SINDy – Sparse Identification of Nonlinear Dynamics

Imperial Aeronautics DPSA & EmTech lecture 2025

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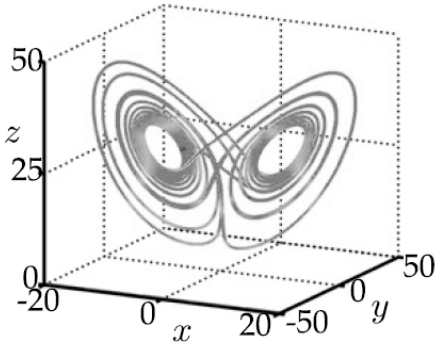
SINDy

Data | *Dynamics (assumptions)* | *Model structure & coefficients*



$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x})$$

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = x(\rho - z) - y$$
$$\dot{z} = xy - \beta z$$

Lecture outline

Part 1: SINDy – Sparse Identification of Nonlinear Dynamics

- **Intro**: identifying ODEs
- SINDy – **applications**

Part 2: Coding example

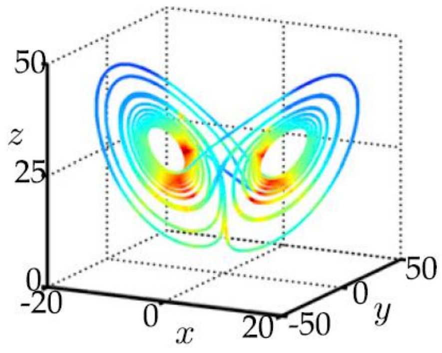
- Matlab / Python **example**: *Lorenz system*
- **PySINDy**

Part 3: SINDy with Control

- My SINDy **research**
- SINDy **Reinforcement Learning**

Identifying ODEs and PDEs from data – SINDy

Data → **Dynamics (assumptions)** → **Model structure & coefficients**



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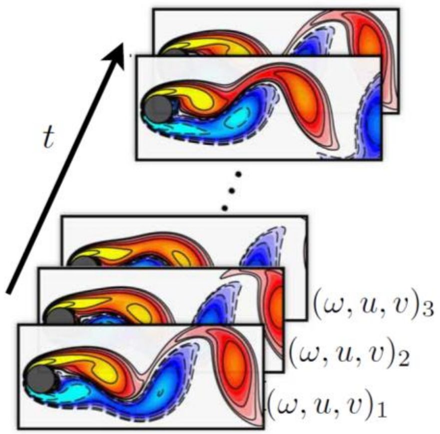


$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{N}(\mathbf{u})$$

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \frac{1}{Re}\nabla^2\omega$$

SL Brunton, JL Proctor, JN Kutz (2016) Discovering governing equations from data by sparse identification of nonlinear dynamical systems.

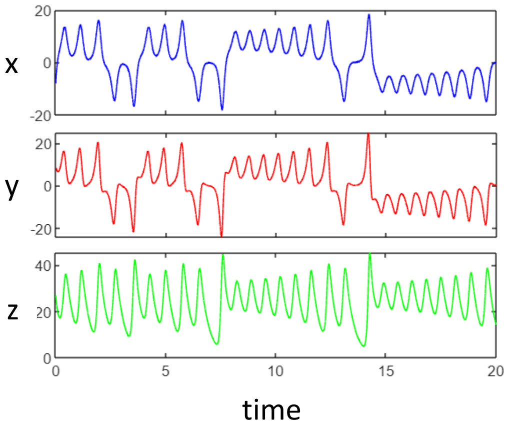
SINDy

1) True Lorenz System

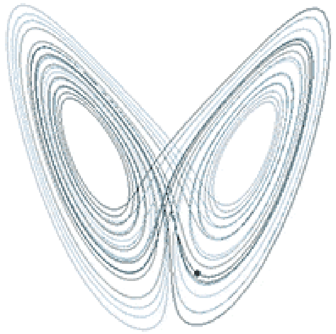
$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= x(\rho - z) - y\\ \dot{z} &= xy - \beta z.\end{aligned}$$

Collect time series data *(ρ = 28, σ = 10, β = 8/3)*

Time series



Phase space (attractor)



SL Brunton, JL Proctor, JN Kutz (2016) [Discovering governing equations from data by sparse identification of nonlinear dynamical systems].

Sequential thresholded least squares algorithm

Sparse regression: penalised least squares

$$\rightarrow \hat{\boldsymbol{\xi}}_k = \operatorname{argmin}_{\boldsymbol{\xi}_k} \left\|\dot{\mathbf{X}}_k - \boldsymbol{\Theta}(\mathbf{X})\boldsymbol{\xi}_k\right\|_2^2 + \lambda R(\boldsymbol{\xi}_k)$$

