SINDy – Sparse Identification of Nonlinear Dynamics

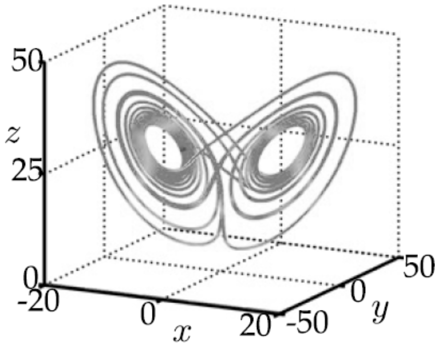
Imperial Aeronautics DPSA & EmTech lecture 2025

Urban Fasel

Imperial College London

SINDy

Data → *Dynamics (assumptions)* → *Model structure & coefficients*



$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x})$$

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

Lecture outline

Part 1: SINDy – Sparse Identification of Nonlinear Dynamics

- **Intro**: identifying ODEs
- SINDy – **applications**

Part 2: Coding example

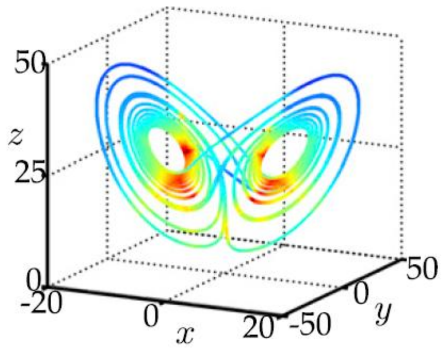
- Matlab / Python **example**: *Lorenz system*
- **PySINDy**

Part 3: SINDy with Control

- My SINDy **research**
- SINDy **Reinforcement Learning**

Identifying ODEs and PDEs from data – SINDy

Data → **Dynamics (assumptions)** → **Model structure & coefficients**



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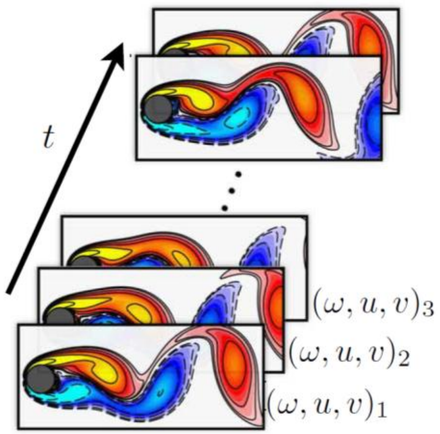


$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{N}(\mathbf{u})$$

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \frac{1}{Re}\nabla^2\omega$$

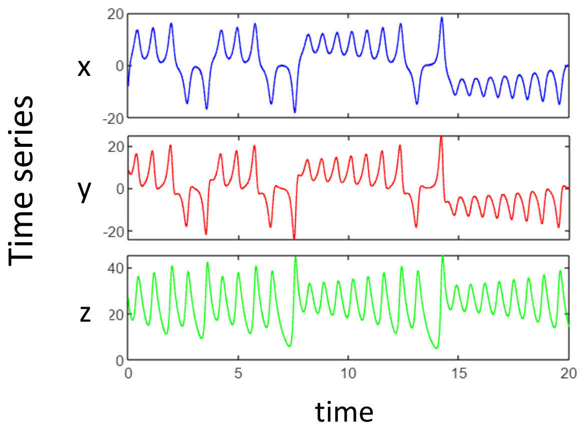
SL Brunton, JL Proctor, JN Kutz (2016) Discovering governing equations from data by sparse identification of nonlinear dynamical systems.

