

Chapter 23

One-particle transfer

23.1 Fractional parentage coefficients

23.1.1 One orbit

Observables for the removal or addition of a nucleon from a specific initial state to a specific final state are related to the matrix elements of the creation and destruction operators. In this section we will study the basic properties of these matrix elements and their sum-rules. The reduced matrix elements of the creation and destruction operators are used to define the spectroscopic factors associated with nuclear reactions. They are also the building blocks for the more complicated operators associated with one-body (such as electromagnetic and beta decay) and two-body transition amplitudes.

The creation operator a_{km}^+ is a tensor of rank j since it creates the single-particle state $| km \rangle$ (k stands for the set of single-particle quantum numbers $n\ell j$). The destruction operator a_{km} is not a tensor of rank j , however, by Eq. [13.34],

$$\tilde{a}_{km} \equiv (-1)^{j+m} [a_{k,-m}^+]^+ = (-1)^{j+m} a_{k,-m} \quad (23.1)$$

is a tensor of rank j . The inverse of Eq. (23.1) is

$$a_{km} = (-1)^{j-m} \tilde{a}_{k,-m}. \quad (23.2)$$

Eq. [13.35] (with the matrix elements being real and $p = j$) can be used to relate the reduced matrix elements involving \tilde{a} to those involving a^+

$$\langle k^{n-1} \omega' J' | \tilde{a}_k | k^n \omega J \rangle = (-1)^{j+J'-J} \langle k^n \omega J | a_k^+ | k^{n-1} \omega' J' \rangle, \quad (23.3)$$

All many-body matrix elements of a^+ can be reduced to these involving a single k -state.

Wave-function expansion relations and sum-rules for the states in the $n - 1$ particle system can be obtained by operating with the number operator

$$\hat{N}_k = \sum_m a_{km}^+ a_{km} \quad (23.4)$$

on the k^n configuration and then inserting a complete set of states with $n - 1$ particles

$$\begin{aligned} \hat{N}_k | k^n \omega JM \rangle &= \sum_m a_{km}^+ a_{km} | k^n \omega JM \rangle = n | k^n \omega JM \rangle \\ &= \sum_{m\omega'J'M'} a_{km}^+ | k^{n-1} \omega' J' M' \rangle \langle k^{n-1} \omega' J' M' | a_{km} | k^n \omega JM \rangle. \end{aligned}$$

The matrix element of $a_{k,m}$ can be reduced with the Wigner-Eckhart theorem

$$\begin{aligned} \langle k^{n-1} \omega' J' M' | a_{km} | k^n \omega JM \rangle &= (-1)^{j-m} \langle k^{n-1} \omega' J' M' | \tilde{a}_{k,-m} | k^n \omega JM \rangle \\ &= (-1)^{j-m+J'-M'} \begin{pmatrix} J' & j & J \\ -M' & -m & M \end{pmatrix} \langle k^{n-1} \omega' J' || \tilde{a}_k || k^n \omega J \rangle, \end{aligned}$$

and by Eq. (23.3) the reduced matrix element of \tilde{a} can be converted in a reduced matrix element of a^+ to obtain the final result:

$$\begin{aligned} \hat{N}_k | k^n \omega JM \rangle &= n | k^n \omega JM \rangle = \sum_{m\omega'J'M'} (-1)^{-m-M'+J} \begin{pmatrix} J' & j & J \\ -M' & -m & M \end{pmatrix} \\ &\quad \times a_{km}^+ | k^{n-1} \omega' J' M' \rangle \langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle. \end{aligned} \quad (23.5)$$

We can thus expand the k^n wavefunction in terms of those in the k^{n-1} basis:

$$| k^n \omega JM \rangle = (-1)^n \sum_{\omega'J'} \frac{\langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle}{n \sqrt{(2J+1)}} [Z^+(k^{n-1} \omega' J') \otimes a_k^+]_M^J | \rangle. \quad (23.6)$$

A phase factor of $(-1)^{n-1}$ arises from commuting a^+ with the $n - 1$ particles in the state k^{n-1} .

A sum-rule for the matrix elements of a^+ can be obtained by multiplying both sides of Eq. (23.5) by $\langle k^n \omega'' J'' M'' |$ to obtain

$$\begin{aligned} n \delta_{JJ''} \delta_{MM''} \delta_{\omega\omega''} &= \sum_{m\omega'J'M'} (-1)^{-m-M'+J} \begin{pmatrix} J' & j & J \\ -M' & -m & M \end{pmatrix} \\ &\quad \times \langle k^n \omega'' J'' M'' | a_{km}^+ | k^{n-1} \omega' J' M' \rangle \langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle \\ &= \sum_{m\omega'J'M'} \begin{pmatrix} J' & j & J \\ -M' & -m & M \end{pmatrix} \begin{pmatrix} J'' & j & J' \\ -M'' & m & M' \end{pmatrix} \\ &\quad \times \langle k^n \omega'' J'' || a_k^+ || k^{n-1} \omega' J' \rangle \langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle \\ &= \frac{\delta_{JJ''} \delta_{MM''}}{(2J+1)} \sum_{\omega'J'} \langle k^n \omega'' J'' || a_k^+ || k^{n-1} \omega' J' \rangle \langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle, \end{aligned} \quad (23.7)$$

where Eq. [13.12] has been used for the summation over $3j$ coefficients. Thus one finds the sum-rule:

$$\sum_{\omega' J'} \langle k^n \omega'' J'' || a_k^+ || k^{n-1} \omega' J' \rangle \langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle = n(2J+1) \delta_{\omega \omega''} \delta_{JJ''}. \quad (23.8)$$

Thus:

$$\sum_{\omega' J'} |\langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle|^2 = \sum_{\omega' J'} |\langle k^{n-1} \omega' J' || \tilde{a}_k || k^n \omega J \rangle|^2 = n(2J+1). \quad (23.9)$$

The matrix elements in which the sum over final states is normalized to unity are historically called coefficients of fractional parentage [1] (CFP) defined by:

$$\langle j^n \omega J || j^{n-1} \omega' J' \rangle \equiv \frac{\langle k^n \omega J || a_k^+ || k^{n-1} \omega' J' \rangle}{\sqrt{n(2J+1)}}. \quad (23.10)$$

The j is used in the CFP rather than k , in order to emphasize the fact that these coefficients depend only on j and not on n and ℓ . In terms of CFP the sum-rule of Eq. (23.9) becomes:

$$\sum_{\omega' J'} |\langle j^n \omega J || j^{n-1} \omega' J' \rangle|^2 = 1. \quad (23.11)$$

Thus, the square of the one-particle CFP is the probability of taking one particle in the state k from the configuration $|k^n \omega J\rangle$ leaving and leaving it in the configuration $|k^{n-1} \omega' J'\rangle$.