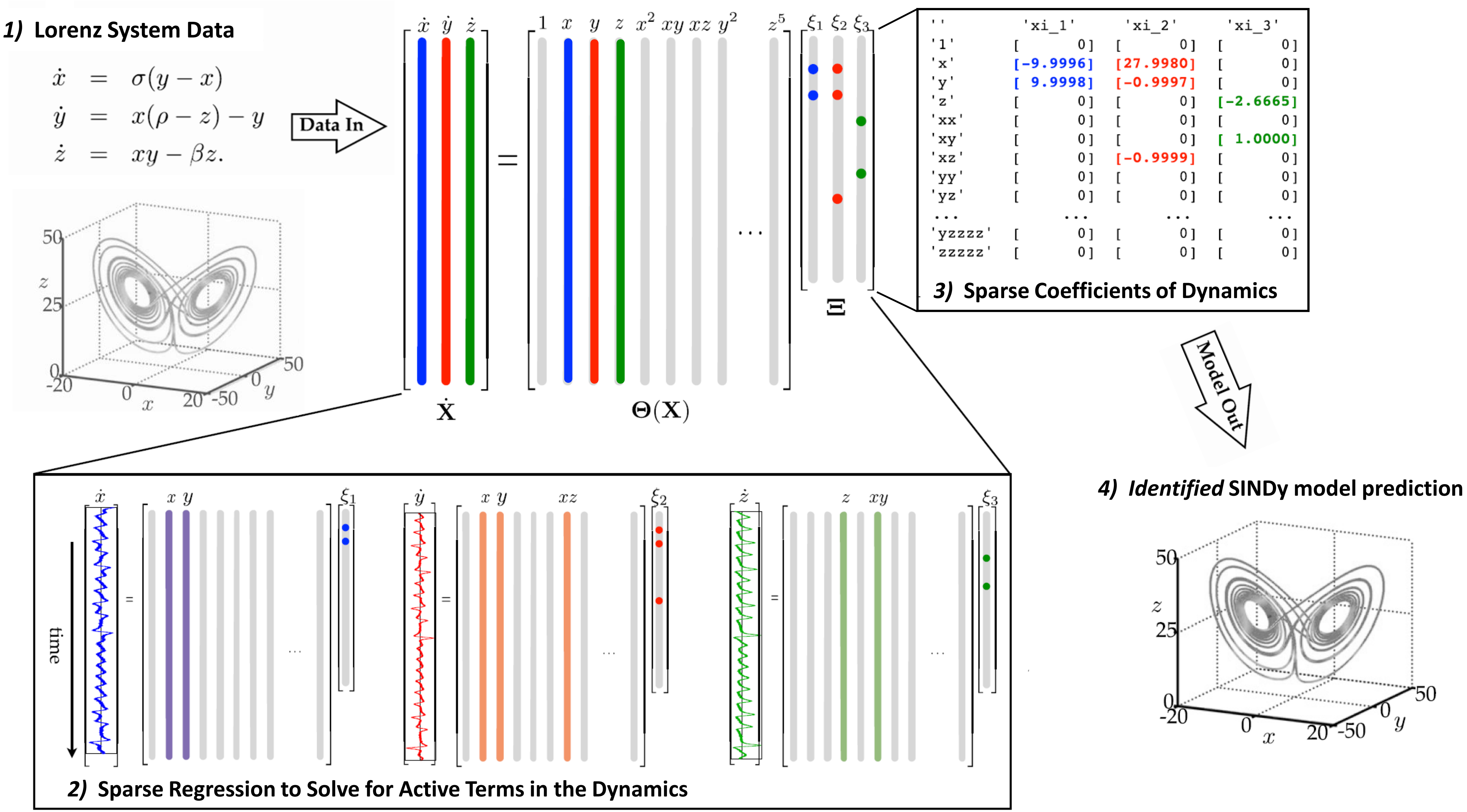
```
function Xi = sparsifyDynamics(Theta,dXdt,lambda,n)
% Compute Sparse regression: sequential least squares
Xi = Theta\dXdt;  % Initial guess: Least-squares

% Lambda is our sparsification knob.
for k=1:10
    smallinds = (abs(Xi)<lambda);   % Find small coefficients
    Xi(smallinds)=0;                % and threshold
    for ind = 1:n                   % n is state dimension
        biginds = ~smallinds(:,ind);
% Regress dynamics onto remaining terms to find sparse Xi
        Xi(biginds,ind) = Theta(:,biginds)\dXdt(:,ind);
    end
end
```

Sparse Regression to Solve for Active Terms in the Dynamics

SL Brunton, JL Proctor, JN Kutz (2016) Discovering governing equations from data by sparse identification of nonlinear dynamical systems.