SINDy

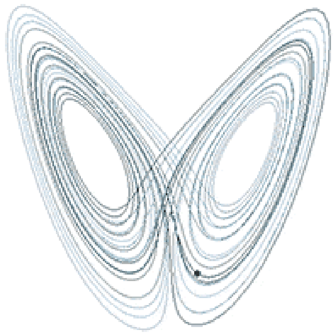
1) **True Lorenz System**

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{aligned}
$$

Collect time series data *(ρ = 28, σ = 10, β = 8/3)*



Phase space (attractor)



SL Brunton, JL Proctor, JN Kutz (2016) [Discovering governing equations from data by sparse identification of nonlinear dynamical systems]().

Sequential thresholded least squares algorithm

Sparse regression: penalised least squares

$$\rightarrow \hat{\xi}_k = \mathrm{argmin}_{\xi_k} \left\|\dot{\mathbf{X}}_k - \mathbf{\Theta}(\mathbf{X})\xi_k\right\|_2^2 + \lambda R(\xi_k)$$

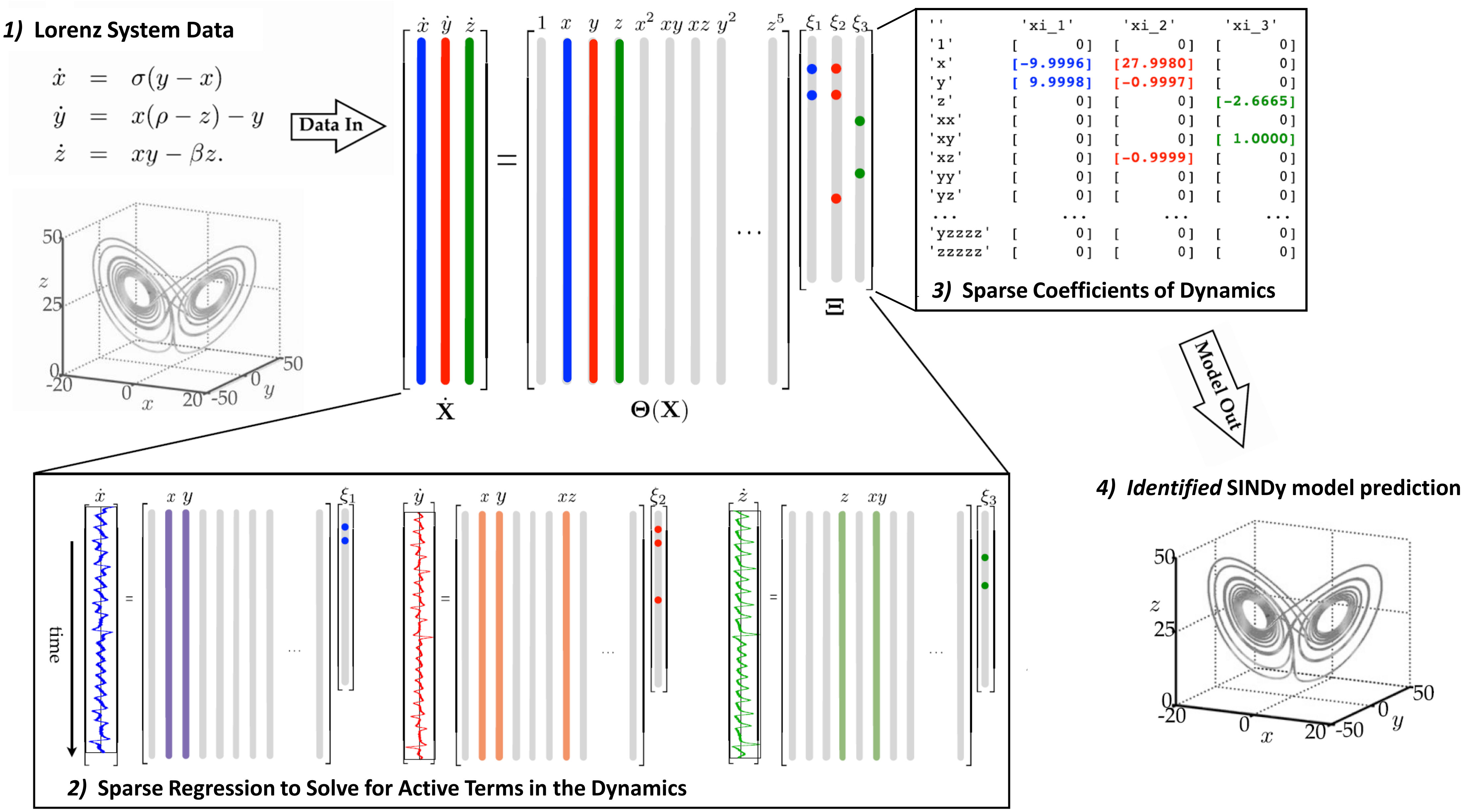
```
function Xi = sparsifyDynamics(Theta,dXdt,lambda,n)
% Compute Sparse regression: sequential least squares
Xi = Theta\dXdt;  % Initial guess: Least-squares

% Lambda is our sparsification knob.
for k=1:10
    smallinds = (abs(Xi)<lambda);   % Find small coefficients
    Xi(smallinds)=0;                % and threshold
    for ind = 1:n                   % n is state dimension
        biginds = ~smallinds(:,ind);
% Regress dynamics onto remaining terms to find sparse Xi
        Xi(biginds,ind) = Theta(:,biginds)\dXdt(:,ind);
    end
end
```

