Numerical Analysis: Midterm (**60 marks**)

Urbain Vaes

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Question 1 (Floating point arithmetic, **12 marks**). True or false? (+1/0/-1)

1. Let $(\bullet)_2$ denote binary representation. It holds that $(0.1111)_2 + (0.0001)_2 = 1$.
2. It holds that $(1000)_{16} \times (0.001)_{16} = 1$.
3. It holds that
$$(0.\overline{1})_3 = \frac{1}{2}.$$
4. In base 16, all the natural numbers from 1 to 300 can be represented using 2 digits.
5. If x is finite `Float64` number, then x is a rational number.
6. In Julia, `eps(2.0)` returns the machine epsilon for the Float64 format.
7. Storing the matrix obtained with the Julia command `zeros(10^5, 10^5)` would require more than 50GB of memory.
8. The spacing (in absolute value) between successive double-precision (`Float64`) floating point numbers is constant.
9. It holds that $(0.\overline{10101})_2 = (1.2345)_{10}$.
10. Machine addition $\widehat{+}$ is an associative operation. More precisely, given any three double-precision floating point numbers x, y and z, the following equality holds:
$$(x \mathbin{\widehat{+}} y) \mathbin{\widehat{+}} z = x \mathbin{\widehat{+}} (y \mathbin{\widehat{+}} z).$$
11. The machine epsilon is the smallest strictly positive number that can be represented in a floating point format.
12. In Julia, the command `nextfloat(2.)` returns the next `Float32` number after 2.

⚙ **Question 2** (Interpolation and approximation, **10 marks**). Throughout this exercise, we use the notation $x_i^n = i/n$ and assume that $u \colon \mathbf{R} \to \mathbf{R}$ is a continuous function. The notation $\mathbf{P}(n)$ denotes the set of polynomials of degree less than or equal to n. We proved in class that, for all $n \geqslant 0$, there exists a unique polynomial $p_n \in \mathbf{P}(n)$ such that

$$\forall i \in \{0, \dots, n\}, \qquad p_n(x_i^n) = u(x_i^n). \tag{1}$$

1. The degree of p_n is exactly n or $n - 1$.
2. Suppose that $u \in \mathbf{P}(m)$. Then $p_n = u$ if $n \geqslant m$.
3. Fix $u(x) = \sin(3\pi x)$. Then $p_3(x) = 0$.
4. Fix $u(x) = \cos(\pi x)$. Then $p_2(x) = (2x - 1)^2$.
5. For all u that is smooth, it holds that
$$\max_{x \in [0,1]} \bigl|u(x) - p_n(x)\bigr| \xrightarrow[n \to \infty]{} 0.$$
6. Fix $u(x) = \cos(2x)$. Then
$$\max_{x \in [0,1]} \bigl|u(x) - p_n(x)\bigr| \xrightarrow[n \to \infty]{} 0.$$
7. Fix $u(x) = \sin(x)$. Then
$$\max_{x \in \mathbf{R}} \bigl|u(x) - p_n(x)\bigr| \xrightarrow[n \to \infty]{} 0.$$
8. Suppose that $p(x) \in \mathbf{P}(n)$ and let $q(x) = p(x + 1) - p(x)$. Then $q \in \mathbf{P}(n - 1)$.
9. Let $(f_0, f_1, f_2, \dots) = (1, 1, 2, \dots)$ denote the Fibonacci sequence. There exists a polynomial p such that
$$\forall n \in \mathbf{N}, \qquad f_n = p(n).$$
10. For any matrix $\mathsf{A} \in \mathbf{R}^{20 \times 10}$, the linear system
$$\mathsf{A}^T \mathsf{A} \boldsymbol{\alpha} = \mathsf{A}^T \boldsymbol{\alpha}$$
admits a unique solution.

Question 3 (Numerical integration, **9 marks**)**.** True or false? (+1/0/-1)

1. The degree of precision of the following rule is equal to 2:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx 2u(0).$$

2. The degree of precision of the following rule is equal to 3:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx u\left(-\frac{1}{3}\right) + u\left(\frac{1}{3}\right).$$

3. For any natural number $N > 0$, there exists a quadrature rule with a degree of precision equal to $2N+1$ of the form

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \sum_{n=0}^{N} w_n u(x_n).$$

4. Legendre polynomials are orthogonal for the inner product

$$\langle f, g\rangle := \int_{-1}^{1} f(x)\,g(x)\,\mathrm{d}x.$$

5. Fix $u(x) = \cos(x)$ and let

$$I_N^{MC} = \frac{1}{N}\sum_{n=1}^{N} u(X_n), \qquad X_n \overset{\text{i.i.d.}}{\sim} \mathcal{U}([0,1]). \tag{2}$$

The expectation of $\widehat{I}_N$ is independent of N.