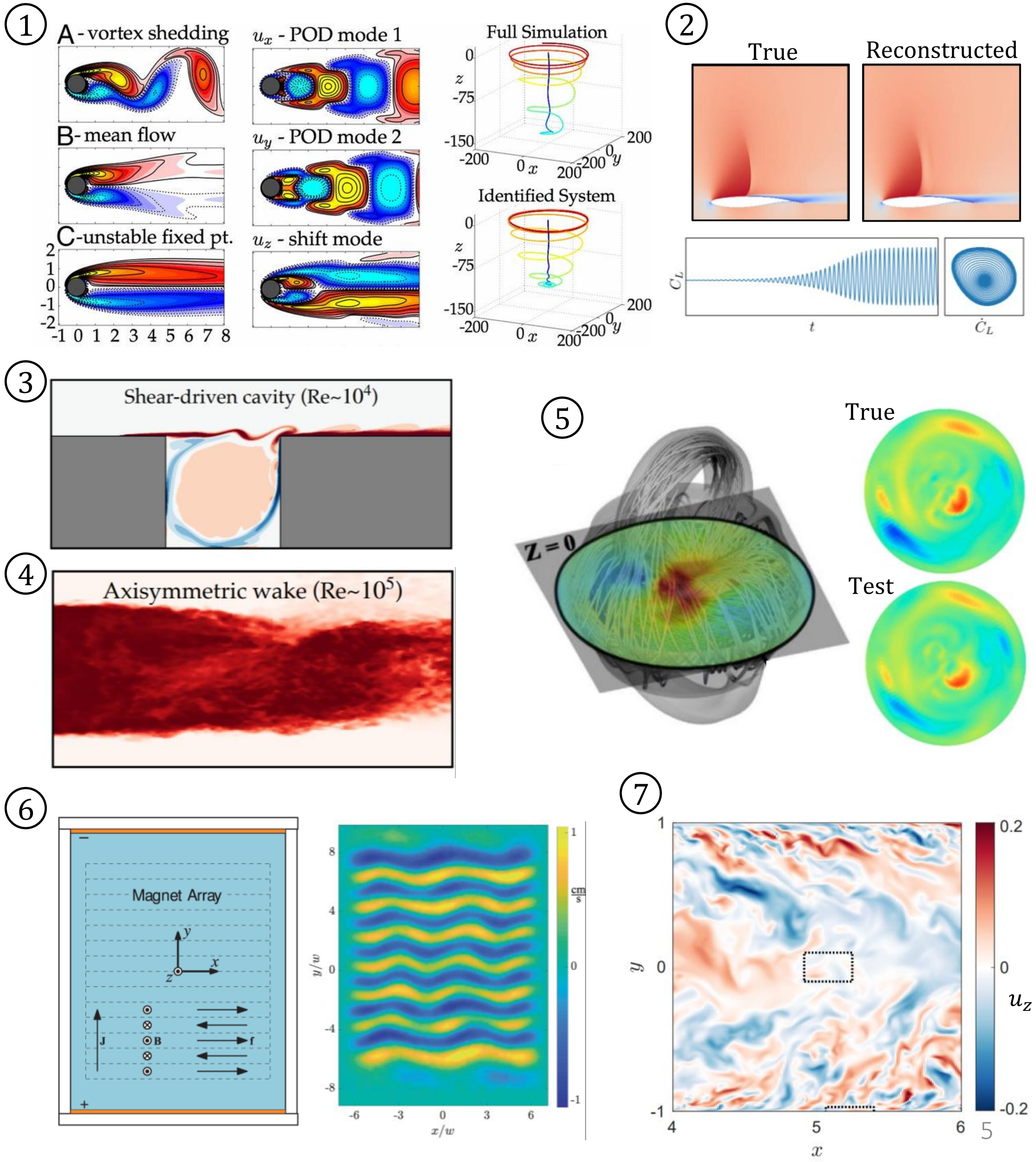
SINDy



SL Brunton, JL Proctor, JN Kutz (2016) [Discovering governing equations from data by sparse identification of nonlinear dynamical systems]().

SINDy – applications

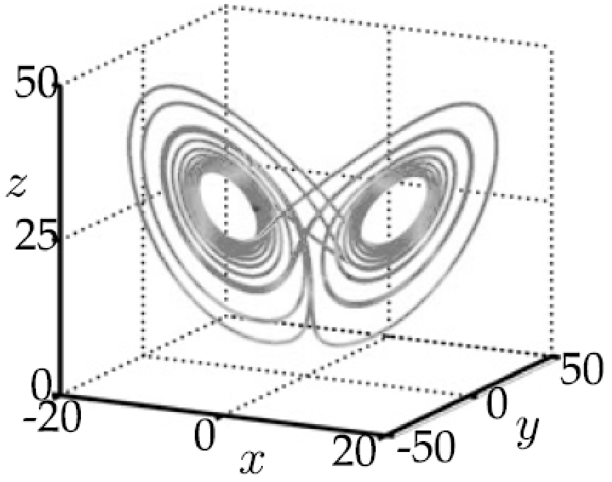
1. **Vortex shedding past a cylinder**
   - *Time history of POD coefficients:*
     - $\dot{x} = \mu x - \omega y + Axz$
     - $\dot{y} = \omega x + \mu y + Ayz$
     - $\dot{z} = -\lambda(z - x^2 - y^2)$
2. **Shock wave dynamics 2D airfoil transonic buffet conditions**
   - *Parametric c_L model for different α*
     - $c_L(r,\phi) = c_0 + c_1 r + c_2 r\cos(\varphi) + c_3 r\sin(\varphi) + c_4 r^2 \cos(2\varphi) + c_5 r^2 \sin(2\varphi)$
3. **Cavity flow**
   - *Coefficients of 2 active DMD modes*
     - $\dot{\alpha}_1 = \lambda_1\alpha_1 - \mu_1\alpha_1|\alpha_1|^2$
     - $\dot{\alpha}_5 = \lambda_5\alpha_5 - \mu_5\alpha_5|\alpha_5|^2$
4. **Experimental measurements turbulent bluff body wake**
   - *Statistical behavior of the CoP (learning drift and diffusion of SDE)*
     - $\dot{r} = \lambda r - \mu r^3 + \frac{\sigma^2}{2r} + (\sigma_0 + \sigma_1^2)w(t)$
5. **Plasma dynamics (magnetohydrodynamics): 3D spheromak sim**
   - *Dominant POD coefficient dynamics*
     - $\dot{a}_1 = 0.091\mathrm{a}_2 + 0.009\mathrm{a}_5$
     - $\dot{a}_2 = -0.091\mathrm{a}_1 + 0.008\mathrm{a}_5 - 0.011\mathrm{a}_6$
     - …
6. **Experimental weakly turbulent fluid flow in a thin electrolyte layer**
   - *Measured velocity field, identify PDE: form similar to N-S*
     - $\partial_t \mathbf{u} = c_1(\mathbf{u}\cdot\nabla)\mathbf{u} + c_2\nabla^2\mathbf{u} + c_3\mathbf{u} - \rho^{-1}\nabla p + \rho^{-1}\mathbf{f}$
7. **Turbulent 3D channel flow ($Re = 1000$) Johns Hopkins database**
   - *Identify PDEs: N-S, continuity equation, boundary conditions*
     - $\partial_t \mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - 0.995\nabla p + 4.93\cdot 10^{-5}\nabla^2\mathbf{u}$



