

# MIDTERM EXAM LINEAR ALGEBRA Monday October 30<sup>th</sup>, 2023

Read all of the following information before starting the test.

- The answers to the questionnaire below must be given **exclusively** on the separate sheet, provided for that purpose.
- In order to select a box, you must blacken the box and not just check it. Hence

 $ot\!\!\!/$  ou  $\boxtimes$ 

will not be considered as selected boxes. On the contrary,

will be considered as selected box.

- Questions with symbol ♣ may have zero, one or several right answers. The other questions have only one right answer.
- Negative points may be given to very bad answers.
- This questionnaire will be treated by optic reading.
  - Any non-compliance with the rules given above, that would lead to a manual intervention will result in delay in the correction process and may be penalized.
- This questionnaire is closed book. You are not allowed to use a calculator nor consult any notes while taking the test.
- You are required to give the notes you wrote during this test in order to justify your reasoning. Show all work, clearly, if you want to get full credit. We reserve the right to take off points if we cannot see how you arrived at your answer (even if your final answer is correct).
- This present document contains  ${\bf 17}$  questions. It is worth 100 points.
- The present document is: 11 pages long. It is your responsibility to make sure that you have all of the pages!
- This exam time limit is 1 hour and 30 minutes. Good luck!

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#### Part I Solving linear Systems

#### Question 1.

We want to solve the following linear system, defined for every real m, by setting:

$$(\mathscr{S}_m) \begin{cases} x + y + (1 - m)z = 1 - m \\ (1 + m)x - y + 2z = 2m \\ 2x - my + 3z = -m^2. \end{cases}$$

Define, for all real number m,

$$B_m := \begin{pmatrix} 1 - m \\ 2m \\ -m^2 \end{pmatrix} \quad \text{and} \quad X := \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Denote  $A_m$  the matrix such that the System  $(\mathscr{S}_m)$  can be written under the form:

$$A_m \cdot X = B_m. \tag{E_m}$$

$$\boxed{\mathbf{D}} \quad A_m := \begin{pmatrix} 1 & 1 & 1 - m \\ 1 + m & -1 & 2 \\ 2 & -m & 3 \end{pmatrix}.$$

# Question 2. 4 (Q1 Part II)

We keep the notations of Question 1 We want to write down  $(E_{-2})$ , i.e.  $(E_m)$ in the particular case where m=-2. Check the right answer(s) among the following ones.

A None of these answers are correct  $\begin{cases} 2x + 2y + 6z = 6 \\ -3x - 3y + 6z = -12 \\ -10x - 10y - 15z = 20. \end{cases}$   $\begin{bmatrix} (E_{-2}): & \begin{cases} x + y + 3z = 3 \\ -x - y + 2z = -4 \\ 2x + 2y + 3z = -4. \end{cases}$   $\begin{bmatrix} 2x + 2y + 3z = 3 \\ -3x - 2y + z = -4 \\ x + 2y + z = -6. \end{cases}$ 

$$\begin{array}{c}
\boxed{D} \quad (E_{-2}): \\
\end{array} \begin{cases}
2x + 2y + 3z = 3 \\
-3x - 2y + z = -4 \\
x + 2y + z = -6.
\end{cases}$$



#### Question 3. (Q1 Part III)

We keep the notations of Questions 1 and 2 We are now looking for the PLU factorization of  $A_{-2}$  (i.e. the matrices  $P, \overline{L}$  and  $\overline{U}$  such that  $PA_{-2} = LU$ ). Check the right answer(s) among the following ones.

$$\begin{array}{|c|c|c|c|c|}\hline \mathbf{A} & P = -I_3, L := \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2/3 & -1/5 & 1 \end{pmatrix} \& \ U := \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \text{ is a } PLU \text{ factorization of } A_{-2}. \\ \hline \mathbf{B} & P = I_3, L := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix} \& \ U := \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is a } LU \text{ factorization of } A_{-2}.$$

B 
$$P = I_3, L := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix} \& U := \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$
 is a  $LU$  factorization of  $A_{-2}$ .

C None of these answers are correct

# Question 4. 4 (Q1 Part IV)

We keep the notations of Questions 2 and 3 We now want to solve the system  $(\mathscr{S}_{-2})$ , i.e. the equation  $A_{-2}X = B_{-2}$ , in X. Using exclusively the two previous questions, one can write:

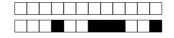
- The system  $A_{-2}X = B_{-2}$  has no solution.  $\mathscr{S}_{(E_{-2})} = \emptyset.$ 
  - $lue{C}$  The system  $A_{-2}X = B_{-2}$  has a unique solution.
    - $\mathbb{D}$   $\mathscr{S}_{(E_{-2})} = \{(x, x 2y, 3y) | (x, y) \in \mathbb{R}^2 \}.$ E None of these answers are correct
  - The system  $A_{-2}X = B_{-2}$  has infinitely many solutions.
    - G  $\mathscr{S}_{(E_{-2})} = \{(2,1,-1)\}.$

**I-Solving**  $(E_0)$ 

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## Question 5. (Q1 Part V)

We keep the notations of Question 1 We want to write down  $(E_0)$ , i.e.  $(E_m)$  in the particular case where m=0. Check the right answer(s) among the following ones.

