the spatial correlation function for the bound state of Be. The single maximum of the correlation density function is located at the coordinates . In contrast with the 6He and 6Li nuclei the 9Be nucleus has only one pronounced spatial configuration corresponding to the two -clusters (Be ground state) glued by the valence neutron. A very similar picture has been found in Ref. .

## Density Function of Nuclear Matter

On the basis of the three-body wave function [[totwf]](#totwf), the nuclear matter density distributions are calculated for He, Li, and Be in the ground states using Equations [[rho\_nk]](#rho_nk), [[rho\_alphak]](#rho_alphak), [[rho\_nq]](#rho_nq), and [[rho\_alpha\_q]](#rho_alpha_q). The results are shown in the Figures [3.1](#he0_theDD), [3.2](#li0_theDD), and [3.3](#be0_theDD) respectively. The plotted the density functions are normalized to their own atomic masses.

A distinctive feature of the obtained results is in the extended tail of the density function for the 6He and 6Li nuclei in Figures [3.1](#he0_theDD) and [3.2](#li0_theDD). This is caused by the properties of the valence nucleons in the three-body system. In particular, the density function of the core tends to zero rapidly as the radius increases in comparison with the nucleon density function . As regards of 6Li, it testifies stronger contribution of the deuteron-cluster component, in contrast, the 6He has less contribution of di-neutron cluster. The main contribution to the density functions beginning from is due to two valence nucleons. Another point of behaviour of density function of nuclei with is a maximum in the density functions at . In particular, the bump of 6Li is sufficiently remarkable. This character of function explains that the valence nucleons are moved apart from the cm. The view disappears, if the plot of function is built as . It should be noted that the similar plot are absent elsewhere.

The nuclear matter density distribution of 6He with the cluster component contributions;

The same caption as in Fig. [3.1](#he0_theDD), but for Li.

The same caption as in Fig. [3.1](#he0_theDD), but for 9Be.

Comparison of the total nuclear matter density distribution function of 6He with the LSSM calculations .

The density function of 9Be has a different shape than the other two previous nuclei (see Fig. [3.3](#be0_theDD)). The alpha particle density function has become much more voluminous. The specific feature of the obtained density function is an extended tail at large distances determined by the valence neutron. The neutron is thus mainly located far from the cm and determines the low binding energy of 9Be nucleus. The density function begins from zero, getting the maximum at , what is characteristic for -shell nucleons. Another peculiarity of the 9Be structure is in minimum of the function at the start. Such minima are strongly manifested in stable carbon, nitrogen, and oxygen isotopes. For 9Be considered within the three body model, the minimum testifies to its cluster structure and noticeable deformation, reflecting that both valence neutron and -clusters are displaced with respect to the cm. Such kind of feature for 9Be were obtained also with in the framework of the AMD model .

A comparison of the total density function for 6He with the density function calculated in the framework of the Large-Scale Shell-Model (LSSM), (for more, see Ref. ), is illustrated in Fig.[3.4](#he0_compPlot). Both functions are in good agreement. However, starting from the three body model overestimates slightly.

Comparison of the calculated rms charge and matter radii obtained within the three-body model with experimental data.

|  | He | Li | Be |
| --- | --- | --- | --- |
|  | 2.03 | 2.57 | 2.48 |
|  | 2.066 | 2.57(10) | 2.50(9) |
|  | 2.75 | 2.49 | 2.58 |
|  | 2.75 0.01 | 2.54 0.03 | 2.49 0.01 |

The density function of nuclear matter is an excellent subject for studying rms matter radii. Corresponding formulae for calculating are given by Eq.([[rms\_general]](#rms_general)). The calculated results are presented in Tab.[3.2](#tab:rms_matter). Here, it should be reminded that the rms radii for -cluster are taken as: for nuclear matter, for charge radius . Obtained results are in good agreement with the experimental data. Except that the rms matter radius of 9Be overestimates for . It should be noted that similar theoretical estimation was obtained by means of the AMD model . The rms matter radius within this model is .

It worth to note that matrix elements of the rms radii may be obtained also in the following way

where . This method gives results in easier way, but doesn’t go into the density distribution function of the -clusters. But, eventually, the rms radii calculated with both Eq. ([[rms\_general]](#rms_general)) and ([[rms\_r\_square]](#rms_r_square)) give the same results. More detailed derivation of Eq. ([[rms\_r\_square]](#rms_r_square)) is provided in Ref. .

In the further calculations the components of three body wave function, contributing less than , have been omitted (see Tab.[3.1](#tab:variational_data)), since their influence on the results is insignificant.

# + 6He nuclear reactions

## Elastic scattering

The differential cross section of elastic scattering ([[es\_diff\_cross\_section]](#es_diff_cross_section)) for the 6He +  nuclear reaction has been calculated within the framework of the OM. The calculations were carried out by means of the code [[2]](#footnote-46) .

The optical potential for the 6He +  system was obtained within the DF model ([[df\_potential]](#df_potential)) using the density functions of nuclear matter (see Eq. [[rho\_alphak]](#rho_alphak) and [[rho\_nq]](#rho_nq)). For the NN-force the density depended DDM3Y-Paris potential was chosen . The DF potential may have terms depending on the interaction of projectile with the clusters of the three body system. In particular, for the 6He +  system, the DF potential can be defined as

The calculated results are demonstrated in Fig. [4.1](#he_df). The folding potential of -core and two valence nucleon with the projectile He show specific feature of the interaction. The potential of the -cluster with the projectile provides strong central part, while the interaction with halo neutrons is localized at the peripheral region.

The folding potential calculated with the density functions of nuclear matter of 6He. The designation is for the potential, while is for the potential.