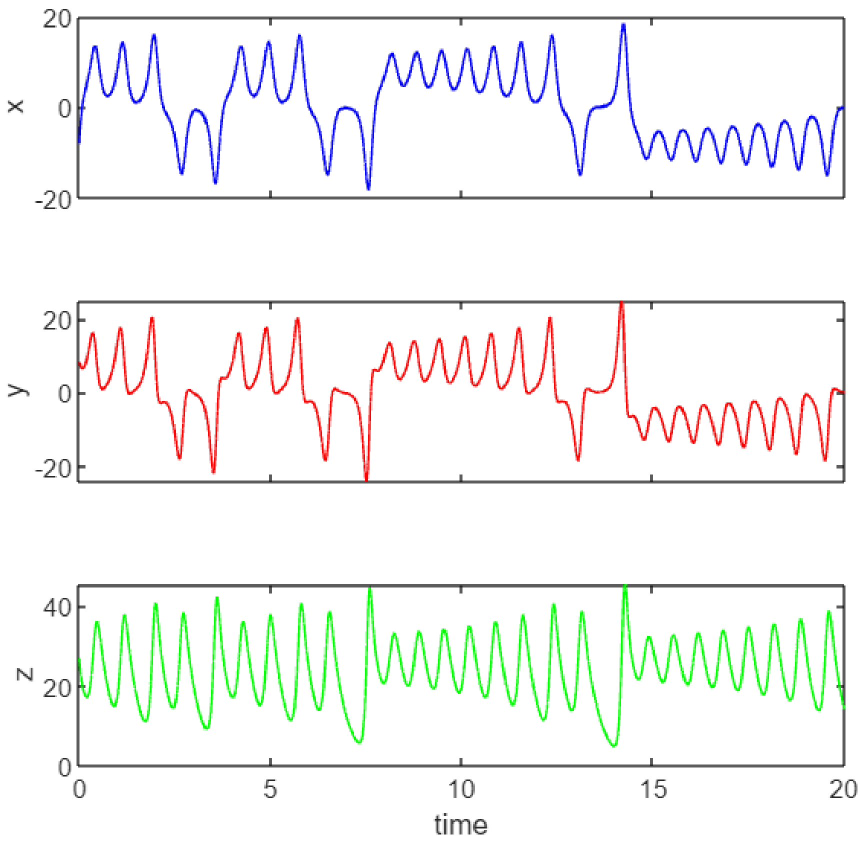
Tutorial → https://github.com/urban-fasel/Imperial_DPSA_EmTech_2025

Lorenz system

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{aligned}
$$



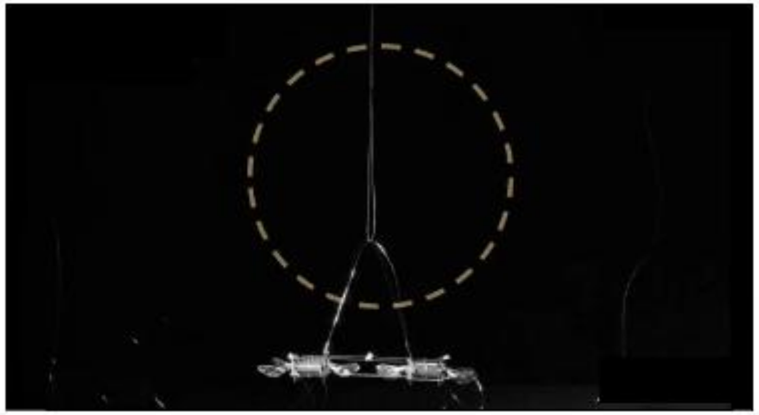
Data: time series x, y, z

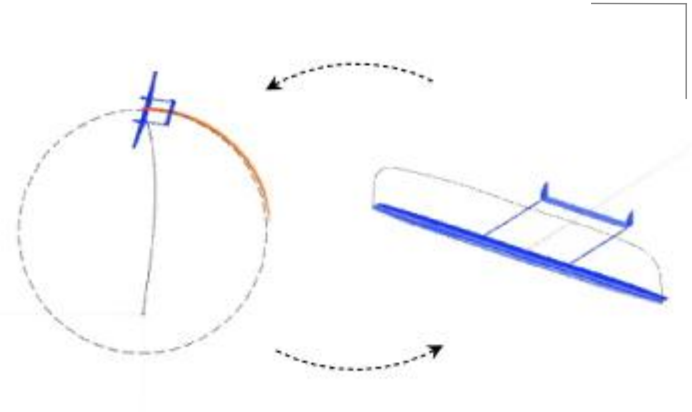


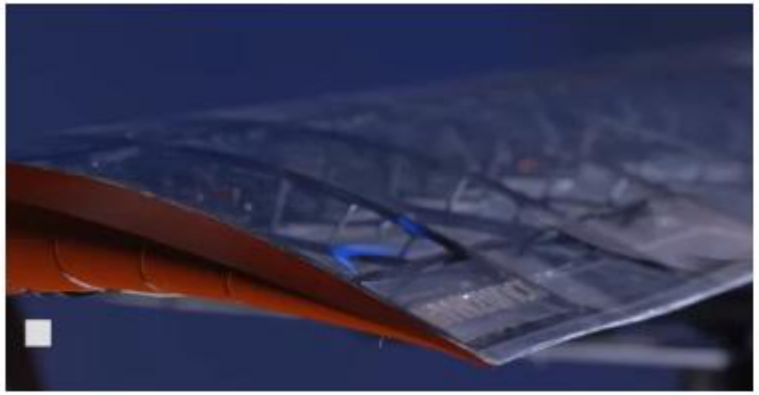
Additional MATLAB tutorials → https://github.com/urban-fasel/FiltonWorkshop2024