A 
$$(E_0)$$
: 
$$\begin{cases} x + 2y + z = 1 \\ 2x - 2y + 2z = 0 \\ 2x + 3z + 1 = 0 \end{cases}$$

B None of these answers are correct

A 
$$(E_0)$$
: 
$$\begin{cases} x + 2y + z = 1 \\ 2x - 2y + 2z = 0 \\ 2x + 3z + 1 = 0 \end{cases}$$
B None of these answers are considered as 
$$\begin{cases} x + y + z = 1 \\ x - y + 2z = 0 \\ 2x + 3z = 0 \end{cases}$$
D  $(E_0)$ : 
$$\begin{cases} x + 2y + z = 1 \\ x - 2y + 2z = 0 \\ 2x + y + 3z = 0 \end{cases}$$

$$\begin{array}{l}
\boxed{\mathbf{D}} \quad (E_0): \qquad \begin{cases}
 x + 2y + z = 1 \\
 x - 2y + 2z = 0 \\
 2x + y + 3z = 0
\end{cases}$$

# Question 6. 4 (Q1 Part VI)

We keep the notations of Question 1 We are now looking for the LU factorization of  $A_0$ . Check the right answer(s) among the following ones.

$$\boxed{ \textbf{A} } \quad L := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \& \ U := \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ is a } LU \text{ factorization of } A_0.$$

D None of these answers are correct

# Question 7. 4 (Q1 Part VII)

We keep the notations of Questions and 6 We now want to solve the system  $(\mathscr{S}_0)$ . Denote  $\mathscr{S}_{(E_0)}$  the set of all solutions of the equation  $(E_0)$ . Using only the results obtained at Questions and one can solve directly the system  $A_0X = B_0$ , and write:

- A The system  $A_0X = B_0$  has a unique solution.
  - B | None of these answers are correct
- C The system  $A_0X = B_0$  has no solution.

The system 
$$A_0X=B_0$$
 has infinitely many solutions.

E  $\mathscr{S}_{(E_0)}=\{(1+z,-z,z)|z\in\mathbf{R}\}.$  F  $\mathscr{S}_{(E_0)}=\{(1,1,1)\}.$  G  $\mathscr{S}_{(E_0)}=\emptyset.$  H  $\mathscr{S}_{(E_0)}=\{(x+1,\frac{x}{2},-1)|x\in\mathbf{R}\}.$ 

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#### I-Solving $(E_2)$

# Question 8. (Q1 Part VIII)

We keep the notations of Question 1 We want to write down  $(E_2)$ , i.e.  $(E_m)$  in the particular case where m=2. Check the right answer(s) among the following ones.

B 
$$(E_2)$$
: 
$$\begin{cases} x+y+z=1\\ x-2y=-2\\ 2z=-1 \end{cases}$$

$$\boxed{\mathbf{D}} \quad (E_2): \qquad \begin{cases} x+y-z=1\\ -2y+z=-1\\ 0=-1 \end{cases}$$

#### Solving $(\mathscr{S}_2)$

# Question 9. 4 (Q1 Part IX)

We keep the notations of Questions 1 and 8. We now want to solve the system  $(\mathscr{S}_2)$ . Denote  $\mathscr{S}_{(E_2)}$  the set of all solutions of the equation  $(E_2)$ . Using a LUfactorization of  $A_2$ , one can solve directly the system  $A_2X=B_2$ , and write:

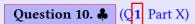
 $\square$  The system  $A_2X = B_2$  has infinitely many solutions. E The system  $A_2X = B_2$  has no solution.

Study of  $(\mathcal{S}_1)$ 

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$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

We keep the notations of Question 1 We now want to solve the system  $A_1X = B_1$ . To do so one will invert, if it is possible, Matrix  $A_1$ . Besides, one recall that  $\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$ , for any matrix  $A := (a_{ij})_{1 \leq i,j \leq n}$  of  $\mathcal{M}_n(\mathbf{R})$ . Check the right answer(s) among the following one(s).

$$\boxed{\mathbf{A}} \quad \operatorname{tr}(A_1^{-1}) = \frac{1}{3}. \qquad \boxed{\mathbf{B}} \quad (A_1^{-1})^2 = \begin{pmatrix} \frac{7}{9} & \frac{2}{3} & -\frac{10}{9} \\ -\frac{4}{9} & 1 & -\frac{8}{9} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

$$\begin{array}{ccc}
E & A_1^{-1} = \begin{pmatrix} \frac{1}{3} & 1 & -\frac{4}{3} \\ \frac{2}{3} & -1 & \frac{2}{3} \\ 0 & -1 & 1 \end{pmatrix}.
\end{array}$$

#### Solving $(\mathscr{S}_1)$

# Question 11. 🌲 (Q1 Part XI)

We keep the notations of Questions 10 We now want to solve the system  $(\mathcal{S}_1)$ . Denote  $\mathscr{S}_{(E_1)}$  the set of all solutions of the equation  $(E_1)$ . After solving directly the system  $A_1X = B_1$ , we can write:

A The system 
$$A_1X = B_1$$
 has no solution.  
B  $\mathscr{S}_{(E_1)} = \{(8/3, -8/3, -3)\}.$ 

B  $\mathscr{S}_{(E_1)} = \{(8/3, -8/3, -3)\}.$ C The system  $A_1X = B_1$  has infinitely many solutions.