A sum-rule related to the connection between states in the n and $n+1$ particle systems can be obtained with the number operator written in the form:

$$\hat{N}_j = \sum_m (1 - a_{jm} a_{jm}^+). \quad (23.12)$$

Multiplying this on the state k^n and inserting a complete set of states of k^{n+1} gives:

$$\sum_{\omega' J'} |\langle k^{n+1} \omega' J' || a_k^+ || k^n \omega J \rangle|^2 = (2J+1)(2j+1-n). \quad (23.13)$$

Combining Eqs. (23.9) and (23.13) we obtain a sum-rule for the sum over all states in both k^{n-1} and k^{n+1} :

$$\begin{aligned} \sum_{\omega' J'} |\langle k^{n-1} \omega' J' || \tilde{a}_k || k^n \omega J \rangle|^2 + \sum_{\omega' J'} |\langle k^{n+1} \omega' J' || a_k^+ || k^n \omega J \rangle|^2 \\ = (2J+1)(2j+1). \end{aligned} \quad (23.14)$$

When there is only one initial and final state, the CFP are unity. The basic examples are:

$$|\langle j^1, J=j || j^0, J'=0 \rangle| = 1, \quad (23.15)$$

$$|\langle j^2 J || j^1, J'=j \rangle| = 1, \quad (23.16)$$

where J is even, and

$$|< j^{2j+1}, J=0 | \{j^{2j}, J'=j >| = 1. \quad (23.17)$$

The wavefunctions for a closed-shell minus n particles can be expressed in terms of those for holes:

$$|k^{-n}\omega J > \equiv |k^{2j+1-n}\omega J >. \quad (23.18)$$

The matrix elements of \tilde{a} for the hole-configurations are algebraically the same as the matrix elements of a^+ for particle configurations:

$$< k^{-n}\omega' J' | \tilde{a}_k | k^{-(n-1)}\omega J > = < k^n\omega' J' | a_k^+ | k^{n-1}\omega J >. \quad (23.19)$$

Thus from Eq. (23.3) we can relate the matrix elements of a^+ for hole states to those for particle states:

$$\begin{aligned} < k^{-(n-1)}\omega J | a_k^+ | k^{-n}\omega' J' > = (-1)^{j+J'-J} < k^{-n}\omega' J' | \tilde{a}_k | k^{-(n-1)}\omega J > \\ = (-1)^{j+J'-J} < k^n\omega' J' | a_k^+ | k^{n-1}\omega J >, \end{aligned} \quad (23.20)$$

or

$$< k^{2j+1-(n-1)}\omega J | a_k^+ | k^{2j+1-n}\omega' J' > = (-1)^{j+J'-J} < k^n\omega' J' | a_k^+ | k^{n-1}\omega J >. \quad (23.21)$$

In terms of the one-particle CFP this becomes

$$\begin{aligned} < j^{-(n-1)}\omega J | \{j^{-n}\omega' J' > \equiv < j^{2j+1-(n-1)}\omega J | \{j^{2j+1-n}\omega' J' > \\ = (-1)^{j+J'-J} \sqrt{\frac{n(2J'+1)}{(2j+2-n)(2J+1)}} < j^n\omega' J' | \{j^{n-1}\omega J >. \end{aligned} \quad (23.22)$$

We can convert the special results of Eqs. (23.15) and (23.16) for particles to those for holes. When $n=1$, $J'=j$, and $J=0$ one obtains the same result as in Eq. (23.17). For the case when $n=2$ and $J=j$ one obtains

$$|< j^{2j}, J=j | \{j^{2j-1} J' >| = \sqrt{\frac{(2J'+1)}{j(2j+1)}} |< j^{2J'} | \{j^1, J=j >| = \sqrt{\frac{(2J'+1)}{j(2j+1)}}, \quad (23.23)$$

where J' is even. For example when we take one particle from the $[(j=5/2)^5, J=5/2]$ configuration the probability to leave the system in the states $[(j=5/2)^4, J']$ is $\frac{2}{30}$, $\frac{10}{30}$ and $\frac{18}{30}$, for $J'=0, 2$ and 4 , respectively.

There are a variety of methods and computer programs for calculating the one-particle CFP [2], [3], [4], [5], [6], [7], and they are tabulated in the literature [1], [8], [9], [10], [11], [12] for small values of n and j . Also the projection operator method discussed in the next section can be used to calculate these CFP. All matrix elements in the J -scheme can be reduced to equations involving these one-particle CFP, and thus they are a primary input to any calculation or computer code that utilizes a J -coupled basis. As an example, the

Table (23.1). One-particle CFP for the $(j = 5/2)^3$ configuration.

$J \backslash J'$	0	2	4
3/2	0	$-\sqrt{\frac{5}{7}}$	$\sqrt{\frac{2}{7}}$
5/2	$-\sqrt{\frac{2}{9}}$	$\sqrt{\frac{5}{18}}$	$\sqrt{\frac{1}{2}}$
9/2	0	$\sqrt{\frac{3}{14}}$	$-\sqrt{\frac{11}{14}}$

Table (23.2). One-particle CFP for the $(j = 7/2)^3$ configuration.

$J \backslash J'$	0	2	4	6
3/2	0	$\sqrt{\frac{3}{14}}$	$-\sqrt{\frac{11}{14}}$	0
5/2	0	$\sqrt{\frac{11}{18}}$	$\sqrt{\frac{2}{33}}$	$-\sqrt{\frac{65}{198}}$
7/2	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{5}{36}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{13}{36}}$
9/2	0	$\sqrt{\frac{13}{126}}$	$-\sqrt{\frac{50}{77}}$	$\sqrt{\frac{49}{198}}$
11/2	0	$-\sqrt{\frac{5}{18}}$	$\sqrt{\frac{13}{66}}$	$-\sqrt{\frac{52}{99}}$
15/2	0	0	$\sqrt{\frac{5}{22}}$	$-\sqrt{\frac{17}{22}}$

one-particle CFP for the $(j = 5/2)^3$ and $(j = 7/2)^3$ configuration [1] are given in Tables (23.1) and (23.2).