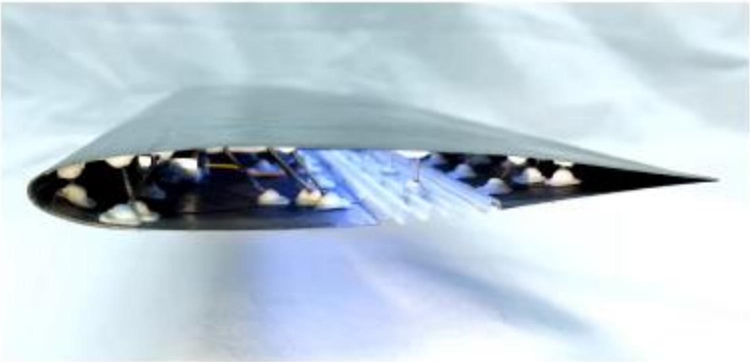
Additional Python tutorials → https://github.com/urban-fasel/I-X_workshop_2025

Research related to SINDy









(Adaptive) flight systems

1. Flapping wing MAV
2. Renewable energy systems
3. Composite additive manufacturing morphing wing drones
4. Morphing wings
5. Airborne wind energy

Methods

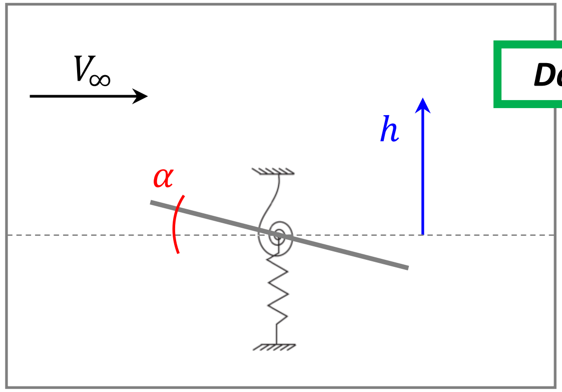
1. Co-design optimization
2. **Data driven modeling & control**
   - DMDc, SINDyC, SINDy-RL
   - E-SINDy, B-SINDy, SINDy-CP
   - Poincaré SINDy, Slow Manifolds

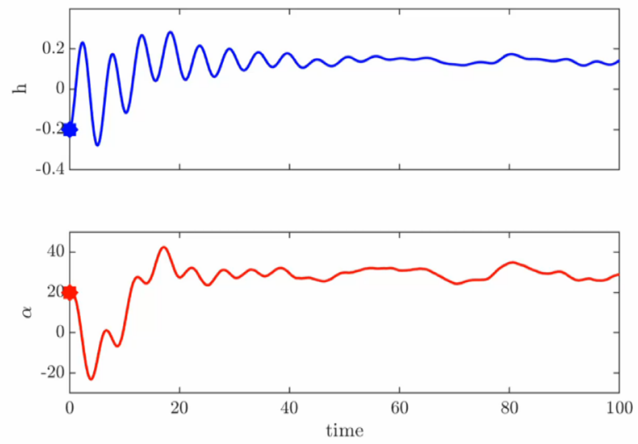
SINDy – toy problem in aeroelasticity

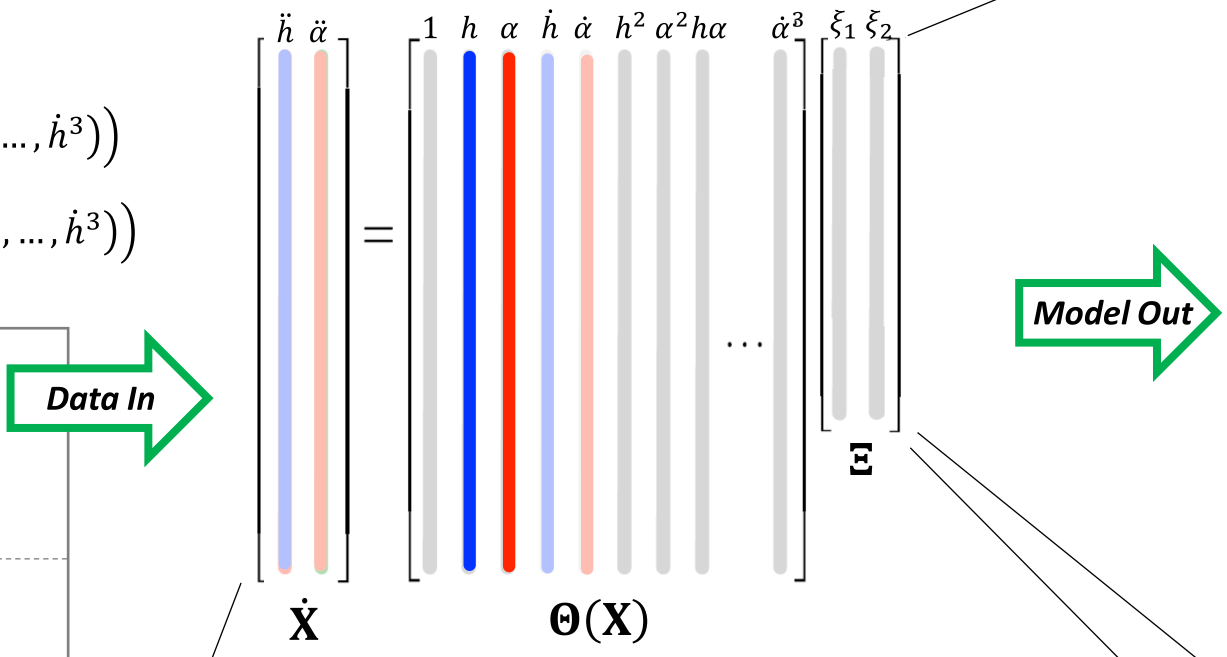
1) 2-DoF Elastically Supported Airfoil

$$\ddot{h} = \frac{1}{m}\left(-k_h h + L\left(V_\infty^2, \alpha, \dot{h}, \alpha^2, \ldots, \dot{h}^3\right)\right)$$

$$\ddot{\alpha} = \frac{1}{I_p}\left(-k_\alpha \alpha + M\left(V_\infty^2, \alpha, \dot{h}, \alpha^2, \ldots, \dot{h}^3\right)\right)$$