Sparse Regression to Solve for Active Terms in the Dynamics

SL Brunton, JL Proctor, JN Kutz (2016) Discovering governing equations from data by sparse identification of nonlinear dynamical systems.

SINDy

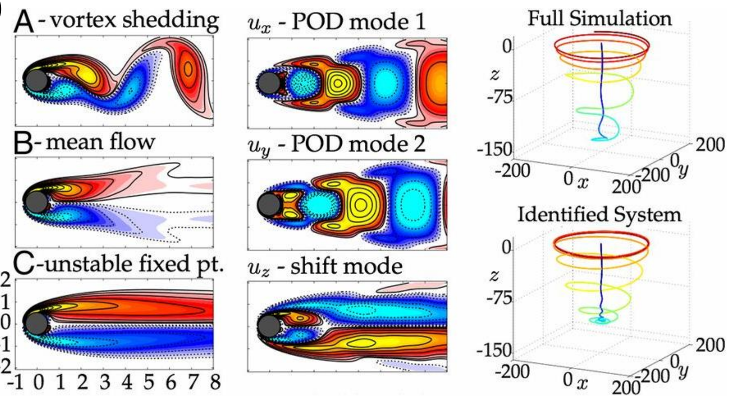


SL Brunton, JL Proctor, JN Kutz (2016) [Discovering governing equations from data by sparse identification of nonlinear dynamical systems](#).

