MIDTERM EXAM
LINEAR ALGEBRA
Monday October 30$^{\text{th}}$, 2023

Read all of the following information before starting the test.

- The answers to the questionnaire below must be given **exclusively** on the separate sheet, provided for that purpose.
- In order to select a box, you must blacken the box and not just check it. Hence

  ☑ ou ☒

  will not be considered as selected boxes. On the contrary,

  ■

  will be considered as selected box.
- Questions with symbol ♣ may have zero, one or several right answers. The other questions have only one right answer.
- Negative points may be given to *very bad* answers.
- This questionnaire will be treated by optic reading.

  Any non-compliance with the rules given above, that would lead to a manual intervention will result in delay in the correction process and may be penalized.
- This questionnaire is closed book. You are not allowed to use a calculator nor consult any notes while taking the test.
- You are required to give the notes you wrote during this test in order to justify your reasoning. Show all work, clearly, if you want to get full credit. We reserve the right to take off points if we cannot see how you arrived at your answer (even if your final answer is correct).
- This present document contains **17** questions. It is worth 100 points.
- **The present document is : 11 pages long. It is your responsibility to make sure that you have all of the pages!**
- This exam time limit is **1** hour and **30** minutes. Good luck!

Part I Solving linear Systems

Question 1.

We want to solve the following linear system, defined for every real m, by setting:

$$(\mathscr{S}_m)\begin{cases} x+y+(1-m)z = 1-m \\ (1+m)x-y+2z = 2m \\ 2x-my+3z = -m^2. \end{cases}$$

Define, for all real number m,

$$B_m := \begin{pmatrix} 1-m \\ 2m \\ -m^2 \end{pmatrix} \qquad \text{and} \qquad X := \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Denote A_m the matrix such that the System $(\mathscr{S}_m)$ can be written under the form:

$$A_m \cdot X = B_m. \qquad (E_m)$$

A None of these answers are correct

B $A_m := \begin{pmatrix} 1-m & 1 & 1 \\ 1 & -1 & 1+m \\ 2-m & m & -3 \end{pmatrix}$.

C $A_m := \begin{pmatrix} 2 & 2 & 2-2m \\ 1+2m & -1 & 4 \\ 2 & -2m & 3 \end{pmatrix}$.

D $A_m := \begin{pmatrix} 1 & 1 & 1-m \\ 1+m & -1 & 2 \\ 2 & -m & 3 \end{pmatrix}$.

Question 2. ♣ (Q1, Part II)

We keep the notations of Question 1. We want to write down (E_{-2}), *i.e.* (E_m) in the particular case where $m = -2$. Check the right answer(s) among the following ones.

A None of these answers are correct

B (E_{-2}): $\begin{cases} 2x+2y+6z = 6 \\ -3x-3y+6z = -12 \\ -10x-10y-15z = 20. \end{cases}$.

C (E_{-2}): $\begin{cases} x+y+3z = 3 \\ -x-y+2z = -4 \\ 2x+2y+3z = -4. \end{cases}$.

D (E_{-2}): $\begin{cases} 2x+2y+3z = 3 \\ -3x-2y+z = -4 \\ x+2y+z = -6. \end{cases}$.

Question 3. (Q1. Part III)

We keep the notations of Questions 1 and 2. We are now looking for the PLU factorization of A_{-2} (*i.e.* the matrices P, L and U such that $PA_{-2} = LU$). Check the right answer(s) among the following ones.

A $P = -I_3, L := \begin{pmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2/3 & -1/5 & 1 \end{pmatrix}$ & $U := \begin{pmatrix} 1 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is a PLU factorization of A_{-2}.

B $P = I_3, L := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}$ & $U := \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ is a LU factorization of A_{-2}.

C None of these answers are correct

D $P = I_3, L := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3/5 & 1 \end{pmatrix}$ & $U := \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ is a PLU factorization of A_{-2}.