The results for specific CFP given in the text and tables implicitly have associated a choice for the phase factors of the initial and final wavefunctions. It is important to keep these phase factors consistent throughout the intermediate stages of a calculation, especially when the CFP are combined with the outputs of other programs.

23.2 Many orbits

The reduced matrix elements of the creation and destruction operators for the full many-body wave functions have the form:

$$\langle \Psi_f^{A-1} \omega_f J_f | \tilde{a}_k | \Psi_i^A \omega_i J_i \rangle = (-1)^{j+J_f-J_i} \langle \Psi_i^A \omega_i J_i | a_k^+ | \Psi_f^{A-1} \omega_f J_f \rangle. \quad (23.24)$$

One can use the number operator to derive the sum-rules:

$$\sum_{\omega_f J_f} |\langle \Psi_f^{A-1} \omega_f J_f | \tilde{a}_k | \Psi_i^A \omega_i J_i \rangle|^2 = (2J_i + 1) \langle n_k \rangle_i, \quad (23.25)$$

and

$$\sum_{\omega_f J_f} |\langle \Psi_f^{A+1} \omega_f J_f | a_k^+ | \Psi_i^A \omega_i J_i \rangle|^2 = (2J_i + 1)[(2j + 1) - \langle n_k \rangle_i]. \quad (23.26)$$

In these expressions $\langle n_k \rangle_i$ is the number of particle in orbit k in state Ψ_i . If one has a pure configuration this number is an integer, but for a mixed configuration this is the average number of particles in orbit k . For example, if one has a basis states from a single partition, e.g.

$$| n \omega J \rangle_i = | [(k_1^{n_1} \omega_1 J_1)(k_2^{n_2} \omega_2 J_2) J_{12}] [k_3^{n_3} \omega_3 J_3] J \rangle_i,$$

then

$$\langle n_{k_a} \rangle_i = n_a,$$

for $a = 1, 2$ and 3 . If one has a wavefunction that contains two or more different partitions, e.g.

$$| \Psi \rangle_i = c | \Phi \rangle_i + c' | \Phi' \rangle_i,$$

where

$$| \Phi \rangle_i = | [(k_1^{n_1} \omega_1 J_1)(k_2^{n_2} \omega_2 J_2) J_{12}] [k_3^{n_3} \omega_3 J_3] J \rangle,$$

and

$$| \Phi' \rangle_i = | [(k_1^{n'_1} \omega'_1 J'_1)(k_2^{n'_2} \omega'_2 J'_2) J'_{12}] [k_3^{n'_3} \omega'_3 J'_3] J \rangle, \quad (23.27)$$

then

$$\langle n_{k_a} \rangle_i = c^2 n_a + c'^2 n'_a,$$

for $a = 1, 2$ and 3 .

23.3 Spectroscopic factors

23.3.1 Basic definitions and sum-rules

The spectroscopic factor is defined in terms for the reduced matrix elements of a^+ by:

$$S = \frac{|\langle \Psi^A \omega J || a_k^+ || \Psi^{A-1} \omega' J' \rangle|^2}{(2J+1)} = \frac{|\langle \Psi^{A-1} \omega' J' || \tilde{a}_k || \Psi^A \omega J \rangle|^2}{(2J+1)}, \quad (23.28)$$

where the $(2J+1)$ factor is by convention associated with the heavier mass A . S implicitly depends upon all quantities in the expressions (explicit labels are added when they are needed).

When the wavefunction is restricted to one orbital k with angular momentum j the spectroscopic factor is related to the coefficient of fractional parentage given in Eq. [20.10] by:

$$S = n |\langle j^n \omega J || j^{n-1} \omega' J' \rangle|^2. \quad (23.29)$$

The cross section for reactions involving the removal of a particle (proton or neutron) from nucleus A is proportional to the full matrix element of \tilde{a} summed over m and the final M -states M_f , and averaged over the initial M -states M_i :

$$\begin{aligned} \sigma^- &\sim \frac{1}{2J_i+1} \sum_{M_i, M_f, m} |\langle \Psi_f^{A-1} \omega_f J_f M_f | \tilde{a}_{k,m} | \Psi_i^A \omega_i J_i M_i \rangle|^2 \\ &= \sum_{M_i, M_f, m} \begin{pmatrix} J_f & j & J_i \\ -M_f & m & M_i \end{pmatrix} \begin{pmatrix} J_f & j & J_i \\ -M_f & m & M_i \end{pmatrix} \frac{|\langle \Psi_f^{A-1} \omega_f J_f || \tilde{a}_k || \Psi_i^A \omega_i J_i \rangle|^2}{(2J_i+1)} \\ &= \frac{|\langle \Psi_f^{A-1} \omega_f J_f || \tilde{a}_k || \Psi_i^A \omega_i J_i \rangle|^2}{(2J_i+1)} = S. \end{aligned} \quad (23.30)$$

From the reaction theory one usually calculates a “single-particle” cross section σ_{sp} (i.e. assuming that $S = 1$) for the removal of one particle (proton or neutron) in orbit k and thus:

$$\sigma^- = S \sigma_{sp}. \quad (23.31)$$

Similarly the cross section for the addition of a particle to nucleus A is obtained from the full matrix element of a^+ summed over m and M_f and averaged over M_i :

$$\begin{aligned} \sigma^+ &\sim \frac{1}{2J_i+1} \sum_{M_i, M_f, m} |\langle \Psi_f^{A+1} \omega_f J_f M_f | a_{k,m}^+ | \Psi_i^A \omega_i J_i M_i \rangle|^2 \\ &= \sum_{M_i, M_f, m} \begin{pmatrix} J_f & j & J_i \\ -M_f & m & M_i \end{pmatrix} \begin{pmatrix} J_f & j & J_i \\ -M_f & m & M_i \end{pmatrix} \frac{|\langle \Psi_f^{A+1} \omega_f J_f || a_k^+ || \Psi_i^A \omega_i J_i \rangle|^2}{(2J_i+1)} \end{aligned}$$

$$= \frac{|\langle \Psi_f^{A+1} \omega_f J_f || a_k^+ || \Psi_i^A \omega_i J_i \rangle|^2}{(2J_i + 1)} = \frac{(2J_f + 1)}{(2J_i + 1)} S, \quad (23.32)$$

where S is the spectroscopic factor associated with the removal of a particle from $A + 1$ to A . Thus the cross section for the addition of a particle is given by:

$$\sigma^+ = \frac{(2J_f + 1)}{(2J_i + 1)} S \sigma_{sp}, \quad (23.33)$$

where the spectroscopic factor S is for the removal of a nucleon from $(A + 1)$ to A , and the extra J factors take into account the different M state averaging for the two type of reactions.