SINDy – applications

1. [Vortex shedding past a cylinder]
   - *Time history of POD coefficients:*
     - $\dot{x} = \mu x - \omega y + Axz$
     - $\dot{y} = \omega x + \mu y + Ayz$
     - $\dot{z} = -\lambda(z - x^2 - y^2)$
2. [Shock wave dynamics 2D airfoil transonic buffet conditions]
   - *Parametric c_L model for different α*
     - $c_L(r,\phi) = c_0 + c_1 r + c_2 r\cos(\varphi) + c_3 r\sin(\varphi) + c_4 r^2\cos(2\varphi) + c_5 r^2\sin(2\varphi)$
3. [Cavity flow]
   - *Coefficients of 2 active DMD modes*
     - $\dot{\alpha}_1 = \lambda_1\alpha_1 - \mu_1\alpha_1|\alpha_1|^2$
     - $\dot{\alpha}_5 = \lambda_5\alpha_5 - \mu_5\alpha_5|\alpha_5|^2$
4. [Experimental measurements turbulent bluff body wake]
   - *Statistical behavior of the CoP (learning drift and diffusion of SDE)*
     - $\dot{r} = \lambda r - \mu r^3 + \frac{\sigma^2}{2r} + \left(\sigma_0 + \sigma_1^2\right)w(t)$
5. [Plasma dynamics (magnetohydrodynamics): 3D spheromak sim]
   - *Dominant POD coefficient dynamics*
     - $\dot{a}_1 = 0.091\mathrm{a}_2 + 0.009\mathrm{a}_5$
     - $\dot{a}_2 = -0.091\mathrm{a}_1 + 0.008\mathrm{a}_5 - 0.011\mathrm{a}_6$
     - …
6. [Experimental weakly turbulent fluid flow in a thin electrolyte layer]
   - *Measured velocity field, identify PDE: form similar to N-S*
     - $\partial_t \mathbf{u} = c_1(\mathbf{u}\cdot\nabla)\mathbf{u} + c_2\nabla^2\mathbf{u} + c_3\mathbf{u} - \rho^{-1}\nabla p + \rho^{-1}\mathbf{f}$
7. [Turbulent 3D channel flow ($Re = 1000$) Johns Hopkins database]
   - *Identify PDEs: N-S, continuity equation, boundary conditions*
     - $\partial_t \mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - 0.995\nabla p + 4.93\cdot 10^{-5}\nabla^2\mathbf{u}$

①



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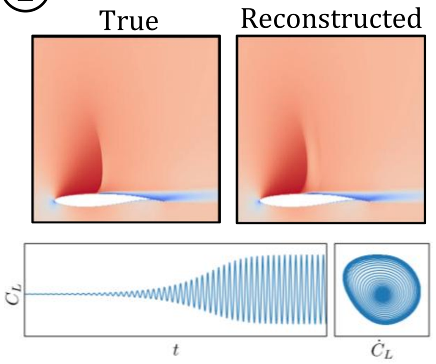

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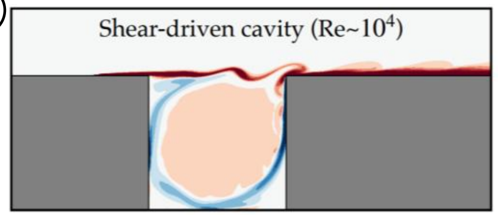

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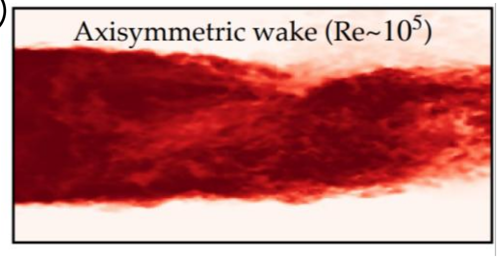

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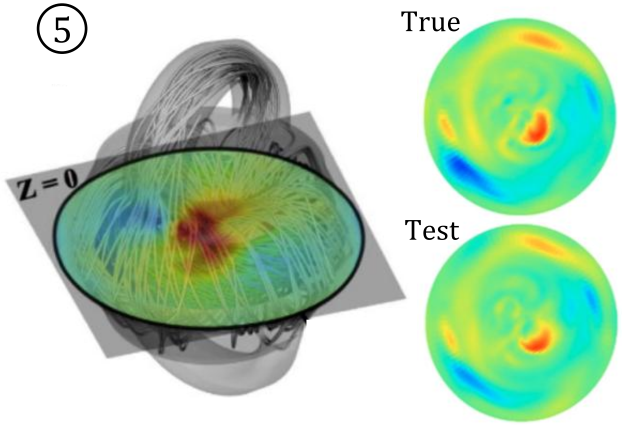