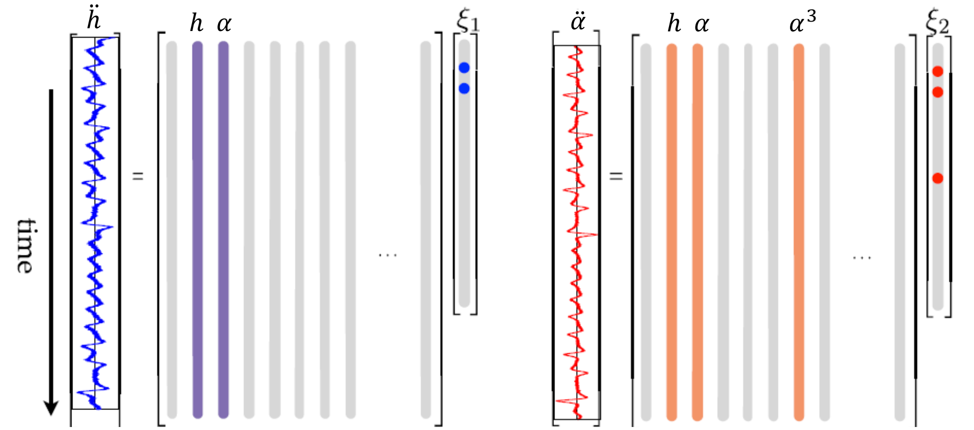






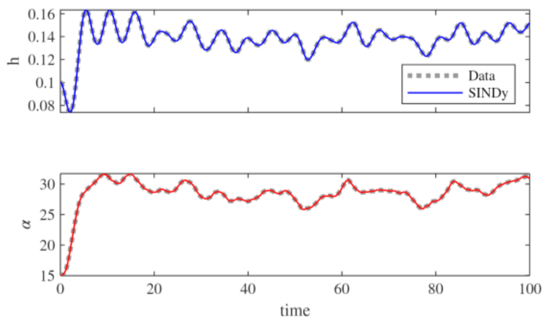
	$'\ddot{h}'$	$'\ddot{\alpha}'$
'1'	[0]	[0]
$'\dot{h}'$	[-0.3912]	[-0.9818]
$'\dot{\alpha}'$	[0]	[0]
$'h'$	[-1.4402]	[0]
$'\alpha'$	[0.3927]	[-0.0182]
$'\dot{h}h'$	[0]	[0]
$'\dot{h}\alpha'$	[0]	[0]
...	...	...
$'\dot{h}\dot{h}h'$	[0.0652]	[0.1633]
$'\dot{h}\dot{h}\alpha'$	[0]	[0]
$'\dot{h}hh'$	[0]	[0]
$'\dot{h}h\alpha'$	[-0.1962]	[-0.4913]
...	...	...
$'\dot{h}\alpha\alpha'$	[0.1959]	[0.4914]
...	...	...
$'hh\alpha'$	[0]	[0]
$'h\alpha\alpha'$	[0]	[0]
$'\alpha\alpha\alpha'$	[-0.0654]	[-0.1639]