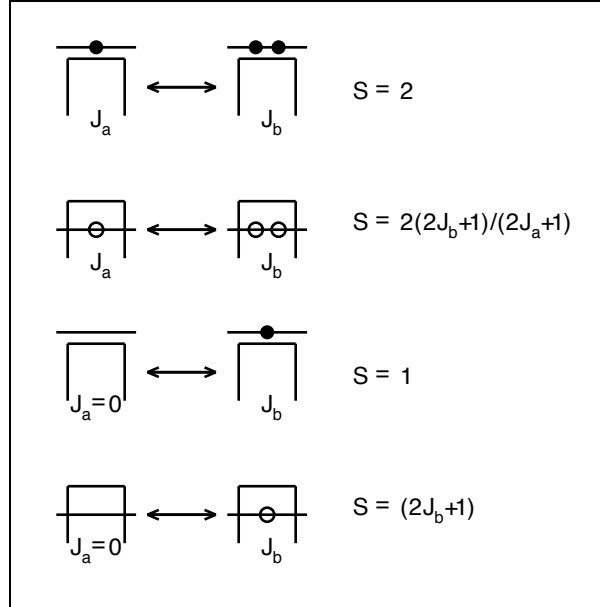


Figure 23.1: Schematic diagram for the spectroscopic factors for nucleon transfer for simple configurations where the sum-rules can be used to get the results. The box indicates a closed-shell configuration with holes (open circles) and particle (filled circles) added to one orbital.

The particle removal sum-rule for spectroscopic factors is obtained from Eq. [20.26] by summing over all states in the $(A - 1)$ nucleus (denoted by $f -$):

$$\sum_{f-} S_{i,f,k} = \langle n_k \rangle_i \quad [A \rightarrow (A - 1)], \quad (23.34)$$

and the particle addition sum-rule for spectroscopic factors is obtained from Eq. [20.27] by summing over all states in the $(A + 1)$ nucleus (denoted by $f +$):

$$\sum_{f+} \frac{(2J_f + 1)}{(2J_i + 1)} S_{i,f,k} = (2j + 1) - \langle n_k \rangle_i \quad [A \rightarrow (A + 1)], \quad (23.35)$$

where $\langle n_k \rangle_i$ is the average occupation of protons or neutrons in orbit k in the initial state i [with a maximum possible value of $(2j + 1)$]. The total sum-rule is:

$$\sum_{f-} S_{i,f,k} + \sum_{f+} \frac{(2J_f + 1)}{(2J_i + 1)} S_{i,f,k} = (2j + 1). \quad (23.36)$$

If we also sum Eq. (23.34) over all orbits we have a sum-rule for the total number of particles (protons or neutrons) in the nucleus:

$$\sum_{f-,k} S_{i,f,k} = Z \text{ or } N \quad [A \rightarrow (A - 1)]. \quad (23.37)$$

For the situations where there is only one term in the summation in Eqs. (23.34) or (23.35) one obtains the analytic results given in Fig. 1. If we have a k^n configuration with n =even and spin $J_a = 0$ then the final state for $(n + 1)$ particles must have $J_b = j_k$ and the sum rules give $S = (2J_b + 1 - n)/(2J_b + 1)$.

23.3.2 Isospin dependence

The spectroscopic factors S as defined in Sec. 21.1 apply to protons or neutrons without regard to isospin. One can generalize the results to nucleons in the state (J, T) by adding the isospin labels. Eq. (23.28) becomes:

$$S(T) = \frac{|\langle \Psi^A \omega J T || a_k^+ || \Psi^{A-1} \omega' J' T' \rangle|^2}{(2J + 1)(2T + 1)} = \frac{|\langle \Psi^{A-1} \omega' J' T' || \tilde{a}_k || \Psi^A \omega J \rangle|^2}{(2J + 1)(2T + 1)}, \quad (23.38)$$

where the triple bar matrix element indicates that it is reduced in both space and isospin. The sum-rules can be applied to the nucleon occupancies, e.g.

$$\sum_{f-} S_{i,f,k}(T) = \langle n_k \rangle_i \quad [A \rightarrow (A - 1)], \quad (23.39)$$

where $\langle n_k \rangle_i$ is the average occupation of nucleons in orbit k in the initial state i (with a maximum possible value of $2(2j + 1)$), and

$$\sum_{f-,k} S_{i,f,k}(T) = A \quad [A \rightarrow (A - 1)], \quad (23.40)$$

When isospin is introduced we can also make explicit relationships between proton and neutron spectroscopic factors and nucleon spectroscopic factors. Starting with Eq. (23.28) but adding the explicit T_z dependence:

$$S(t_z) = \frac{|\langle \Psi^A \omega J T T_z || a_{k,t_z}^+ || \Psi^{A-1} \omega' J' T' T'_z \rangle|^2}{(2J + 1)}, \quad (23.41)$$

where our convention is $(t, t_z) = (\frac{1}{2}, \frac{1}{2})$ for a neutron and $(t, t_z) = (\frac{1}{2}, -\frac{1}{2})$ for a proton, we use the Wigner-Eckhart theorem in isospin space to obtain:

$$\begin{aligned} S(t_z) &= \left(\begin{array}{ccc} T & \frac{1}{2} & T' \\ -T_z & t_z & T'_z \end{array} \right)^2 \frac{|\langle \Psi^A \omega J T || a_k^+ || \Psi^{A-1} \omega' J' T' \rangle|^2}{(2J+1)} \\ &= |\langle T T_z | T' T'_z t t_z \rangle|^2 \frac{|\langle \Psi^A \omega J T || a_k^+ || \Psi^{A-1} \omega' J' T' \rangle|^2}{(2J+1)(2T+1)}, \end{aligned} \quad (23.42)$$

for the removal of a nucleon and:

$$S(t_z) = |\langle T T_z | T' T'_z t t_z \rangle|^2 \frac{|\langle \Psi^{A+1} \omega J T || a_k^+ || \Psi^A \omega' J' T' \rangle|^2}{(2J+1)(2T+1)}, \quad (23.43)$$

for the addition of a nucleon. Thus in shorthand notation:

$$S(t_z) \equiv C^2 S(T), \quad (23.44)$$

where C^2 is the square the isospin Clebsch and $S(T)$ is the nucleon spectroscopic factor given by Eq. (23.38). In these expressions (T, T_z) are always associated with the heavier nucleus.