Obtained potential was used to calculate the differential cross section for elastic scattering of 6He by -particles at energy . As the real part of the optical potential the folding potential with the parameter was used. For the imaginary part of the optical potential we used again the folding potential, but with the parameter . The corresponding results are shown in Fig. [4.2](#he_elastic). In addition to the folding potential, two other potentials (WS1-6He and WS2-6He) are presented as the Woods-Saxon potential taken from . The potential parameters used in the OM calculations are presented in Tab. [4.1](#tab:he_elastic). Calculated results with the both double folding and Woods-Saxon potentials are in good agreement with the experimental data at forward scattering angles. Except the potential WS2-6He, it underestimates the cross section in the range of .

The advantage of folding potential in comparison with others is in a less number of adjustable optical parameters. Instead of six parameters fitted in the case of Woods-Saxon potentials, the folding potential allows to reproduce data within two parameters.

The differential cross section of the elastic scattering of 6He by -particles at calculated with the different potentials: the double folding potential (blue solid), the WS1-6He(gray dashed) and WS2-6He(gray dotted) . Experimental data were taken from .

The cross sections of the elastic channel taking into account few possible reaction mechanisms: elastic scattering (solid gray), simultaneous transfer of di-neutron (gray dashed), sequential transfer of di-neutron (dotted) and their coherent sum (solid blue). Experimental data were taken from .

## Elastic transfer

Theoretical curves in Fig. [4.2](#he_elastic) could show a good result in describing the experimental data on elastic scattering within the framework of the optical model. However, this is true only at the front scattering angles, while the backscattering angles are far from description of optical calculations. Therefore, in order to explain the disagreement, we propose the following transfer mechanisms of two nucleons: the sequential and simultaneous transfer.

The schematic representation of mechanisms can be seen in Fig. [4.4](#he_transfer_scheme). Taking into account these mechanisms, the differential cross section can be written as follows

where the amplitude – an amplitude of the finite-range transfer, which may be calculated within the DWBA method ([[dwba\_amplitude]](#dwba_amplitude)), – an amplitude of two step transfer mechanisms (see, i.e. ).

Obtained calculation results for the differential cross section of the elastic transfer were carried out within the framework of the CRC method. Potential for the input and output channels was chosen to be the double folding potential 6He-DF. The potential for the intermediate channel was taken as the optical potential with global optical parametrizations for particles . Trial calculations have shown that the results depend insignificantly on the selected potential for the intermediate channel. The wave function of the bound states was chosen by fitting the potential depth to the binding energy of the composite systems. In particular, the binding energy of one neutron with He was chosen 1.8 MeV, neutron with He – 0.1 MeV, and two neutrons with – 0.9 MeV.

Schematic representation of the elastic transfer nuclear reaction, on the left - the simultaneous transfer of two nucleons, on the right - the sequential transfer of two nucleons.

The parameters of the potentials used in OM calculations for both + 6He and + 6Li nuclear reactions. For more details, see the text

|  | , MeV | , fm | , fm | , MeV | , fm | , fm |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| DF-6He | =1.4 | | | =0.5 | | | 7.32 |
| DF-6Li | =2.0 | | | =1.8 | | | 11.61 |
| WS1-6He | 102.5 | 1.78 | 0.920 | 13.0 | 3.85 | 0.5 | 5.75 |
| WS2-6He | 102.5 | 1.54 | 0.904 | 7.0 | 4.28 | 0.569 | 6.31 |
| WS1-6Li | 102.5 | 1.78 | 0.820 | 11.8 | 4.11 | 0.950 | 15.43 |

The results of the CRC calculations for elastic transfer are shown in Fig. [4.3](#he_elastic_transfer). As mentioned in the previous section, the elastic collision mechanism prevails at the forward scattering angles. Beginning from the angle 90, the contribution to the cross section is mainly caused by the simultaneous transfer of two nucleons. It is worth noting here that two neutrons have the configuration, which, possibly, leads to an oscillatory cross section. One magnitude less is the contribution of the sequential transfer of di-neutron. In this case both neutrons have configuration. For the best reproduction of the experimental data, the SA of di-neutrons was taken . The extracted value correspond with the value from Ref. .

Thus, it was possible to achieve good agreement between the calculated differential cross section, using the double-folding potential DF-6He and the proposed transfer mechanisms, with the experimental data .

# + 6Li nuclear reactions

## Elastic scattering

To calculate the elastic scattering cross section we used again the folding potential with the density depended DDM3Y-Paris effective NN-potential . The three body density functions of nuclear matter used to calculate the potential similarly to Eq. ([[df\_in\_terms]](#df_in_terms)). Therefore, it was possible to look into the interaction of the -projectile with deuteron and clusters inside the nucleus 6Li(see, Fig. [5.1](#li_df)). The function of the interaction potential of the projectile with the internal alpha particle rapidly tends to zero, while the function of the interaction potential of the projectile with the internal deuteron slowly decreases. It should also be noted that, starting from the distance , the main contribution is due to the interaction of the projectile with the internal deuteron.

The folding potential calculated with the density functions of nuclear matter of 6Li. The designation is for the potential, while is for the potential.

The folding potential was chosen to be the real part of the optical potential with the fitting parameter. As regards the imaginary part, the same shape of real part was used, but with the parameters . The Woods-Saxon potential with six parametrizations was taken from as comparison to DF-6Li. The results of calculating the differential cross section of elastic collision of the + 6Li system are shown in Fig. [5.2](#li_elastic). The theoretical curves based on the potentials DF-6Li and WS1-6Li show good agreement with the experimental data. It should be noted that the double folding potential DF-6Li for the elastic scattering + 6Li reaction depends only on two parameters rather than the Woods-Saxon potential WS1-6Li with six parameters.

The 6Li structure is very similar to 6He. Therefore, as in the case of 6He, the calculations made in the framework of the OM are well described only at the forward scattering angles. The potential parameters are listed in Tab. [4.1](#tab:he_elastic).