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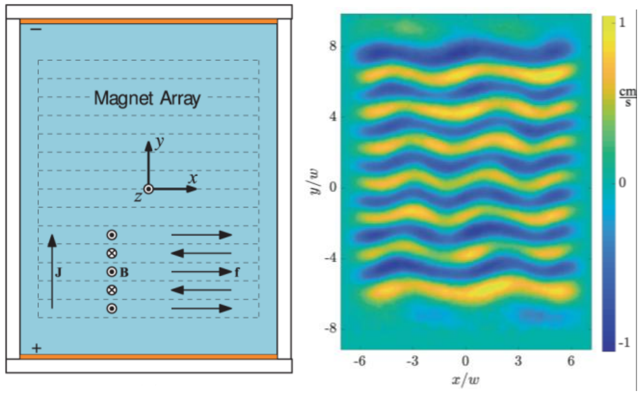

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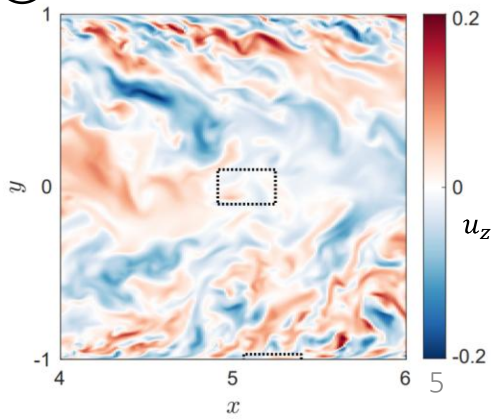
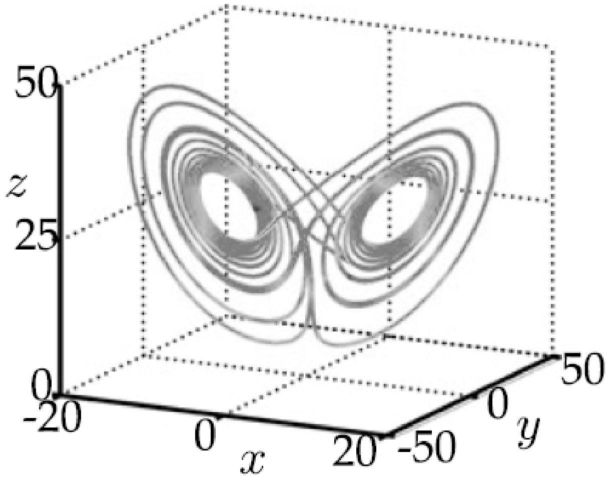


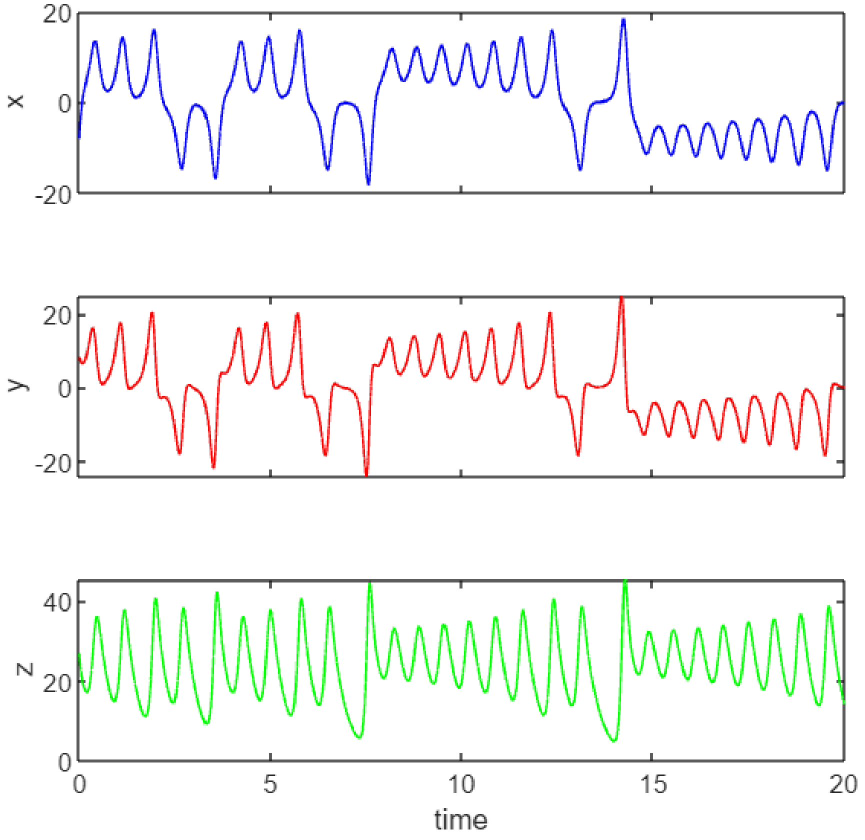
Tutorial → https://github.com/urban-fasel/Imperial_DPSA_EmTech_2025

