# Numerical Analysis: Midterm (**50 marks**)

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*You are not required to complete every question.Although the total marks on the exam sum to 55, your midterm grade will be calculated out of 50.*

**Question 1** (Floating point arithmetic, **10 marks**). True or false?

1. Let $(\bullet)_2$ denote binary representation. Then $(0.1101)_2 + (0.0011)_2 = (1.0)_2$.
2. Let $\varepsilon_{64}$ denote the machine epsilon for the `Float64` format, i.e. `eps(Float64)`. Then the number 2 is representable exactly in this format, and the next representable number is $2 + \varepsilon_{64}$.
3. It holds that $(10000)_2 \times (0.1010)_2 = (1010)_2$.
4. In Julia, `Float64(.6) == Float32(.6)` evaluates to `true`.
5. The spacing (in absolute value) between successive single-precision (`Float32`) floating point numbers is constant.
6. Infinitely many distinct real numbers can be represented exactly in the `Float64` format, but only finitely many can be represented exactly in the `Float32` format.
7. It holds that $(0.\overline{101})_2 = \frac{5}{7}$.
8. Machine addition $\widehat{+}$ is an operation that is *associative* but not *commutative*.
9. The machine epsilon is the smallest number of the form $2^{-n}$ with $n \in \mathbf{N}$ that can be represented exactly in a floating point format.
10. In Julia, the expression `1 + eps()/3 == 1 + eps()` evaluates to `true`.
11. **Bonus.** In Julia, the expression `exp(eps()/2) == 1 + eps()` evaluates to `true`.

    *Explain briefly:*

12. **Bonus.** In Julia, the expression `cos(eps()) == 1` evaluates to `true`.

    *Explain briefly:*

**Question 2** (Interpolation and approximation, **10 marks**). Are the following statements true or false? Prove or disprove. Recall that $\mathcal{P}_d$ denotes the set of polynomials of degree at most $d$.

1. Assume that $x_0 < x_1 < x_2 < x_3$ and $y_0, y_1, y_2, y_3$ are given real numbers. Then there exists a polynomial $p \in \mathcal{P}_3$ such that $p(x_i) = y_i$ for all $i \in \{0, 1, 2, 3\}$.

   *Justification:*

2. Assume that $x_0 < x_1 < x_2$ and $y_0, y_1, y_2$ are given real numbers. Then, there can exist *at most one* polynomial $p \in \mathcal{P}_2$ such that $p(x_i) = y_i$ for all $i \in \{0, 1, 2\}$.

   *Justification:*

3. Let $p \in \mathcal{P}_d$ be a polynomial of degree $d > 0$, and let $q \colon \mathbf{R} \to \mathbf{R}$ be given by $q(x) = p(x+1) - p(x)$. Then it holds that $q \in \mathcal{P}_{d-1}$.

   *Justification:*

4. For $n \in \mathbf{N}$, let $x_i^n = i/n$ for $i = 0, 1, \ldots, n$. Assume that $u \colon \mathbf{R} \to \mathbf{R}$ is the smooth function given by $u(x) = \sin(3x) + x^3$, and let $p_n \in \mathcal{P}_n$ denote the polynomial interpolation of $u$ at the points $x_0^n, x_1^n, \ldots, x_n^n$. Then it holds that

$$\lim_{n \to \infty} \left( \max_{x \in [0,1]} \left| u(x) - p_n(x) \right| \right) = 0 \,.$$

   *Justification:*

⚙ **Question 3** (Interpolation, open-ended question, **5 marks**). In polynomial interpolation, the error depends both on the function being interpolated and the choice of interpolation nodes. Consider two families of nodes on the interval $[-1, 1]$:

- Equally spaced nodes,
- Chebyshev nodes.

Discuss qualitatively (and illustrate with examples or plots if you wish) how the interpolation error behaves as the degree $n$ increases in each case. Why does one choice of nodes perform better for large $n$, and what is the mathematical motivation for using Chebyshev nodes? *You may refer to Runge's phenomenon, but go beyond merely stating it.*

**Implementation exercise 1** (Interpolation, **5 marks**). Write Julia code that computes and plots the interpolating polynomial $p \in \mathcal{P}_3$ through the following points: $(0,0)$, $(1,4)$, $(2,15)$, $(3,40)$. The plot should display both the interpolation points and the graph of the interpolating polynomial over an appropriate range. Do not use any other library than the ones already imported.

```julia
using LinearAlgebra
using Plots
# Write your code here
```

**Question 4** (Numerical integration, **5 marks**). Are the following statements true or false? Justify briefly.

1. The degree of precision of the following quadrature rule is 2:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx 2u(0)\,.$$

*Justification:*

2. The degree of precision of the following rule is equal to 3:

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx u\left(-\frac{1}{3}\right) + u\left(\frac{1}{3}\right).$$

*Justification:*

3. For any natural number $N > 0$, there exists a quadrature rule with a degree of precision equal to $2N - 1$ of the form