Differential cross sections belonging to the backward angles have the character of another mechanism - the transfer of from 6Li. How exactly this subsystem is transferred will be discussed in the next section devoted to elastic transfer for the + 6He reaction.

## Elastic transfer

An increase in the cross section for the elastic scattering + 6Li reaction starting from 90 suggests the existence of additional reaction mechanisms. In this case, in order to register the same particle as the elastically scattered projectile, a deuteron particle must be transferred to the target at angles above 90. The transfer mechanisms are shown schematically in Fig [5.4](#li_transfer_scheme).

As in the case of 6He, calculations for obtaining the differential cross section of the elastic transfer were carried out within the framework of the CRC method. For the input and output channels the double folding potential was taken, and for the intermediate channel the optical potential with global optical parametrizations for particles was chosen. The calculations have shown that the cross sections depend insignificantly on the selected potential for the intermediate channel. The wave function of the bound states was chosen by fitting the potential depth to the binding energy of the composite systems. In particular, the binding energy of one neutron with Li was 5.663 MeV, as neutron with He taken 0.9 MeV, and as with taken 1.474 MeV. The differential cross section for this reaction is determined using the same formula [[eq:he\_cs]](#eq:he_cs).

The differential cross section of the elastic scattering of 6Li by -particles at with the different potentials: the double folding potential DF-6Li(blue solid), the WS1-6Li(gray dashed) . Experimanetal data taken from .

The cross sections of the elastic transfer reaction are represented in terms of the transfer mechanisms: elastic scattering (solid gray), simultanious transfer of (gray dashed), sequential transfer of (dotted) and their coherent sum (solid blue). Experimanetal data taken from .

The results of calculations for elastic transfer are shown in Fig. [5.3](#li_elastic_transfer). The mechanism of elastic collision in the reaction of + 6Li prevails over other mechanisms at the forward scattering angles. However, starting from 90 we see that the main contribution to the cross section is caused by the simultaneous transfer of two nucleons of 6Li, that is, the deuteron. The difference from the reaction + 6He is in the spin structures of the composite nucleus ${\rm \textsuperscript{6}Li}(1^+)~=~ \alpha (0^+) + d (1^+)$. Accordingly, deuteron can be transferred by and configurations.

At the backward scattering angles the domination of the simultaneous transfer on sequential transfer mechanism is seen. That makes total sense because deuteron is less bound than neutron with 6Li- and . Moreover, two step mechanisms are typically few order magnitude lower that the one step transfers. The best agreement with the experimental data is obtained if one uses the SA for the configuration , and for the configuration . As for the sequential transfer of the system, it has a contribution much less than the simultaneous transfer by one order of magnitude. Taken results based on the three body model are in good agreement with the experimental data.

Schematic representation of the elastic transfer nuclear reaction, on the left - the simultaneous transfer of two nucleons, on the right - the sequential transfer of two nucleons.

# + 9Be nuclear reactions

## Elastic channel

The DF potential was calculated using the effective DDM3Y-Paris nucleon-nucleon potential and the nuclear-matter-densities of projectile and target nuclei (see Eq. ([[rho\_nk]](#rho_nk)) and ([[rho\_alpha\_q]](#rho_alpha_q))). The matter density distribution of the deuteron projectile was chosen to be of the form

where $\Psi^d({\bf r})$ – a wave function of deuteron, ${\bf r}$ – a relative distance between proton and neutron. The calculated results of the DF potential for + 9Be are plotted in Fig. [6.1](#dbe_df). It is seen that the the main contribution of the + 9Be interaction starting at is caused by .

The folding potential calculated with the density functions of nuclear matter of 9Be. The designation is for the potential, while is for the potential, see Eq. ([[df\_in\_terms]](#df_in_terms)).

For convenience, in the OM and CC (CRC) calculations the DF potentials have been fitted by means of the sum of three Woods-Saxon potentials:

The angular distribution of elastic scattering data of from Be at laboratory energy 19.5 MeV in comparison with theoretical calculations within the OM (solid curve).

The cross sections of inelastic scattering Be()Be\* (E=2.43 MeV) at laboratory energies 19.5 MeV (full circle) and 35 MeV (full triangle). Theoretical curves are described in the text.

Here, the parameters of the imaginary part of the optical potential were obtained by fitting the theoretical cross sections to the experimental data at 19.5 MeV and 35 MeV incident energies. As a starting point, the same parameterizations of the real part were used. The obtained potential parameters after fitting are listed in Table [[dbe\_potpar]](#dbe_potpar) for both 19.5 MeV and 35.0 MeV incident energies.

The comparison of the results of the theoretical calculations with the measured data for elastic scattering at 19.5 MeV and 35.0 MeV energies are plotted in Fig. [6.2](#dbe_fig3). The cross sections obtained in the framework of the OM with the DF potential are shown as solid curves. Theoretical results obtained by means of the OM give an excellent agreement, , with the experimental data. The parameters of parameterized double-folding potential are listed in Table [[dbe\_potpar]](#dbe_potpar).

|  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| E, | i | V, | r, | a, | W, | r, | a, | V, | N, | r, |  |
| MeV |  | MeV | fm | fm | MeV | fm | fm | MeV |  | fm |  |
| 19.5 | 1 | 6.18 | 0.328 | 0.308 | 3.99 | 0.328 | 0.127 | 3.275 | 1.22 | 0.809 | 2.490 |
|  | 2 | 70.97 | 0.746 | 0.831 | 25.50 (17.5) | 0.746 | 0.766 |  |  |  |  |
|  | 3 | 0.605 | 1.491 | 1.724 | 0.924 | 1.491 | 2.238 |  |  |  |  |
| 35.0 | 1 | 5.941 | 0.328 | 0.308 | 7.07 | 0.612 | 0.108 | 3.275 | 1.17 | 0.809 | 2.503 |
|  | 2 | 68.68 | 0.746 | 0.831 | 22.50 (17.5) | 0.838 | 0.731 |  |  |  |  |
|  | 3 | 0.58 | 1.491 | 1.724 | 0.999 | 1.377 | 1.856 |  |  |  |  |

Radii are defined as .  
 The values are used in CRC calculations.