Lorenz system

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{aligned}
$$



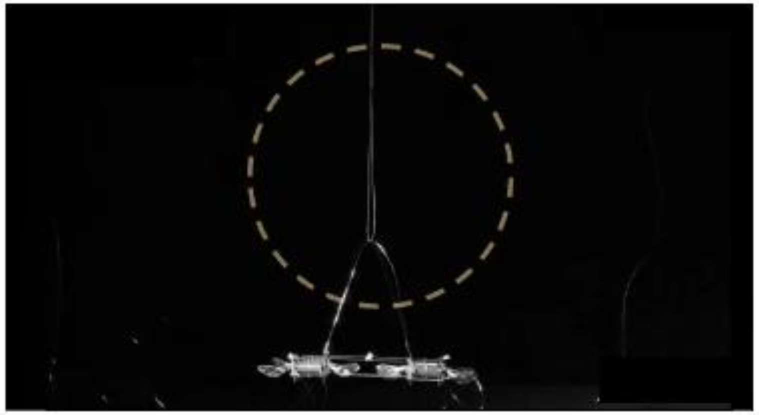
Data: time series x, y, z

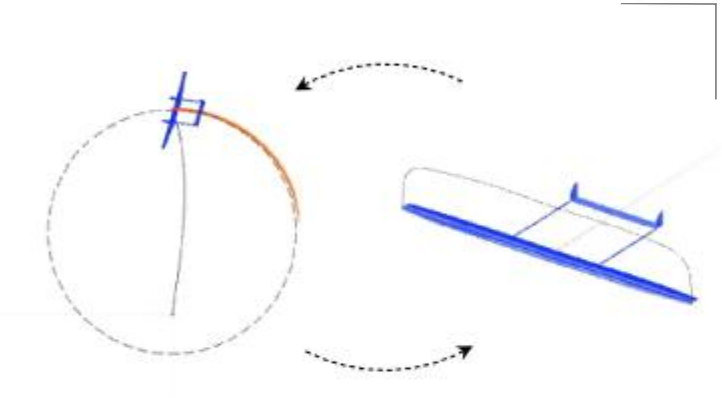


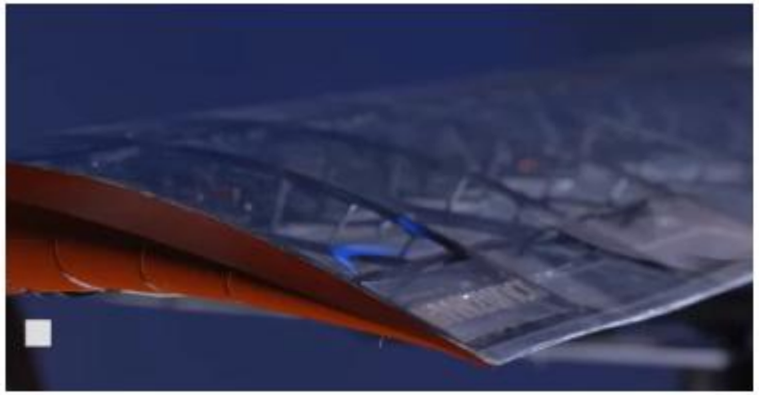
Additional MATLAB tutorials → https://github.com/urban-fasel/FiltonWorkshop2024