Although $S(t_z)$ are always the basic quantities measured, for nuclei not too far from $N = Z$ it is often convenient to calculate $S(T)$ in isospin formalism and then use Eq. (23.44) to convert to $S(t_z)$. In addition, some tables of experimental values and compilations [13] are given in terms of $S(T)$ by dividing the experimental $S(t_z)$ value by C^2 . With good isospin there is an isospin selection rule given by the triangle condition $\Delta(T_f, \frac{1}{2}, T_i)$.

For an initial nucleus with $N = Z$, $S(t_z) = S(-t_z)$ by mirror symmetry. If we add a nucleon to $(T = 0, T_z = 0)$ then we must go to $T_f = 1/2$ with $C^2 = 1$. If we remove a nucleon from $(T = 0, T_z = 0)$ then we go to $T_f = 1/2$ with $C^2 = \frac{1}{2}$.

In a neutron-rich nucleus, proton removal involves only one isospin value in the final state:

$$(T, T_z) \text{ proton} \rightarrow (T + \frac{1}{2}, T_z + \frac{1}{2}) = (T_>, T_z + \frac{1}{2}), \quad (23.45)$$

(transitions to the final state with higher isospin are forbidden by the triangle condition). Neutron removal can go to two isospin values:

$$(T, T_z) \text{ neutron} \rightarrow (T + \frac{1}{2}, T_z - \frac{1}{2}) = (T_>, T_z - \frac{1}{2}), \quad (23.46)$$

that are the isobaric analogues of the $(T + \frac{1}{2}, T_z + \frac{1}{2})$ states, and

$$(T, T_z) \text{ neutron} \rightarrow (T - \frac{1}{2}, T_z - \frac{1}{2}) = (T_<, T_z - \frac{1}{2}). \quad (23.47)$$

The $T_>$ and $T_<$ are referred to as the “ T -upper” and “ T -lower” states, respectively. The C^2 values for these three types of transitions are:

$$|\langle T, T_z | T + \frac{1}{2}, T_z + \frac{1}{2}, t, -\frac{1}{2} \rangle|^2 = \left(\frac{2T+1}{2T+2} \right), \quad (23.48)$$

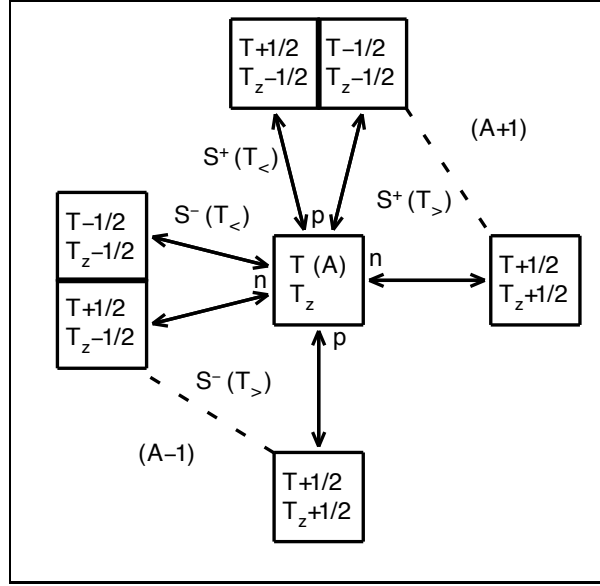


Figure 23.2: Schematic diagram for nucleon transfer from a neutron-rich target nucleus (A) to nuclei with (A-1) and (A+1). The dashed line connects isobaric analogue states.

$$|< T, T_z | T + \frac{1}{2}, T_z - \frac{1}{2}, t, +\frac{1}{2} >|^2 = \left(\frac{1}{2T+2} \right), \quad (23.49)$$

and

$$|< T, T_z | T - \frac{1}{2}, T_z - \frac{1}{2}, t, +\frac{1}{2} >|^2 = 1. \quad (23.50)$$

The explicit relations between $S(t_z)$ and $S(T)$ are:

$$S(\text{proton}) = \left(\frac{2T+1}{2T+2} \right) S(T_>), \quad (23.51)$$

$$S(>, \text{neutron}) = \left(\frac{1}{2T+2} \right) S(T_>) = \left(\frac{1}{2T+1} \right) S(\text{proton}), \quad (23.52)$$

and

$$S(<, \text{neutron}) = S(T_<). \quad (23.53)$$

Thus, in neutron-rich nucleus one can measure “proton” spectroscopic factors, by neutron removal to the $T_>$ states. Although these $T_>$ states lie at a high excitation energy and are often unbound to proton decay, their widths are small because their nucleonic decay is isospin forbidden. These type of transitions have only been observed in nuclei not too far from $N = Z$ since where the $(2T+1)$ factor is not too large.

French-Macfarlane sum-rules [14] are obtained from these results. For proton removal to $T_> = T_i + \frac{1}{2}$ ($T_z = T_i + \frac{1}{2}$) states:

$$\sum_f S(\text{proton}) = < n_{k,\text{proton}} >_i, \quad (23.54)$$

for neutron removal to the isobaric analogues of $T_>$ ($T_z = T_i - \frac{1}{2}$) states:

$$\sum_f S(>, \text{neutron}) = \frac{\langle n_{k,\text{proton}} \rangle_i}{2T + 1}. \quad (23.55)$$

For the total neutron removal:

$$\sum_f S(\text{neutron}) = S(<, \text{neutron}) + S(>, \text{neutron}) = \langle n_{k,\text{neutron}} \rangle_i, \quad (23.56)$$

and thus for neutron removal to the $T_<$ ($T_z = T_i - \frac{1}{2}$) states:

$$\sum_f S(<, \text{neutron}) = \langle n_{k,\text{neutron}} \rangle_i - \frac{\langle n_{k,\text{proton}} \rangle_i}{2T + 1}. \quad (23.57)$$

23.3.3 Simple situations

If we make a simple model for the initial and final states the values for the spectroscopic factors are sometimes easy to calculate. The results given in Sec. 20 for the special values of CFP in simple situations can be used to obtain the spectroscopic factors for the corresponding cases.