## Inelastic scattering

The CC and DWBA approaches have been applied to analyse the measured inelastic scattering data corresponding to the Be( MeV) excitation. Calculations were performed employing the FRESCO code and the DWUCK5 code which are available in the NRV knowledge-base .

In order to describe the measured experimental data one has to consider the Be target having a quadrupole deformation. Thus, the Be spectrum consists of the rotational band including the ground state, state at 2.43 MeV and state at 6.38 MeV. Couplings to these states were taken into account within the CC approach. The spin reorientations were also taken into account. The coupling interaction has the usual form:

where is the deformation parameter of multipole describing the target-nucleus form. Here, we neglect as usual the contribution of the Coulomb interaction.

The target coupling schemes in the Be()Be (upper) and the Be()Be (lower) nuclear reactions. The bold two-headed arrows indicate E transitions. The spin re-orientation effects are indicated as back-pointing arrows.

The calculated cross sections for inelastic scattering to the state at 2.43 MeV are shown in Fig. [6.3](#dbe_fig4). The solid curves correspond to the results obtained within the CC approach, while the dashed and dotted curves were obtained within the DWBA approach using different values of the deformation parameter . The used potential parameters are listed in Table [[dbe\_potpar]](#dbe_potpar).

All the results in Fig. [6.3](#dbe_fig4) are in good agreement with the experimental data, except for the cross sections around 60 at 19.5 MeV incident energy. The quadrupole deformation parameter extracted within the coupled-channel model is consistent with the previous studies .

In the case of DWBA calculations, one uses the DF potential (see Table [[dbe\_potpar]](#dbe_potpar)) for both the entrance and the exit channels. The DWBA angular distributions very well reproduce the structure of experimental data but clearly underestimate them when the deformation parameter is used (see the dotted curves in Fig. [6.3](#dbe_fig4)). In order to get the best fit the deformation parameter must be increased up to , which is quite close to the values reported in previous studies (see, for example, ).

Thus, one may confirm that channel coupling and the effects of spin reorientation enhance the cross section that results in the reduction of the deformation parameter. However, the DWBA approach takes into account only first-order contributions to the transition amplitude. In particular, it also describes only general features of the angular distributions and overestimates the deformation parameter in order to compensate the difference between the experimental data and the DWBA cross sections.

## One-nucleon transfer reactions

The one-neutron pick-up Be()Be and stripping Be()Be reactions were analyzed here within the framework of the CRC method.

The double-folding potential given in Table [[dbe\_potpar]](#dbe_potpar) was used in the CRC calculations for the entrance channel and the global optical parameterizations from Ref. were used for the exit channels. The coupling schemes of target and daughter nuclei for the Be()Be and Be()Be reactions are illustrated in Fig. [6.4](#dbe_fig5). The states of Be, and , as well as the low-lying excited states of Be, and , were included in the coupling scheme. Also, the schemes take into account the spin reorientations of states on the condition .

In order to construct the bound-state wave functions of the transferred particle in the entrance and exit channels, the common method, i.e. fitting the depth of the corresponding Woods-Saxon potential to the known binding energy, was employed. The reduced radius and diffuseness in this case are set to be fm and = 0.65 fm, respectively. If the transfer takes place to a final unbound state, the depth of the potential for this state was adjusted to yield a binding energy equal to MeV in accordance with the procedure used in Ref. .

Differential cross sections for the Be()Be reactions at 19.5 MeV leading to different final states (labelled in the figure) in Be. The experimental data are shown in comparison with theoretical results obtained within the CRC method.

If the core and the composite nuclei have internal excitation energies, a renewed binding energy of the transferred particle is expressed by the formula:

where the binding energy of the transferred particle, excitation energies of the composite and core nuclei, respectively.

Differential cross sections for the Be()Be reactions at 19.5 and 35 MeV leading to different final states (labelled in the figure) in Be. The experimental data are shown in comparison with theoretical results obtained within the CRC method.

The SA ([[sa\_shell\_model]](#sa_shell_model)) for the addition of a particle to a core with angular momentum to form a composite with is related to the matrix element of the creation operator :

$$\begin{aligned}
\label{eq:SA}
\mathcal{A}\_{Nlj} = \frac{\langle J\_{com} \| \hat{a}^\dagger \_{Nlj} \| J\_{core} \rangle}{\sqrt{2J\_{com}+1}}
%= (-)^{j+J-J'} \frac{\langle J\_{core} \| \hat{a} \_{NL\_J} \| J\_{com} \rangle}{\sqrt{2J+1}}\end{aligned}$$

where is the set of particle quantum numbers. The spectroscopic amplitudes for one particle states were calculated by means of the code using the effective Cohen-Kurath interaction for -shell nuclei . The calculated SA for the one-nucleon transfer reactions are listed in Table [[dbe\_SA]](#dbe_SA).