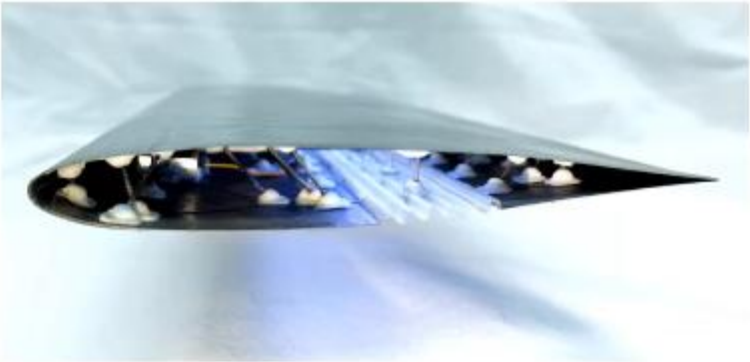
Additional Python tutorials → https://github.com/urban-fasel/I-X_workshop_2025

Research related to SINDy









(Adaptive) flight systems

1. Flapping wing MAV
2. Renewable energy systems
3. Composite additive manufacturing morphing wing drones
4. Morphing wings
5. Airborne wind energy

Methods

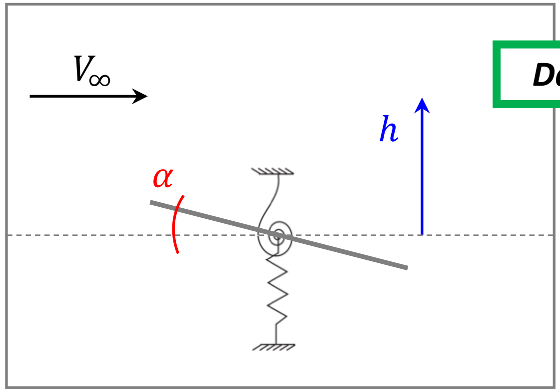
1. Co-design optimization
2. **Data driven modeling & control**
   - DMDc, SINDyC, SINDy-RL
   - E-SINDy, B-SINDy, SINDy-CP
   - Poincaré SINDy, Slow Manifolds