For example, ^{16}O might be assumed to be the closed-shell configuration $(0s_{1/2})^4 (0p_{3/2})^8 (0p_{1/2})^4$. One nucleon removal to $A = 15$ would then go to only three states each of which has a definite sum: $1/2^-$ with $S(0p_{1/2}) = 4$, $3/2^-$ with $S(0p_{3/2}) = 8$ and $1/2^+$ with $S(0s_{1/2}) = 4$. One proton or neutron removal for $T_i = 0$ going to $T_f = 1/2$ has $C^2 = 1/2$ and the sum rules are 2, 4 and 2, respectively, the same result one would obtain from the proton and neutron occupancies. (Center-of-mass corrections for these quantities will be discussed below.)

The spectroscopic factor for adding a $0d_{5/2}$ neutron to the closed-shell configuration for ^{16}O to make ^{17}O is $S_{0d_{5/2}} = 1$ and the stripping sum rule of Eq. (23.35) ($J_f = 5/2$ and $J_i = 0$) gives 6 which is the number of $0d_{5/2}$ neutron holes outside of ^{16}O .

23.3.4 Center-of-mass corrections

The center-of-mass (CM) correction to spectroscopic factors is closely associated with the problem of spurious states. Lets consider again the ^{16}O to $A = 15$ transition with a closed-shell configuration $(0s_{1/2})^4 (0p_{3/2})^8 (0p_{1/2})^4$ for ^{16}O . We take a single-particle basis that corresponds to the motion of the nucleons around a fixed center. In actuality, we should take them with respect to the CM of the $A - 1$ nucleons. However, with harmonic-oscillator wave functions, the CM of the nucleus is in a $0s$ state as long as the nucleons fill the lowest possible oscillator states. There are effects due to CM motion, but they are the simple ones associated with the CM being on its $0s$ state. We will refer to these as “non-spurious” states.

All oscillator states of the form $(0s_{1/2})^4(0p_{3/2})^n(0p_{1/2})^m$ are non-spurious and in particular the $A = 15$ the $1/2^-$ and $3/2^-$ states formed in this way are non-spurious.

Starting from a non-spurious state we can make $1\hbar\omega$ excited states either by an intrinsic $1\hbar\omega$ excitation leaving the CM in the $0s$ state, or by the $1\hbar\omega$ excitation of the CM state from $0s$ to $1p$. These latter are spurious states in the nuclear shell model. A method for generating spurious states is to construct a two-body hamiltonian corresponding to the oscillator CM motion and then diagonalize this in the shell-model basis. If the coefficient in front of the CM hamiltonian is large enough, the spurious states will be pushed to a high excitation energy and are prevented from mixing into the low-lying (non-spurious) states.

The lowest $1\hbar\omega$ excitation for $A = 15$, $1/2^+$ corresponds to a mixture of $(0s)^3(0p)^8$ and $(0s)^4(0p)^7(1s0d)^1$ configurations (e.g. the notation $0p$ stands for all possible arrangements involving the $0p_{1/2}$ and $0p_{3/2}$). There are a total of 18 basis states and diagonalization of the CM hamiltonian results in two spurious states that are linear combinations of these basis states.

We can calculate the summed spectroscopic strength from the closed-shell configuration of ^{16}O to the non-spurious $A = 15$ states. The result is $S = 12$ for the $0p$ shell and $S = 3.2$ for the $0s$ shell – the remaining $S = 0.8$ for the $0s$ shell is in the two spurious states. However, the sum rule of $S = 16$ still applies to the wave-functions in the intrinsic frame, and in order to recover this result we must make the following correction to the $0p$ spectroscopic factors [15], [16]:

$$S_{0p} \rightarrow \frac{A}{(A-1)} S_{0p}. \quad (23.58)$$

where $A = 16$ in this example. This correction is the largest at the lower end of the $0p$ shell, e.g. $(7/6)$ for the $0p$ -shell $A=7$ to $A=6$ spectroscopic factors. The general derivation and result is given in [15], [16]:

$$S_k \rightarrow \left[\frac{A}{(A-1)} \right]^{2n_r + \ell} S_k. \quad (23.59)$$

For example, for the removal of the $0d_{5/2}$ in ^{17}O to ^{16}O , the CM correction factor is $(17/16)^2 = 1.129$, and for the removal of the $0f_{7/2}$ in ^{48}Ca to ^{47}Ca the CM correction factor is $(48/47)^3 = 1.065$. [The first Variational Monte Carlo (VMC) calculations [17] for the $^{16}\text{O}(e,e'p)$ reaction indicated that the CM correction had the effect of reducing the $0p$ spectroscopic factor by about 12 percent, however subsequently an error was found [18] and the new VMC result is consistent with the 7 percent enhancement expected from the CM correction.]

23.3.5 Computation of shell-model spectroscopic factors

In simple situations, the spectroscopic factors can be obtained from the sum rules. In particular, pickup from a closed shell configuration has a value of $2(2j+1)$ for nucleons or $(2j+1)$ for protons or neutrons. Pickup from a state which is represented by single-particle outside of a closed shell is unity. Often the j^n configuration allows only a single state with

a given J value and the spectroscopic factor in this situation can be calculated with the use of tabulated coefficients of fractional parentage. For complicated situation one must use a shell-model computer code. For example, in Oxbash [19] one first calculates the wave functions and then takes the one-particle overlaps. The outputs are in files labeled *.LSA for the spectroscopic amplitudes and *.LSF for the spectroscopic factors.