| Com | 2J | Core | 2J | N | 2J | SA |  | Com | 2J | Core | 2J | N | 2J | SA |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| Be | 3 | Be | 0 |  | 3 | 0.761 |  | Be | 3 | Li | 2 |  | 1 | 0.444 |
| Be | 3 | Be | 4 |  | 3 | 0.816 |  | Be | 3 | Li | 6 |  | 3 | 0.592 |
| Be | 3 | Be | 4 |  | 1 | 0.242 |  | Be | 3 | Li | 2 |  | 3 | 0.236 |
| Be | 5 | Be | 4 |  | 3 | 0.986 |  | Be | 3 | Li | 2 |  | 1 | 0.036 |
| Be | 5 | Be | 4 |  | 1 | 0.417 |  | Be | 5 | Li | 4 |  | 3 | 0.593 |
| Be | 5 | Be | 8 |  | 3 | 0.374 |  | Be | 5 | Li | 4 |  | 1 | 0.515 |
| Be | 7 | Be | 4 |  | 3 | 0.457 |  | Be | 5 | Li | 2 |  | 3 | 0.672 |
| Be | 7 | Be | 8 |  | 3 | 0.919 |  | Be | 5 | Li | 6 |  | 3 | 0.571 |
| Be | 7 | Be | 8 |  | 1 | 0.429 |  | Be | 5 | Li | 6 |  | 1 | 0.171 |
| Be | 0 | Li | 3 |  | 3 | 1.204 |  | Be | 5 | Li | 2 |  | 3 | 0.200 |
| Be | 0 | Li | 1 |  | 1 | 0.736 |  | Be | 7 | Li | 4 |  | 3 | 0.323 |
| Be | 4 | Li | 3 |  | 3 | 0.748 |  | Be | 7 | Li | 6 |  | 3 | 0.899 |
| Be | 4 | Li | 3 |  | 1 | 0.612 |  | Be | 7 | Li | 6 |  | 1 | 0.564 |
| Be | 4 | Li | 1 |  | 3 | 0.667 |  | Li | 3 | Li | 2 |  | 3 | 0.657 |
| Be | 4 | Li | 7 |  | 3 | 0.624 |  | Li | 3 | Li | 2 |  | 1 | 0.538 |
| Be | 4 | Li | 5 |  | 3 | 0.079 |  | Li | 3 | Li | 6 |  | 3 | 0.744 |
| Be | 4 | Li | 5 |  | 3 | 0.146 |  | Li | 3 | Li | 4 |  | 3 | 0.032 |
| Be | 8 | Li | 7 |  | 3 | 0.864 |  | Li | 3 | Li | 4 |  | 1 | 0.399 |
| Be | 8 | Li | 7 |  | 1 | 0.687 |  | Li | 1 | Li | 2 |  | 3 | 0.925 |
| Be | 8 | Li | 5 |  | 3 | 0.374 |  | Li | 1 | Li | 2 |  | 1 | 0.197 |
| Li | 4 | Li | 3 |  | 3 | 0.988 |  | Li | 1 | Li | 4 |  | 3 | 0.555 |
| Li | 4 | Li | 3 |  | 1 | 0.237 |  | Li | 7 | Li | 6 |  | 3 | 0.936 |
| Li | 4 | Li | 1 |  | 3 | 0.430 |  | Li | 7 | Li | 6 |  | 1 | 0.645 |
| Li | 4 | Li | 7 |  | 3 | 0.496 |  | Li | 7 | Li | 4 |  | 3 | 0.456 |
| Li | 4 | Li | 5 |  | 3 | 0.665 |  | Li | 5 | Li | 2 |  | 3 | 0.650 |
| Li | 4 | Li | 5 |  | 1 | 0.275 |  | Li | 5 | Li | 6 |  | 3 | 0.732 |
| Li | 2 | Li | 3 |  | 3 | 0.567 |  | Li | 5 | Li | 6 |  | 1 | 0.549 |
| Li | 2 | Li | 3 |  | 1 | 0.351 |  | Li | 5 | Li | 4 |  | 3 | 0.200 |
| Li | 2 | Li | 1 |  | 3 | 0.905 |  | Li | 5 | Li | 4 |  | 1 | 0.114 |
| Li | 2 | Li | 1 |  | 1 | 0.331 |  | Li | 2 |  | 2 |  | 0 | 0.907 |
| Li | 2 | Li | 5 |  | 3 | 0.767 |  | Li | 2 |  | 2 |  | 4 | 0.077 |
| Li | 6 | Li | 3 |  | 3 | 0.581 |  | Li | 6 |  | 2 |  | 4 | 0.943 |
| Li | 6 | Li | 5 |  | 3 | 0.660 |  | Li | 6 |  | 2 |  | 8 | 0.028 |
| Li | 6 | Li | 5 |  | 1 | 0.541 |  | Li | 4 |  | 2 |  | 4 | 0.929 |
| Li | 6 | Li | 7 |  | 3 | 0.973 |  | Be | 3 | He | 3 |  | 0 | 0.925 |
| Li | 6 | Li | 7 |  | 1 | 0.404 |  | Be | 3 | He | 3 |  | 4 | 0.784 |
| Li | 2 | Li | 3 |  | 3 | 0.617 |  | Be | 5 | He | 3 |  | 4 | 0.974 |
| Li | 2 | Li | 3 |  | 1 | 0.841 |  | Be | 5 | He | 3 |  | 8 | 0.260 |
| Li | 2 | Li | 1 |  | 3 | 0.178 |  | Be | 7 | He | 3 |  | 4 | 0.882 |
| Li | 2 | Li | 1 |  | 1 | 0.331 |  | Be | 7 | He | 3 |  | 8 | 0.737 |
| Li | 2 | Li | 5 |  | 3 | 0.231 |  | Li | 3 |  | 1 |  | 1 | 0.970 |
| Be | 3 | Li | 4 |  | 3 | 0.947 |  | Li | 1 |  | 1 |  | 1 | 0.961 |
| Be | 3 | Li | 4 |  | 1 | 0.319 |  | Li | 7 |  | 1 |  | 3 | 0.952 |
| Be | 3 | Li | 2 |  | 3 | 0.454 |  | Li | 5 |  | 1 |  | 3 | 0.223 |

Angular distributions of the Be()Be nuclear reaction at E=19.5 MeV are shown in comparison with the theoretical curves calculated in the framework of the CRC method in Fig. [6.5](#dbe_fig6).