6. The variance of I_N^{MC} in (2) tends to 0 in the limit $N \to \infty$.

7. Let $x_i^N = i/N$ and consider the following approximation of $\int_0^1 u(x)\,\mathrm{d}x$:

$$I_N^T = \frac{1}{2N}\Big(u\left(x_0^N\right) + 2u\left(x_1^N\right) + 2u\left(x_2^N\right) + \ldots + u\left(x_{N-2}^N\right) + 2u\left(x_{N-1}^N\right) + u\left(x_N^N\right)\Big). \tag{3}$$

Fix $u(x) = (1+25x^2)^{-1}$. Then I_N^T diverges in the limit $N \to \infty$.

8. Fix $u(x) = \cos(x)$ and let I_N^T be as in (3). Then there exists $C \in (0,\infty)$ such that

$$\forall N \geqslant 2, \qquad \left| I_N^T - \int_0^1 u(x)\,\mathrm{d}x \right| \leqslant \frac{C}{N}.$$

9. Fix $u(x) = 2x - 1$ and let I_N^T be as in (3). Then $I_N^T = 0$ for $N \geqslant 2$.

💻 **Computer exercise 1** (Floating point arithmetic, **10 marks**)**.** Read the documentation of the `nextfloat` function. Using this function, plot on the same graph the spacing between successive `Float16`, `Float32` and `Float64` numbers in the range $[1, 10^4]$. Use a logarithmic scale for the x and y axes. You may find it useful to use `LinRange{Type}(a, b, n)` to create an array of n equidistant numbers of type `Type` between a and b.

💻 **Computer exercise 2** (Interpolation, **10 marks**)**.** Consider the data

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \qquad y = \begin{pmatrix} 12 \\ 7 \\ 2 \\ 4 \end{pmatrix}.$$

Find $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ such that

$$\widehat{u}(x) := \alpha_1 \cos(\pi x) + \frac{\alpha_2}{x} + \alpha_3 \, 2^x + \alpha_4 \, x^2.$$

satisfies

$$\forall i \in \{1, 2, 3, 4\}, \qquad \widehat{u}(x_i) = y_i.$$

Plot on the same graph the function $\widehat{u}$ and the data points.

💻 **Computer exercise 3** (Approximation, **10 marks**)**.** Write, without using the `Polynomials` library, a function `approx(x, y, d, X)` to obtain, given data points

$$(x_0, y_0), \ldots, (x_N, y_N)$$

and a nonnegative integer $0 \leqslant d \leqslant N$, the polynomial $p \in \mathbf{P}(d)$ minimizing the total error

$$E := \frac{1}{2} \sum_{n=0}^{N} \bigl|p(x_n) - y_n\bigr|^2.$$

Your function should a vector containing the values $p(X_0), \ldots, p(X_M)$, where $X_0, \ldots, X_M$ are the elements of the vector `X`. Within the function, proceed in 3 steps:

- First create the following matrix and vector:

$$\mathsf{A} \begin{pmatrix} 1 & x_0 & \ldots & x_0^d \\ \vdots & \vdots & & \vdots \\ 1 & x_N & \ldots & x_N^d \end{pmatrix}, \qquad \boldsymbol{b} := \begin{pmatrix} y_0 \\ \vdots \\ y_N \end{pmatrix}.$$

- Then solve the normal equations using the backslash operator:

$$\mathsf{A}^T \mathsf{A} \boldsymbol{\alpha} = \mathsf{A}^T \boldsymbol{b}.$$

- Finally, evaluate the polynomial

$$p(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_d x^d.$$

  at all the points in `X` and return the result in a vector.

Use the data given in the notebook, of the altitude of a marble in free fall as a function of time, to test your code. The experiment was performed on a different planet. Can you find which one? See https://en.wikipedia.org/wiki/Gravitational_acceleration.

💻 **Computer exercise 4** (Numerical integration, **10 marks**). Milne's integration rule reads

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \frac{2}{3}\left(2f\left(-\frac{1}{2}\right) - f(0) + 2f\left(\frac{1}{2}\right)\right)$$

- Write a function `composite_milne(u, a, b, N)`, which returns an approximation of the integral

  $$\int_{a}^{b} u(x)\,\mathrm{d}x$$

  obtained by partitioning the integration interval $[a, b]$ into N cells, and applying Milne's rule within each cell.

- Take $u(x) = \cos(x)$, $a = -1$ and $b = 1$. Plot the evolution of the error for N varying from 1 to 1000.

- Estimate the order of convergence with respect to N, i.e. find α such that

  $$|\widehat{I}_N - I| \propto CN^{-\alpha},$$

  where I denotes the exact value of the integral and $\widehat{I}_N$ denotes its approximation. In order to find α, use the function `fit` from the `Polynomials` package to find a linear approximation of the form

  $$\log|\widehat{I}_N - I| \approx \log(C) - \alpha\log(N).$$