23.4 Overlap functions

23.4.1 Definition and properties

Spectroscopic factors are related to the expansion of the wave function for a specific state Ψ_i^A in the initial nucleus with A nucleons in terms of a summation over the complete set of states Ψ_f^{A-1} in the final nucleus with $A - 1$ nucleons:

$$\Psi_i^A = \sum_{f,\ell,j} \theta_{i,f,\ell,j}(\vec{r}) \Psi_f^{A-1}, \quad (23.60)$$

In the reaction for the removal of particles from state Ψ_i to a specific state Ψ_f one requires the overlap function θ :

$$\langle \Psi_f^{A-1} | \Psi_i^A \rangle = \sum_{\ell,j} \theta_{i,f,\ell,j}(\vec{r}), \quad (23.61)$$

where an explicit summation over all possible ℓ and j values of the single-particle overlap function θ is made. The spectroscopic amplitude A is defined by the normalization of the overlap function:

$$A_{i,f,\ell,j} = \int \theta_{i,f,\ell,j}(\vec{r}) d\tau, \quad (23.62)$$

and the related spectroscopic factor is:

$$S_{i,f,\ell,j} = |A_{i,f,\ell,j}|^2. \quad (23.63)$$

The explicit dependence on ℓ and j is kept because the angular distributions in a particular reaction depend upon ℓ and j , and thus is it possible to separate each of these components in a given transition. Often the dependence on j is small in which case one measures the sum over the two possible j values for a given ℓ :

$$S_{i,f,\ell} = \sum_j S_{i,f,\ell,j}. \quad (23.64)$$

For example, for a $J_i \rightarrow J_f = \frac{3}{2}^+ \rightarrow 2^+$ transition both $0d_{5/2}$ and $0d_{3/2}$ could contribute and one will measure:

$$S_{\ell=2} = S_{0d_{5/2}} + S_{0d_{3/2}}$$

The radial size of the spectroscopic amplitude can be characterized in terms of its mean-square radius:

$$\langle r^2 \rangle_A = \frac{\int \theta(\vec{r}) r^2 d\tau}{\int \theta(\vec{r}) d\tau}. \quad (23.65)$$

One can expand the overlap function in terms of a complete set of single-particle wave functions:

$$\theta_{i,f,\ell,j}(\vec{r}) = \sum_{n_r} B_{i,f,k} \phi_k(\vec{r}) \quad (23.66)$$

where the ϕ are normalized to unity and k stands for (n_r, ℓ, j) . The single-particle states are given by:

$$\phi_k(\vec{r}) = \frac{R_k(r)}{r} [Y^{(\ell)}(\hat{r}) \otimes \chi^s]^j, \quad (23.67)$$

where χ^s is the intrinsic-spin wave function. For the single-particle states ϕ_k , one could take a basis of states generated from a mean-field potential. In practical terms one often approximates the mean-field with harmonic-oscillator or Woods-Saxon potentials. With the general expansion:

$$S_{i,f,\ell,j} = \sum_{n_r} |B_{i,f,k}|^2. \quad (23.68)$$

One often assumes that sum in Eq. (23.66) is restricted to only one radial quantum number n_r , in which case $B = A$, and $S = |A|^2 = |B|^2$.

The possible ℓ and j values are restricted by the total angular momenta J and parities of the initial and final states. In general one must use a complete set of single-particle states in the overlap function. But the quantum numbers associated with j are restricted by the spin and parities of the initial and final states. In particular parity is conserved, and if $J_i^\pi = 0^+$ then $j = J_f$ and $\pi_f = (-1)^\ell$.

The overlap function is exact to the extent that the many-body calculation is exact and the sum in Eq. (23.66) extends over all (n_r) values. However, simplified nuclear models usually lead to the use of only a few (n_r) values related to orbits near the fermi surface. For example, a Hartree-Fock calculation for ^{17}O may be based upon a closed-shell configuration for ^{16}O plus a neutron in the $0d_{5/2}$ orbital. Then the overlap function for the ^{17}O to ^{16}O reaction has $A_{0d_{5/2}} = 1$ and is given by $\theta(\vec{r}) = \phi_{0d_{5/2}}(\vec{r})$ (the Hartree-Fock single-particle wave function).

In order to meaningfully use spectroscopic factors we require that the reaction theory reproduce the observed dependence on projectile, energy and angle. The spectroscopic factor is a single number whose value should not depend on the reaction. When this is not the case, the reaction model or the model assumed for the overlap function must be questioned. A theoretical review is given by Bang et al. [20].

23.4.2 Asymptotic properties

We can consider the overlap in Eq. (23.62) as a function of r . At large r where the nuclear interactions are negligible this overlap must be governed by the kinematical asymptotics appropriate for the energy difference $\Delta E = E_f - E_i$ ($E = -\text{BE}$ where BE is the binding energy). For example for $^{17}\text{O}(5/2^+)$ to $^{16}\text{O}(0^+)$, $\Delta E = -4.14$ MeV and there is an exponential decay appropriate for a d wave bound by 4.14 MeV.

In some reactions such as $(^3\text{He}, d)$ and $(d, ^3\text{He})$, the optical potentials are such that only the part of the overlap function near the nuclear surface is important. In the extreme case (e.g. at incident energies far below the Coulomb barrier) it is only the asymptotic part of the overlap function which is important. The assumptions about the potential only influence the overall normalization, $N_{\ell,j}$, of the asymptotic wave function:

$$\theta_{i,f,\ell,j}(\vec{r}) \rightarrow N_{\ell,j} f_{\ell}(r), \quad (23.69)$$

where $f_{\ell}(r)$ is an asymptotic form independent of the strong potential that depends upon ℓ due to the centrifugal barrier. It is usually taken as the Whittaker function [21]. In the single-particle model the asymptotic behaviour is that of the single-particle radial wavefunction

$$\frac{R_k(r)}{r} \rightarrow N_k f_{\ell}(r), \quad (23.70)$$

The radial part of the overlap integral for large r is:

$$\langle \Psi_f^{A-1} | \Psi_i^A \rangle \rightarrow \sum_{\ell,j} A_{i,f,\ell,j} N_{\ell,j} f_{\ell}(r). \quad (23.71)$$

In this case the nuclear structure part of the reaction cross section for a given (ℓ, j) depends only on the square of the asymptotic normalization $A_{i,f,\ell,j} N_{\ell,j}$.