H None of these answers are correct

Solving  $(\mathscr{S}_m)$ , for all m in  $\mathbf{R} \setminus \{-2, 0, 2\}$ 

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We keep the notations of Questions 1 to 11 We now want to solve the system  $(\mathscr{S}_m)$ , for all m in  $\mathbb{R}\setminus\{-2,0,2\}$ . Inverting matrix  $A_m$ , when it is possible, one can write:

$$\boxed{\mathbf{A}} \quad \forall m \in \mathbf{R} \setminus \{-2, 0, 2\}, \, A_m^{-1} = \begin{pmatrix} \frac{2m-2}{m(m^2-4)} & \frac{m^2-m-3}{m(m^2-4)} & -\frac{-3+m}{m(m^2-4)} \\ -\frac{-1+3m}{m(m^{-2})} & \frac{1+2m}{m(m-4)} & -\frac{m^2+1}{m(m^2-4)} \\ -\frac{-1+m}{(m-2)m} & \frac{1}{(m-2)m} & -\frac{1}{(m-2)m} \end{pmatrix} .$$

B None of these answers are correct

 $\square$   $\exists m \in \mathbb{R} \setminus \{-2, 0, 2\}$ , such that:  $A_m$  is not invertible.

$$\forall m \in \mathbf{R} \setminus \{-2, 0, 2\}, A_m^{-1} = \begin{pmatrix} \frac{2m-3}{m(m^2-4)} & \frac{m^2-m-3}{m(m^2-4)} & -\frac{-3+m}{m(m^2-4)} \\ -\frac{-1+3m}{m(m^2-4)} & \frac{1+2m}{m(m^2-4)} & -\frac{m^2+1}{m(m^2-4)} \\ -\frac{-1+m}{(m-2)m} & \frac{1}{(m-2)m} & -\frac{1}{(m-2)m} \end{pmatrix}.$$

#### Part II Another Exercise on Linear Systems

#### Question 13.

Let A be the matrix defined by setting:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

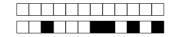
We are interested by the inverse, if it exists, of  $A - 2I_3$ . An easy computation allows us to write that:

$$(A - 2I_3)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

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### Question 14. (Q13 Part II)

We keep the notation from Question [13] From the result of the previous question, one can deduce that:

- $oxed{A}$  A is invertible and  $A^{-1}=rac{1}{4}A^2-rac{3}{2}A+rac{2}{3}I_3$   $oxed{B}$  A is not invertible.

  - $\square$  None of these answers are correct  $\square$  A is invertible and  $A^{-1} = \frac{1}{8}A^2 \frac{3}{4}A + \frac{3}{2}I_3$

# Question 15. (Q13 Part III)

We keep the notation from Questions 13 to 14 From the result of Questions 13 to 14 one can say that:

- A is not invertible . B None of these answers are correct

  - C A is invertible and  $A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{10} & \frac{9}{10} \end{pmatrix}$ D A is invertible and  $A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} & -\frac{3}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & -\frac{3}{3} & \frac{9}{3} \end{pmatrix}$

#### **Part III Vector Spaces**

#### Question 16.

Define the following sets.

$$\Gamma_1 := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \ (\alpha, \beta, \gamma, \delta) \in \mathbf{R}^4 \right\}, \quad \Gamma_2 := \{(x, y, z) \in \mathbf{R}^3, \text{ such that } 2x - y + z = 0\},$$

$$\Gamma_3 := \{A \in \mathcal{M}_2(\mathbf{R}), \ A \text{ is nilpotent } \}$$

Precise, for each of the above sets, which one are vector spaces, when they are endowed with laws + and  $\cdot$ , that can naturally be associated to them

- G  $(\Gamma_1, +, \cdot)$  is a vector space.

<sup>&</sup>lt;sup>a</sup>Recall that an element A in  $\mathcal{M}_n(\mathbf{R})$  is said to be nilpotent if there exists an integer q such that  $A^q =$  $0_{\mathcal{M}_n(\mathbf{R})}$ .



#### Question 17. 🌲

Define  $A:=\{(\alpha,\alpha,\alpha),\ \alpha\in\mathbf{R}\}$  and  $B:=\{(x,y,z)\in\mathbf{R}^3,\ x+y+z=0\}.$  One can say that:

A Only B is a vector subspaces of  $\mathbf{R}^3$ . B None of these answers are correct Both A and B are vector subspaces of  $\mathbf{R}^3$ . D Only A is a vector subspaces of  $\mathbf{R}^3$ .

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