Question 4. ♣ (Q1. Part IV)

We keep the notations of Questions 2 and 3. We now want to solve the system $(\mathscr{S}_{-2})$, *i.e.* the equation $A_{-2}X = B_{-2}$, in X. Using exclusively the two previous questions, one can write:

A The system $A_{-2}X = B_{-2}$ has no solution.

B $\mathscr{S}_{(E_{-2})} = \emptyset$.

C The system $A_{-2}X = B_{-2}$ has a unique solution.

D $\mathscr{S}_{(E_{-2})} = \{(x, x - 2y, 3y) | (x, y) \in \mathbf{R}^2\}$.

E None of these answers are correct

F The system $A_{-2}X = B_{-2}$ has infinitely many solutions.

G $\mathscr{S}_{(E_{-2})} = \{(2, 1, -1)\}$.

I-Solving (E_0)

Question 5. (Q1, Part V)

We keep the notations of Question 1. We want to write down (E_0), *i.e.* (E_m) in the particular case where $m = 0$. Check the right answer(s) among the following ones.

A (E_0): $\begin{cases} x + 2y + z = 1 \\ 2x - 2y + 2z = 0 \\ 2x + 3z + 1 = 0 \end{cases}$

B None of these answers are correct

C (E_0): $\begin{cases} x + y + z = 1 \\ x - y + 2z = 0 \\ 2x + 3z = 0 \end{cases}$

D (E_0): $\begin{cases} x + 2y + z = 1 \\ x - 2y + 2z = 0 \\ 2x + y + 3z = 0 \end{cases}$

Question 6. ♣ (Q1, Part VI)

We keep the notations of Question 1. We are now looking for the LU factorization of A_0. Check the right answer(s) among the following ones.

A $L := \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$ & $U := \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a LU factorization of A_0.

B $L := \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ & $U := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is a LU factorization of A_0.

C $L := I_3$ & $U = A_0$ is a LU factorization of A_0.

D None of these answers are correct

Question 7. ♣ (Q1, Part VII)

We keep the notations of Questions 5 and 6. We now want to solve the system $(\mathscr{S}_0)$. Denote $\mathscr{S}_{(E_0)}$ the set of all solutions of the equation (E_0). Using only the results obtained at Questions 5 and 6, one can solve directly the system $A_0X = B_0$, and write:

A The system $A_0X = B_0$ has a unique solution.

B None of these answers are correct

C The system $A_0X = B_0$ has no solution.

D The system $A_0X = B_0$ has infinitely many solutions.

E $\mathscr{S}_{(E_0)} = \{(1+z, -z, z) | \ z \in \mathbf{R}\}$.

F $\mathscr{S}_{(E_0)} = \{(1, 1, 1)\}$.

G $\mathscr{S}_{(E_0)} = \emptyset$.

H $\mathscr{S}_{(E_0)} = \{(x+1, \frac{x}{2}, -1) | \ x \in \mathbf{R}\}$.

I-Solving (E_2)

Question 8. (Q1. Part VIII)

We keep the notations of Question 1. We want to write down (E_2), *i.e.* (E_m) in the particular case where $m = 2$. Check the right answer(s) among the following ones.

A (E_2): $\begin{cases} x + y + z = 3 \\ 4y + z = -1 \\ 2z = -2 \end{cases}$

B (E_2): $\begin{cases} x + y + z = 1 \\ x - 2y = -2 \\ 2z = -1 \end{cases}$

C None of these answers are correct

D (E_2): $\begin{cases} x + y - z = 1 \\ -2y + z = -1 \\ 0 = -1 \end{cases}$

Solving $(\mathscr{S}_2)$

Question 9. ♣ (Q1. Part IX)

We keep the notations of Questions 1 and 8. We now want to solve the system $(\mathscr{S}_2)$. Denote $\mathscr{S}_{(E_2)}$ the set of all solutions of the equation (E_2). Using a *LU* factorization of A_2, one can solve directly the system $A_2X = B_2$, and write:

A $\mathscr{S}_{(E_2)} = \emptyset$.