23.4.3 The well-depth prescription

In some cases the nuclear model may explicitly contain the correct asymptotic behavior, such as a Hartree-Fock model for ^{17}O in which the last neutron is bound by 4.14 MeV. But usually many-body nuclear models are developed which do not explicitly contain the correct asymptotic form. For example, a typical shell-model calculation for ^{18}O consists of calculating the sd-shell configuration mixing and total energy by evaluating the G matrix elements of a two-body residual interaction in an harmonic-oscillator basis. The implicit radial wave functions used are the same for both ^{17}O and ^{18}O and have the incorrect (oscillator) asymptotics for both the ^{17}O to ^{16}O and the ^{18}O to ^{17}O overlaps.

The general method for generating an overlap function with the correct asymptotic form is to start with a realistic mean-field potential for the single-particle wave functions and then to modify this potential such that the single-particle (ϵ) energy is equal to the actual energy difference (ΔE). One way to do this is to multiply the mean-field potential

by a constant such that the $\epsilon = \Delta E$. This is the “well-depth” prescription. Historically, the well-depth prescription was easy to apply numerically and has become the default method in many codes and analyses. For example, for the $^{18}\text{O}(\text{gs})$ to $^{17}\text{O}(5/2^+ \text{ gs})$ reaction one would increase the potential depth in order to bind the $0d_{5/2}$ orbital by the experimental energy difference $\Delta E = -12.19$ MeV. Likewise for the $^{18}\text{O}(\text{gs})$ to $^{17}\text{O}(1/2^+ 0.87 \text{ MeV excited state})$ reaction one would increase the potential for the $1s_{1/2}$ orbital in order to give the experimental energy difference of $\Delta E = -13.06$ MeV.

In cases where the single-particle wave function generated by the well-depth prescription is close to that of a realistic mean-field (e.g. the multiplying factor is not too different from unity) this method is a reasonable procedure. This usually applies to cases where the spectroscopic factor is near the sum-rule limit. But when the spectroscopic factor is small, this prescription may be questioned. For example, in the $(0d_{5/2})^3$ configuration there are states with $J = 3/2, 5/2$ and $9/2$. The overlap function between $(0d_{5/2})^2, J = 0$ and $(0d_{5/2})^3, J = 3/2$ must vanish ($S = 0$) since the $J = 3/2$ does not match the $j = 5/2$ of the transferred particle. A nonvanishing spectroscopic factor would result from a small $(0d_{5/2})^2(0d_{3/2})$ admixture into the $(0d_{5/2})^3$ wave functions. The single-particle potential needed to give the correct asymptotics for this small component is quite different from the mean-field potential. Thus rather than multiply the mean-field potential it may be more appropriate to add a surface peaked term to the mean-field potential as discussed in the next section.

A common way to implement the well-depth prescription is to generate an overlap function from the single-particle wave functions obtained from a Woods-Saxon potential. For the well-depth prescription, one would adjust the strength of the central potential to obtain a single-particle wave function ϕ_j which has a single-particle energy equal to the energy difference between the initial and final states under consideration. This wave function goes into the reaction theory to calculate a cross section, and then from comparison to experiment one deduces the spectroscopic factor S . However, one should keep in mind that the value for S depends upon the assumptions made about the reaction models and about the overlap function.

23.4.4 Beyond the well-depth prescription

One would like to carry out an experiment which is sensitive to the shape as well as the magnitude of the overlap function. But the nuclear reactions are generally only sensitive to the surface or asymptotic part, and thus the assumption about the fixed Woods-Saxon shape cannot be easily tested. As mentioned above the prescription based on the adjustment of the depth of a central potential is appropriate only when the transition under consideration is actually close to single-particle in nature. When the single-particle strength is fragmented due to residual interactions (beyond the mean-field) one should consider additions to the central potential which are related to the residual interaction.

Pinkston and Satchler [22] discussed the situation for the (p,d) reactions in the $0f_{7/2}$ shell region, for example $^{48}\text{Ti}(\text{p,d})$ leading to the lowest $T_< = 3/2$ and $T_> = 5/2$ states in

^{47}Ti which differ in separation energy by 7.3 MeV. This isospin splitting of the $0f_{7/2}$ orbit can be related to an isovector residual interaction which peaks near the nuclear surface and might be modeled on the derivative of the central Woods-Saxon potential. Thus the shape as well as the strength of the central potential must be considered. In terms of the original Woods-Saxon shape, the surface potential has the effect of effectively increasing the radius r_o for the $T_>$ state (in contrast to the well-depth prescription where the well-depth V_o is increased). This results in a relatively larger radius for the overlap function, a larger DWBA cross section, and hence a smaller spectroscopic factor. Similar considerations regarding the shape of the potential are related to the collective model. [22] More detailed models of these residual-interaction effects have discussed some of which involve solving a set of coupled equations for the bound state [23], [24], [25], [26], [27], [28], [29].

The examples above apply to removal from a partially filled orbit. In this situation the energy difference ΔE for the transition in question is usually larger than the single-particle energy. The well-depth prescription thus gives an rms radius for the overlap function which is smaller than the rms radius of the single-particle state. The surface effect of the residual interaction correction leads to an overlap function which has a relatively larger rms radius (back in the direction the single-particle rms radius). In the $0f_{7/2}$ shell it was found that a practical prescription is to fix the Woods-Saxon well depth and radius to give the correct energy difference and a *state-independent* rms radius for the overlap function [30]. The spectroscopic factors obtained with the residual interaction corrections are typically up to a factor of two smaller than those obtained with the well-depth prescription.

The other extreme is stripping to a nearly filled orbit. For example $^{40}\text{Ca}(\text{d,p})^{41}\text{Ca}$ leading to the $3/2^+$ state (at 2.04 MeV). The spectroscopic factor would be zero in the limit where the $0d_{3/2}$ orbit is filled in ^{40}Ca . The energy difference is $\Delta E = -10.4$ MeV as compared to the single-particle energy of about $\epsilon = -15.6$ MeV (e.g. the $^{40}\text{Ca} - ^{39}\text{Ca}$ binding energy difference). The well-depth prescription thus leads to an rms radius for the overlap function ($\Delta E = -10.4$ MeV) which is larger than that of the single-particle state ($\Delta E = -15.6$ MeV). The surface property of the residual interaction leads to an overlap function which has a relatively smaller rms radius [31] (again back in the direction of the single-particle rms radius).

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