In order to study the couplings of the input channels, the outputs were fixed using the deformation parameter of Be from Ref. , and for Be from Ref. . The direct transition from the ground state is indicated by the dotted line (DWBA). The contributions of the transitions from excited states (CC), and from spin reorientations (SR) are indicated by dashed and solid lines, respectively. During the analysis, it was found that spin reorientation has a significant contribution in the + Be channel, especially in the range of 40-60 degrees.

It is interesting to note that we managed to describe within the CRC method the differential cross section of the Be()Be reaction at all scattering angles, including the range 40-60, where they were not covered in Refs. .

An appreciable contribution of the

transitions was observed in the  + Be channel in the entire range of scattering angles. In the cross section of the  + Be channel, the theoretical calculation underestimates the experimental data starting from 70. Possibly, other higher excited states of Be should be taken into account.

Figure [6.6](#dbe_fig7) displays the cross sections of the Be()Be nuclear reaction at both 19.5 MeV and 35 MeV incident energies. As in the case of the () reactions, the () reactions also show the strong channel-coupling effects. We see the manifestation of spin-reorientation effects in the +Be channels and the significant contribution of the

transitions in the +Be channel. Disagreements around 30 in the +Be channel for both 19.5 MeV and 35 MeV incident energies and around 60 in the +Be channel for 19.5 MeV incident energy are possibly caused by the uncertainty in the +Be interaction potential.

Theoretical calculations made within the CRC method show, in general, good agreement with the experimental data for both () and () reactions. The analysis showed strong coupling effects in both entrance and exit channels. The effects of such couplings were also emphasized in Refs. .

## Cluster-transfer reaction

Differential cross sections for the nuclear reaction Be()Li are of particular interest. This is due to the specific behaviour of the cross section at large scattering angles, which indicates a He transfer. In addition, the cross section calculated within the DWBA approach underestimates the data even at forward scattering angles. Therefore, in order to understand the difference between theory and experiment, the following transfer mechanisms are suggested (see Fig. [6.7](#dbe_fig9)):

* the direct transfer of cluster and of He system;
* the sequential two-step transfer of -, -, - and -;

The scheme illustrates the reaction mechanisms taken into account in CRC calculations of the cross sections for Be()Li reaction.

The resulting differential cross section for the Be(,)Li reaction has the form of a coherent sum of two amplitudes

where the amplitude

describes the transfer of the He nucleus and sequential two-step transfer of n- and -n, and the amplitude

corresponds to the deuteron pick-up and sequential two-step transfer of - and -.

image

The DF potential (see Table [[dbe\_potpar]](#dbe_potpar)) for the entrance channel and global optical potential parameterizations from Refs. for intermediate and exit channels were used in the analysis. The prior form for the first coupling and the post form for the second coupling were chosen for two-step transfer reactions in order to avoid the non-orthogonal terms in the calculations of transition amplitudes.

The spectroscopic amplitudes of the and He systems were taken from Ref. , while the alpha-cluster spectroscopic amplitudes given in Table [[dbe\_SA]](#dbe_SA) were provided by Dr. A. Volya within the method reported in Ref. .

The calculated cross sections are shown in Fig. [[dbe\_fig8]](#dbe_fig8) with the -particle angular distributions formed in the Be(d,)Li reaction at incident energies of 19.5 and 35 MeV and corresponding to the low-lying excitation of the Li nucleus in the exit channels. The transfer of the deuteron (dash-dotted curve) provides the dominant contribution in all the channels. Despite the fact that the spectroscopic amplitude of the deuteron in the Be nucleus is not of great importance, a noticeable cross section is due to the large value of the deuteron spectroscopic amplitude of He.

The angular distribution of deuteron transfer has a significant cross section also at the backward scattering angles, which is mainly caused by the contribution of the wave. This symmetrical behaviour of the cross section of waves is very similar to the cross section of evaporation residues. Tanaka *et al* analyzed the role of the compound process in Be()Li reaction and claimed the domination of the compound nucleus channels at the energies of 12.17 MeV and 14.43 MeV. However, in Ref. the negligible contribution of the compound-nucleus mechanism was shown at 7 MeV using the DWBA analysis. In this regard, our theoretical results based on the CRC method show that there is no need to take into account the mechanism through the compound-nucleus formation at energies of 19.5 and 35.0 MeV.

Starting from scattering angle 120, the transfer of the He system, labeled as He in Fig. [[dbe\_fig8]](#dbe_fig8), has a predominant contribution in all channels. It should be noted that a similar result was reported earlier in Ref. . One-step transfer of the He nucleus was also indicated as a dominant process by Jarczyk *et al* in studying the C(B,Li)O and C(,Li)Be reactions.

Using the CRC method, we are able to estimate the contribution of the sequential transfer of He, which was not studied before. Corresponding cross sections are shown in Fig. [[dbe\_fig8]](#dbe_fig8) as curves labeled and . It turned out that the - and - transfer processes provide indeed a contribution more than one order of magnitude smaller in comparison with the one-step He transfer. Nevertheless, it should be noted that the contribution of the - and the - transfer channels increases with the increase in the Li excitation energy, where they should not be ignored.

The scheme illustrates the energy balance of the different intermediate stages for the two-step mechanisms of Be()Li transfer reaction. The -values for the different intermediate channels are shown near the corresponding lines. The numbers near the arrows correspond to the spectroscopic amplitudes of the heaviest reaction participants. For example, spectroscopic amplitude for the Be = Li + configuration is equal .