3) Sparse Coefficients of Dynamics

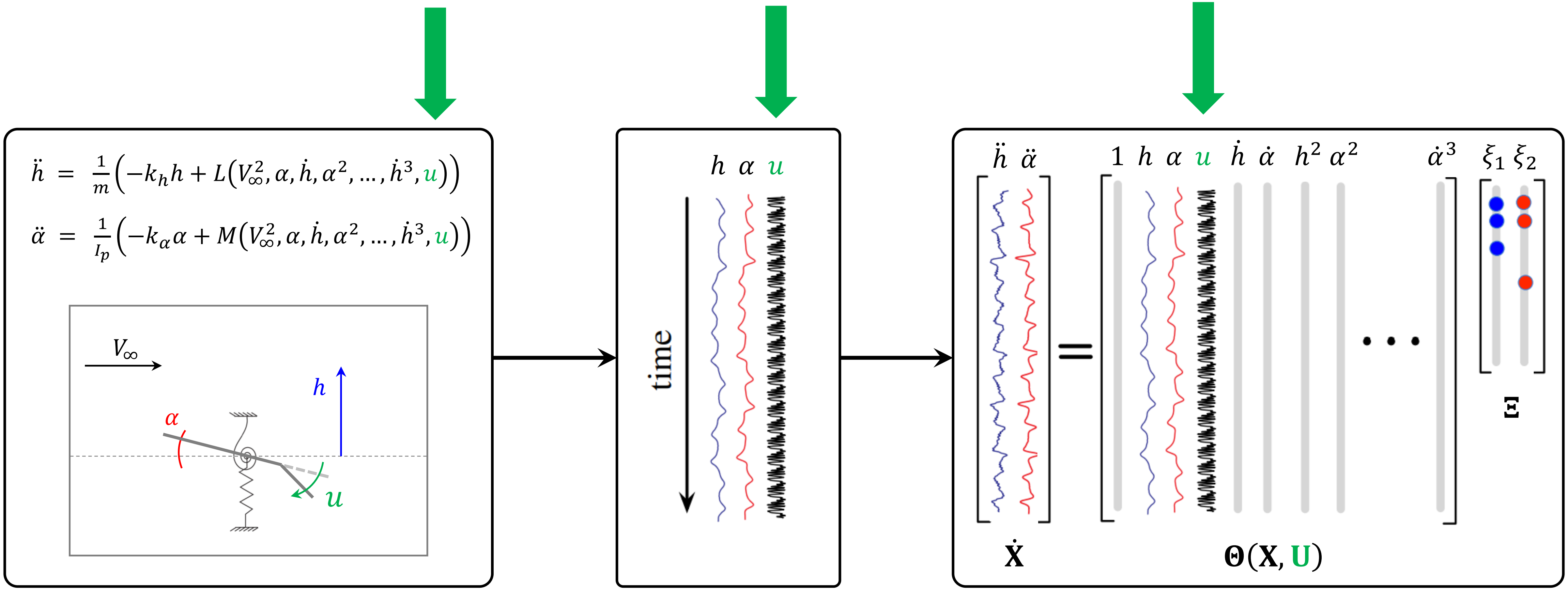


2) Sparse Regression to Solve for Active Terms in the Dynamics

4) SINDy model prediction



SINDy with control



U Fasel, E Kaiser, JN Kutz, BW Brunton, SL Brunton. SINDy with control: A tutorial. IEEE CDC, 2021.

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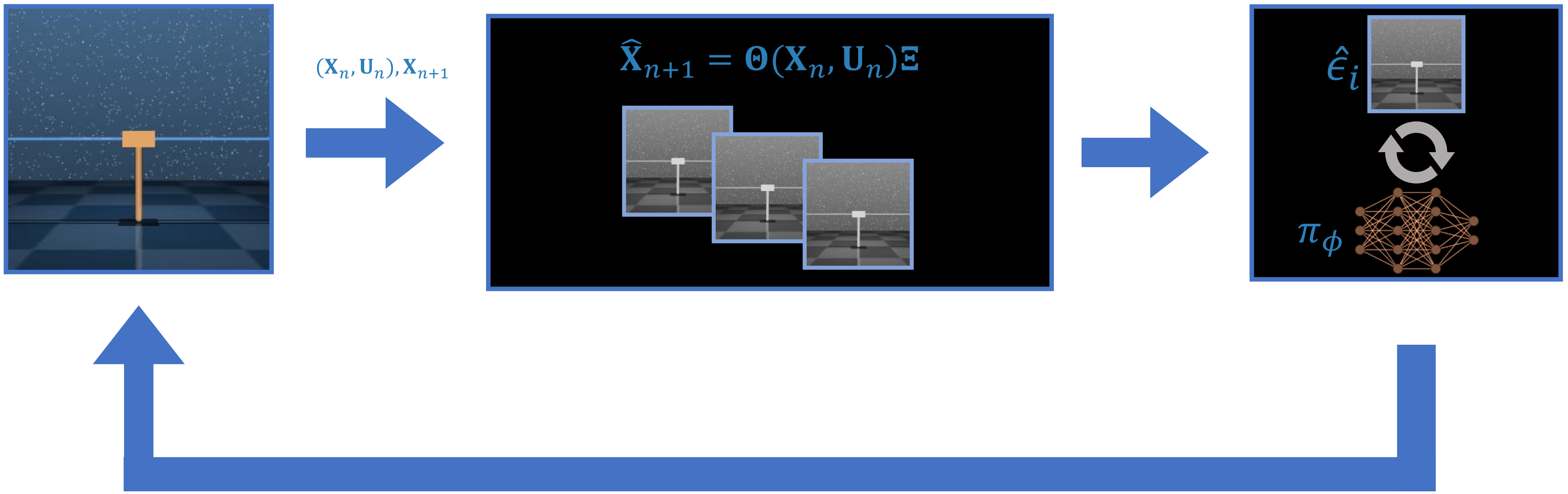
Collect Data in the “real” environment ϵ

Identify Ensemble of SINDy Models $\hat{\epsilon}_i$

Train Policy in SINDy $\hat{\epsilon}_i$



N Zolman, U Fasel, N Kutz, S Brunton. SINDy-RL: Interpretable and Efficient Reinforcement Learning for Fluid Flow Control, arXiv 2403.09110, 2024.

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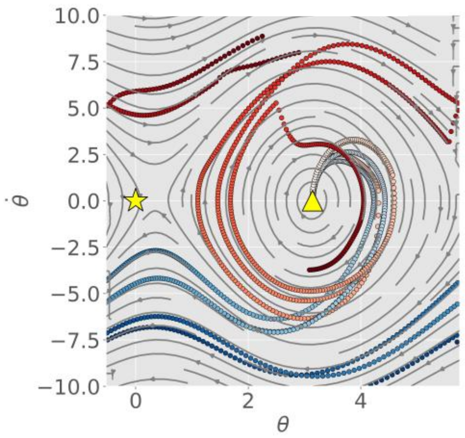
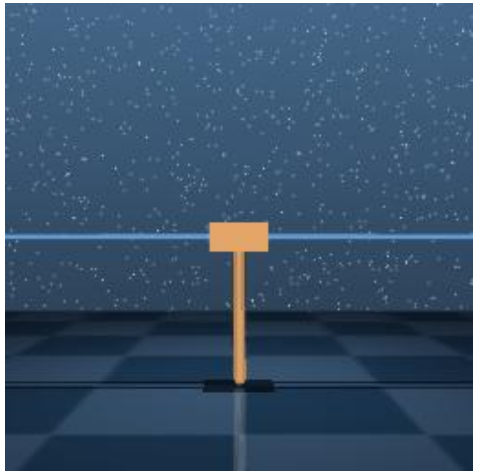
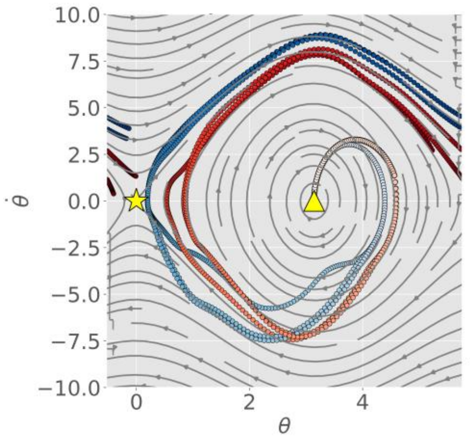
1st Dynamics Fit