B None of these answers are correct

C $\mathscr{S}_{(E_2)} = \{(1, \frac{x}{20}, 20) |\ x \in \mathbf{R}\}$.

D The system $A_2X = B_2$ has infinitely many solutions.

E The system $A_2X = B_2$ has no solution.

F The system $A_2X = B_2$ has a unique solution.

G $\mathscr{S}_{(E_2)} = \{(z, z, z) |\ z \in \mathbf{R}\}$.

H $\mathscr{S}_{(E_0)} = \{(2, 1, 0)\}$.

Study of $(\mathscr{S}_1)$

Question 10. ♣ (Q1 Part X)

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

We keep the notations of Question 1. We now want to solve the system $A_1 X = B_1$. To do so one will invert, if it is possible, Matrix A_1. Besides, one recall that $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$, for any matrix $A := (a_{ij})_{1 \leqslant i,j \leqslant n}$ of $\mathcal{M}_n(\mathbf{R})$. Check the right answer(s) among the following one(s).

A $\operatorname{tr}(A_1^{-1}) = \dfrac{1}{3}$.

B $(A_1^{-1})^2 = \begin{pmatrix} \frac{7}{9} & \frac{2}{3} & -\frac{10}{9} \\ -\frac{4}{9} & 1 & -\frac{8}{9} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$

C $(A_1^{-1})^2 = \begin{pmatrix} \frac{7}{9} & 0 & -\frac{2}{9} \\ -\frac{4}{9} & 1 & -\frac{4}{9} \\ -\frac{2}{3} & 0 & \frac{1}{3} \end{pmatrix}$

D None of these answers are correct

E $A_1^{-1} = \begin{pmatrix} \frac{1}{3} & 1 & -\frac{4}{3} \\ \frac{2}{3} & -1 & \frac{2}{3} \\ 0 & -1 & 1 \end{pmatrix}$.

Solving $(\mathscr{S}_1)$

Question 11. ♣ (Q1 Part XI)

We keep the notations of Questions 10. We now want to solve the system $(\mathscr{S}_1)$. Denote $\mathscr{S}_{(E_1)}$ the set of all solutions of the equation (E_1). After solving directly the system $A_1 X = B_1$, we can write:

A The system $A_1 X = B_1$ has no solution.

B $\mathscr{S}_{(E_1)} = \{(8/3, -8/3, -3)\}$.

C The system $A_1 X = B_1$ has infinitely many solutions.

D $\mathscr{S}_{(E_1)} = \{(x, -1, 2) \mid x \in \mathbf{R}\}$.

E $\mathscr{S}_{(E_1)} = \{(1 - z, -3z, 2z) \mid z \in \mathbf{R}\}$.

F $\mathscr{S}_{(E_1)} = \emptyset$.

G The system $A_1 X = B_1$ has a unique solution.

H None of these answers are correct

Solving $(\mathscr{S}_m)$, for all m in $\mathbf{R} \backslash \{-2, 0, 2\}$

Question 12. ♣ (Q1 Part XII)

We keep the notations of Questions 1 to 11. We now want to solve the system $(\mathscr{S}_m)$, for all m in $\mathbf{R}\backslash\{-2,0,2\}$. Inverting matrix A_m, when it is possible, one can write:

- [A] $\forall m \in \mathbf{R}\backslash\{-2,0,2\}$, $A_m^{-1} = \begin{pmatrix} \frac{2m-2}{m(m^2-4)} & \frac{m^2-m-3}{m(m^2-4)} & -\frac{-3+m}{m(m^2-4)} \\ -\frac{-1+3m}{m(m^{-2})} & \frac{1+2m}{m(m-4)} & -\frac{m^2+1}{m(m^2-4)} \\ -\frac{-1+m}{(m-2)m} & \frac{1}{(m-2)m} & -\frac{1}{(m-2)m} \end{pmatrix}$.
- [B] None of these answers are correct
- [C] $\exists m \in \mathbf{R}\backslash\{-2,0,2\}$, such that: A_m is not invertible.
- [D] $\forall m \in \mathbf{R}\backslash\{-2,0,2\}$, $A_m^{-1} = \begin{pmatrix} \frac{2m-3}{m(m^2-4)} & \frac{m^2-m-3}{m(m^2-4)} & -\frac{-3+m}{m(m^2-4)} \\ -\frac{-1+3m}{m(m^2-4)} & \frac{1+2m}{m(m^2-4)} & -\frac{m^2+1}{m(m^2-4)} \\ -\frac{-1+m}{(m-2)m} & \frac{1}{(m-2)m} & -\frac{1}{(m-2)m} \end{pmatrix}$.
- [E] $\forall m \in \mathbf{R}\backslash\{-2,0,2\}$, A_m is not invertible.

Part II Another Exercise on Linear Systems

Question 13.

Let A be the matrix defined by setting:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ -1 & 3 & -2 \\ -1 & 1 & 0 \end{pmatrix}.$$

We are interested by the inverse, if it exists, of $A - 2I_3$. An easy computation allows us to write that:

- [A] None of these answers are correct
- [B] $(A-2I_3)^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.
- [C] $(A-2I_3)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
- [D] $(A-2I_3)^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- [E] $(A-2I_3)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Question 14. (Q13, Part II)

We keep the notation from Question 13. From the result of the previous question, one can deduce that:

A A is invertible and $A^{-1} = \frac{1}{4}A^2 - \frac{3}{2}A + \frac{2}{3}I_3$

B A is not invertible.

C None of these answers are correct

D A is invertible and $A^{-1} = \frac{1}{8}A^2 - \frac{3}{4}A + \frac{3}{2}I_3$

Question 15. (Q13, Part III)

We keep the notation from Questions 13 to 14. From the result of Questions 13 to 14, one can say that:

A A is not invertible .

B None of these answers are correct

C A is invertible and $A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{3}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{3}{10} & \frac{9}{10} \end{pmatrix}$

D A is invertible and $A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{8} & -\frac{3}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{5}{8} \\ \frac{1}{4} & -\frac{3}{8} & \frac{9}{8} \end{pmatrix}$

Part III Vector Spaces

Question 16. ♣

Define the following sets.

$$\Gamma_1 := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (\alpha, \beta, \gamma, \delta) \in \mathbf{R}^4 \right\}, \quad \Gamma_2 := \{(x, y, z) \in \mathbf{R}^3, \text{ such that } 2x - y + z = 0\},$$

$$\Gamma_3 := \{A \in \mathcal{M}_2(\mathbf{R}), \ A \text{ is nilpotent } \}$$

Precise, for each of the above sets, which one are vector spaces, when they are endowed with laws $+$ and $\cdot$, that can naturally be associated to them[a]

A $(\Gamma_2, +, \cdot)$ is a vector space.

B $(\Gamma_2, +, \cdot)$ is not a vector space.

C None of these answers are correct

D $(\Gamma_3, +, \cdot)$ is a vector space.

E $(\Gamma_1, +, \cdot)$ is not a vector space.

F $(\Gamma_3, +, \cdot)$ is not a vector space.

G $(\Gamma_1, +, \cdot)$ is a vector space.

[a] Recall that an element A in $\mathcal{M}_n(\mathbf{R})$ is said to be nilpotent if there exists an integer q such that $A^q = 0_{\mathcal{M}_n(\mathbf{R})}$.

Question 17. ♣

Define $A := \{(\alpha, \alpha, \alpha),\ \alpha \in \mathbf{R}\}$ and $B := \{(x, y, z) \in \mathbf{R}^3,\ x + y + z = 0\}$. One can say that:

- A Only B is a vector subspaces of $\mathbf{R}^3$.
- B None of these answers are correct
- C Both A and B are vector subspaces of $\mathbf{R}^3$.
- D Only A is a vector subspaces of $\mathbf{R}^3$.