The two-step - transfer is another mechanism providing a noticeable contribution to the cross section. It is due to the prominent cluster structure of the Be nucleus having the weakly bound neutron. This structural feature explains also the weakness of the - sequential transfer contribution to the cross section corresponding to the Li(g.s.) in the exit channel. However, with increasing the Li excitation energy these two mechanisms are interchanged in the significance of their contributions, as depicted by the curves in Fig. [[dbe\_fig8]](#dbe_fig8), and the - transfer begins to play a leading role, providing, in particular, almost 10 times larger contribution in the case of reaction at = 35 MeV with Li(7.46 MeV) in exit channel.

In Fig. [6.8](#dbe_fig10), the possible scenarios for the - and - sequential transfer for the reaction under consideration are shown in respect to the -values. One may see that all the steps of the - sequential transfer have positive -values, while the - transfer goes through the intermediate channel Li + He that has a considerably negative -value. Together with the large values of the spectroscopic amplitudes (shown near to the arrows in Fig. [6.8](#dbe_fig10)), this explains the leading role of the () mechanism in populating the ground state of Li in the exit channel.

Contributions of the different mechanisms to the cross section of the Be()Li reaction. See Fig. [6.7](#dbe_fig9) for explanation of the curve notations.

The situation becomes quite different in the case of the Li(5/2) in the exit channel. First, the population of this state through the - transfer involves the Be = Be +  intermediate configuration where the Be cluster has to be in the excited state. Note that the Be() ground state is inappropriate because of angular-momentum-coupling mismatch in the entrance and exit configurations. Second, the extremely small spectroscopic amplitude of the Be = Li +  configuration, which is , influences the transfer amplitude. These two factors lead to the suppression of the contribution of () mechanism in population of the Li(5/2) state in the exit channel. Therefore, the - sequential transfer prevails over the - one.

Figure [6.9](#dbe_fig11) shows the contributions of all the mechanisms mentioned above to the total cross section of the Be()Li reaction (see Fig. [6.7](#dbe_fig9)) as a function of the deuteron energy. One may conclude that mainly four mechanisms contribute to the cross section of this reaction. The transfer of the deuteron-cluster is predominant channel at all collision energies. The sequential - and - transfers play a significant role at the high energies. The He transfer gives almost 20% of the cross section at low energies and outdoes the sequential - transfer in this energy domain. This allows us to claim that the configurations Be and He provide noticeable contributions to the ground-state wave function of the Be nucleus. These conclusions agree well with the previous experimental studies .

# Summary and Conclusions

This work is dedicated to study the structure of weakly bound nuclei, such as 6He, 6Li, and 9Be, and its impact on the direct nuclear reactions. The nuclei are considered as the three body systems: 6He=, 6Li and 9Be. Their wave functions are chosen to be in the form of Gaussian basis. The three body wave functions are obtained within the stochastic variational method. Variational calculations underestimate the three body eigen-energies when compared to experimental values. Such results point to the presence of additional natures. In particular, the -forces and other cluster configurations are not implemented explicitly in the current model. Taking into account such kind of specificities it may contribute to the properties of the nuclear systems under study.

The density functions of nuclear matter on the basis of the three body wave functions are derived in explicit form. The functions obtained for these nuclei demonstrate drastically different shapes in comparison with the spherical stable nuclei. Using the density functions the rms radii are expressed explicitly. Obtained values on both charge and matter radii are in good agreement with the available estimations from the well-known sources.

In the framework of the double folding model the interaction potentials, 6He, +6Li and +9Be, have been calculated. It should be pointed out that the large diffuseness of the double folding potentials are caused by the valence nucleons. The potential built on the three body density functions was used to calculate the differential cross sections of elastic, inelastic scattering and nuclear transfer reactions.

The nuclear reactions 6He+ and + 6Li, at laboratory energies 151 MeV and 166 MeV, respectively, are excellent tools in terms of the theoretical study. By using them we could extract the optical potential parameters, the spectroscopic information of the three cluster configurations. The obtained optical potentials in calculations of elastic scattering and elastic transfer show the good agreement of corresponding differential cross sections with experimental data. The spectroscopic amplitude of di-neutron in 6He is consistent with the spectroscopic information given in Ref. .

The experimental data on elastic scattering of -particles by 6He and 6Li demonstrate significant growth of cross section at backward angles. Such kind of behaviour is characterized by contribution of the elastic transfer channel. The analysis based on the CRC calculations show that the major contribution to the elastic transfer cross section is resulted from the di-nucleon transfer channel, whereas the sequential transfer is one order of magnitude lower. This confirms validity of the three body model, and the analysis is compatible with the conclusions of other authors from .

The deuteron-induced reactions on a Be target at the collision energies 19.5 and 35 MeV were studied. The calculated double-folding potential is applied successfully to describing the cross sections of elastic and inelastic scatterings, one-nucleon transfer and cluster-transfer reactions. The deformation parameter for the transition

of Be is extracted. The resulting deformation parameter is consistent with the parameters given in Ref. .

The strong coupling effects have been shown for the () and () one-nucleon transfer reactions. In particular, the cross sections of the reaction (, ) obtained within CRC method are in good agreement with the experimental data in wide angular range. It is due to taking into account all possible couplings including the spin-reorientations. The spin reorientation coupling allows, particularly, to improve agreement with data significantly for the 9Be(,)Be reaction. Such kind of effects occurs when a strong deformed and high spin valued nucleus, like 9Be, is involved.

The theoretical analysis of the nuclear reaction Be(, )Li shows that it goes basically through three reaction mechanisms, which are the transfer mechanism of -, the simultaneous and sequential transfer of He. It has become known that the light nucleus He is transferred mainly simultaneously at the backward scattering angles, and the contribution of its sequential transfer is an order of magnitude lower. The - transfer mechanism turned out to be competitive to the transfer. Based on these observations from studying the Be nuclear reactions, it can be concluded that Be nucleus has a cluster structure.