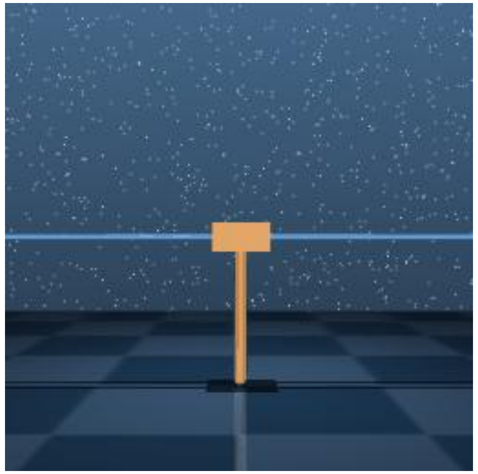


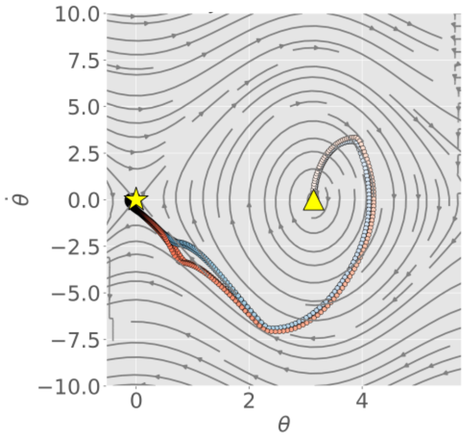
15th Dynamics Fit





25th Dynamics Fit





● Ground Truth

● SINDy

N Zolman, U Fasel, N Kutz, S Brunton. SINDy-RL: Interpretable and Efficient Reinforcement Learning for Fluid Flow Control, arXiv 2403.09110, 2024.

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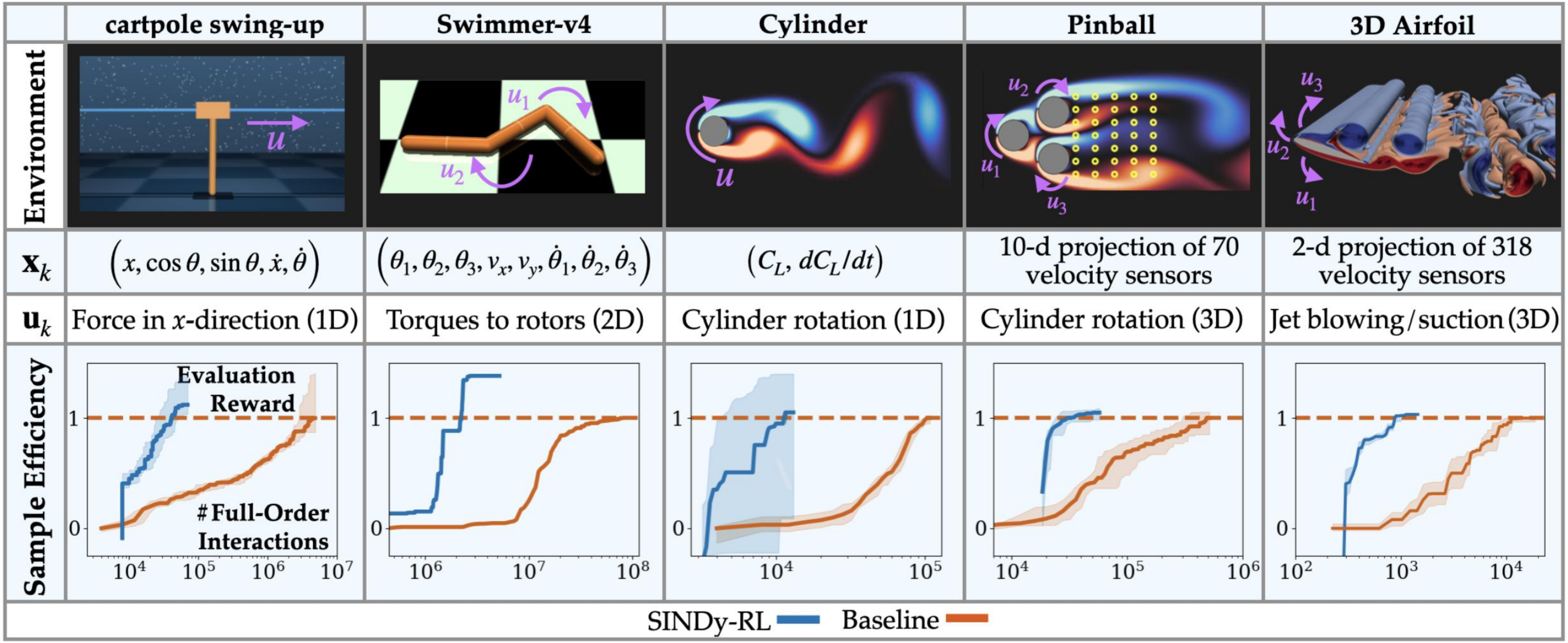

Training speedup → more than 2 orders of magnitude

N Zolman, U Fasel, N Kutz, S Brunton. SINDy-RL: Interpretable and Efficient Reinforcement Learning for Fluid Flow Control, arXiv 2403.09110, 2024.

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