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx \sum_{n=1}^{N} w_n u(x_n).$$

*Justification:*

4. Let $x_i^N = i/N$ and consider the following approximation of $\int_0^1 u(x)\,\mathrm{d}x$:

$$\widehat{I}_N = \frac{1}{2N}\Big(u\left(x_0^N\right) + 2u\left(x_1^N\right) + 2u\left(x_2^N\right) + \ldots + 2u\left(x_{N-2}^N\right) + 2u\left(x_{N-1}^N\right) + u\left(x_N^N\right)\Big). \qquad (1)$$

Suppose first that $u$ is the Runge function, given by $u(x) = (1 + 25x^2)^{-1}$. Then $\widehat{I}_N$ diverges in the limit $N \to \infty$.

*Justification:*

5. Let $u(x) = \cos(3x)$ and let $\widehat{I}_N$ be as in (**??**). Then it holds that

$$\lim_{N\to+\infty}\left(\left|\widehat{I}_N - \int_0^1 u(x)\,\mathrm{d}x\right|\right) = 0.$$

*Justification:*

6. (**Bonus.**) Fix $u(x) = 2x - 1$ and let $\widehat{I}_N$ be as in (**??**). Then $\widehat{I}_N = 0$ for all $N \geqslant 2$.

*Justification:*

**Question 5** (Gaussian–Hermite numerical integration, **10 marks**). The Gauss–Hermite quadrature formula with $n$ nodes is an approximation of the form

$$I(u) := \int_{-\infty}^{\infty} u(x)\, \mathrm{e}^{-x^2}\, \mathrm{d}x \approx \sum_{i=1}^{n} w_i\, u(x_i) =: \widehat{I}_n(u),$$

which is exact when $u$ is a polynomial of degree $\leq 2n-1$. Note that the nodes are numbered $1, \dots, n$. For this question, we take for granted that, for integers $i \geq 0$, it holds that

$$\int_{-\infty}^{\infty} x^i \mathrm{e}^{-x^2}\, \mathrm{d}x = \begin{cases} 0, & \text{if } i \text{ is odd,} \\ (i-1)!!\sqrt{\dfrac{\pi}{2^i}}, & \text{if } i \text{ is even,} \end{cases}$$

where $(i-1)!! := 1 \times 3 \times 5 \times \cdots \times (i-1)$. In particular, with all the integrals being over $(-\infty, \infty)$, the following special cases may be useful in your computations:

$$\int \mathrm{e}^{-x^2}\, \mathrm{d}x = \sqrt{\pi}, \quad \int x^2\, \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2}, \quad \int x^4\, \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{3}{4}\sqrt{\pi}, \quad \int x^6\, \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{15}{8}\sqrt{\pi}\,.$$

1. (**5 marks**) Find the nodes and weights of the Gauss–Hermite rule with $n = 3$ nodes. By symmetry, we expect nodes of the form $(-z, 0, z)$ and weights $(w_1, w_2, w_1)$, which reduces the number of unknowns to three.

   *Your answer:*

2. (**5 marks**) Let $\{H_0, H_1, \dots\}$ denote orthogonal polynomials for the inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x) g(x) \, \mathrm{e}^{-x^2} \, \mathrm{d}x$$

which, in addition, satisfy the following two conditions:

- For all $i \in \mathbf{N}$, the polynomial $H_i$ is of degree $i$.
- The leading coefficient of $H_i$, which multiplies $x^i$, is equal to 1.

Calculate $H_0$, $H_1$, $H_2$ and $H_3$. What is the relationship between $H_3$ and the quadrature rule found in the first item?

*Your answer:*

3. (Bonus, **2 marks**) Calculate $H_4$ and, using this result, deduce the nodes and weights of the Gauss–Hermite quadrature with 4 points.

   *Your answer:*

💻 **Implementation exercise 2** (Numerical integration, **10 marks**). The midpoint quadrature rule reads

$$\int_{-1}^{1} u(x)\,\mathrm{d}x \approx 2u(0)\,.$$

- (**3 marks**) Write a function `midpoint(u, a, b)` that returns, using this quadrature rule, an approximation of the integral

$$\int_{a}^{b} u(x)\,\mathrm{d}x\,. \tag{2}$$

```julia
function midpoint(u, a, b)
    # Write your code here







end
```

- (**4 marks**) Write a function `composite_midpoint(u, a, b, N)` that returns an approximation of the integral (**??**), this time using a composite version of the midpoint rule. More precisely, the approximation should be obtained by partitioning the integration interval $[a, b]$ into $N$ cells, and applying the midpoint rule within each cell.

```julia
function composite_midpoint(u, a, b)
    # Write your code here














end
```

- (**3 marks**) Take $u(x) = \cos(x)$, $a = -1$ and $b = 1$. In this setting, plot the evolution of the error for $N$ varying from 1 to 1000.

```julia
using Plots
# Write your code here
```