# Definitions from the quantum theory of angular momenta

The Appendix presents some definitions used in this work and the table integrals.

A total angular momentum are decomposed into two angular momenta and by means of the Clebsch-Gordan coefficient. For example, to quote a basis with the angular momentum with its -component , the Clebsch-Gordan coefficient can be represented as follow

For non-zero values of the coefficient ([[the\_CG\_coefficient]](#the_CG_coefficient)) vectors , and must satisfy the rule of triangle:

and the condition

If there are three vectors and , one can get a total angular momentum $\bf j$ in two ways

$$\begin{aligned}
\label{6j\_basis1}
\bf j & =\bf \left( j\_1 + j\_2 \right) + j\_3 = j\_{12} + j\_3 \\
\label{6j\_basis2}
& = \bf j\_1 + \left( j\_2 + j\_3 \right) = j\_1 +j\_{23}\end{aligned}$$

The Basis and the basis corresponding to Eq. ([[6j\_basis1]](#Xded43d20728515b6a7edbde1ffb9212f0af4004)) and Eq. ([[6j\_basis2]](#X39f085afadc360944ddc9502bb387ce7a23fdc5)) are related through a factor , which is the Racah coefficient:

Four angular momenta, $\bf j\_1,~j\_2,~j\_3$ and $\bf j\_4$, are added into the total momentum $\bf j$ by

$$\begin{aligned}
\label{9j\_basis1}
\bf j & =\bf \left( j\_1 + j\_2 \right) + ( j\_3 + j\_4) = j\_{12} + j\_{34} \\
\label{9j\_basis2}
& = \bf ( j\_1 + j\_3 ) + \left( j\_2 + j\_4 \right) = j\_{13} +j\_{24}\end{aligned}$$

Two basis and , constructed respectively on the scheme Eq. [[9j\_basis1]](#Xb6e0866fe3cafd5085a3877c67d0bf9a23477e2) and Eq. [[9j\_basis2]](#X425505c41f6c738bc32e7daf84705179c6f4426), are related as follow

where transformation coefficient with square brackets is called a unitary 9j-symbol.

A spacial spherical harmonics is expressed like

$$\mathcal{Y}\_{lm}({\bf r}) = r^{l} Y\_{lm}(\hat{r})$$

where – spherical function, which is a eigenfunction of angular part the Laplace operator. For ${\bf r} = a{\bf r}\_1+b {\bf r}\_2$ a decomposition of the spacial spherical harmonics $\mathcal{Y}\_{lm}({\bf r})$ leads to the following equality

$$\begin{aligned}
\mathcal{Y}\_{lm}({\bf r} = a{\bf r}\_1+b {\bf r}\_2)=& \sum\_{l\_1,l\_2,m\_1,m\_2} a^{l\_1} b^{l\_2}
\langle ~ l\_1 m\_1~l\_2 m\_2~ \vert ~l m~ \rangle \mathcal{D}(l,l\_1,l\_2) \times \nonumber \\
& \times \mathcal{Y}\_{l\_l m\_1}({\bf r\_1}) \mathcal{Y}\_{l\_2 m\_2}({\bf r\_2}) \nonumber \\
= & \sum\_{l\_1,l\_2} a^{l\_1} b^{l\_2}
\mathcal{D}(l,l\_1,l\_2) \left[ \mathcal{Y}\_{l\_l}({\bf r\_1}) \times \mathcal{Y}\_{l\_2}({\bf r\_2}) \right]\_{lm}
\end{aligned}$$

with the condition , and is given by

Spherical harmonics with the momenta and are coupled as follow

where the coefficient reads as

It would be useful also note a coupling between two spherical hyper harmonics kind of (see , p. 269)

$$\left[ Y^{(l\_1l\_2)}\_{l\_{12}}(\hat{{\bf r}}\_1,\hat{{\bf r}}\_2) \times Y^{(l\_3l\_4)}\_{l\_{34}}(\hat{{\bf r}}\_1,\hat{{ \bf r}}\_2) \right]\_{lm}= \sum\_{l\_{13}l\_{24}} {E}^{l\_1l\_2l\_{12}l\_2l\_4l\_{34}l}\_{l\_{13}l\_{24}} Y^{(l\_{13}l\_{24})}\_{lm}(\hat{{\bf r}}\_1,\hat{{\bf r}}\_2)$$

where the coupling coefficient is given as

The modified spherical Bessel function may be expressed as follows (see , p. 98)

where is a modified Bessel function of the first kind.

The integral depending on the parameters and

can be expressed in explicit form as follows

where, – the Gamma function.

An integral kind of

has the analytic form in the following (see , p. 270)

where, – is the associated Laguerre polynomial.

The integral with six parameters

may be expressed explicitly as follow (see , p. 271)

# Parameters of the three body wave function for 6He

@cllr@

& & &

Table  – continued from previous page  
& & &

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& 0.07010120 & 0.17098674 & 0.8287413344E-05  
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3 & 0.07010120 & 1.07956780 & 0.2205912189E-03  
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5 & 0.07010120 & 6.81612260 & 0.1309976070E-03  
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7 & 0.22551676 & 0.55006725 & 0.1457399081E-02  
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15 & 0.44260156 & 6.81612260 & 0.1241336217E-01  
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25 & 2.79447630 & 6.81612260 & -0.7981418030E-03

1. The obtained data of the variational calculations are not the subject of our research, since the results, perhaps better ones, have already been published and discussed. The purpose of the section result of the three-body wave function is to familiarize with what wave function we are dealing with, and its features will be needed in further discussions. [↑](#footnote-ref-40)
2. Further, when it comes to calculations within the framework of the OM, CC and CRC methods, it means that all calculations have been performed using the code FRESCO. [↑](#footnote-ref-46)