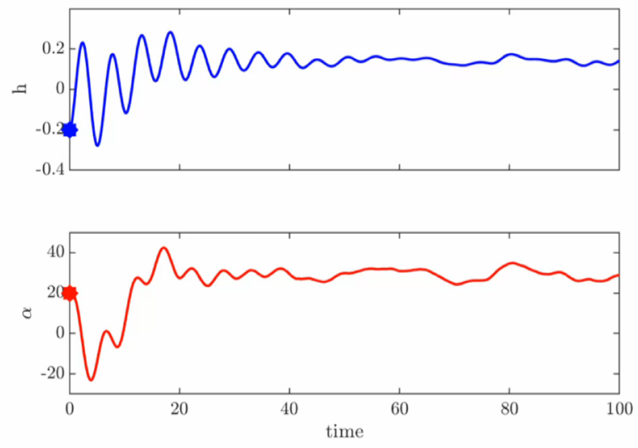
SINDy – toy problem in aeroelasticity

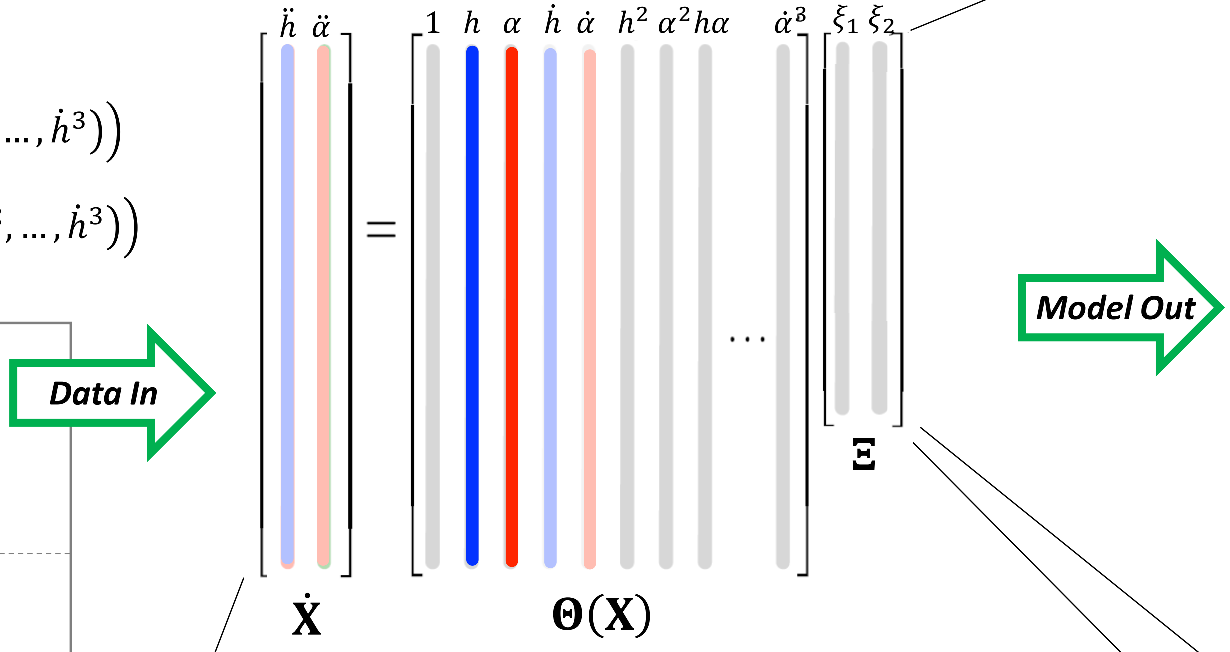
1) 2-DoF Elastically Supported Airfoil

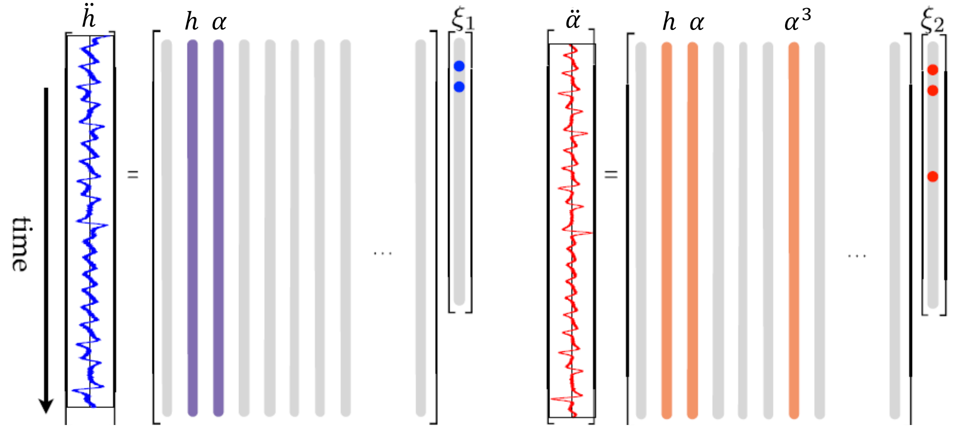
$$\ddot{h} = \frac{1}{m}\left(-k_h h + L\left(V_\infty^2, \alpha, \dot{h}, \alpha^2, \dots, \dot{h}^3\right)\right)$$

$$\ddot{\alpha} = \frac{1}{I_p}\left(-k_\alpha \alpha + M\left(V_\infty^2, \alpha, \dot{h}, \alpha^2, \dots, \dot{h}^3\right)\right)$$







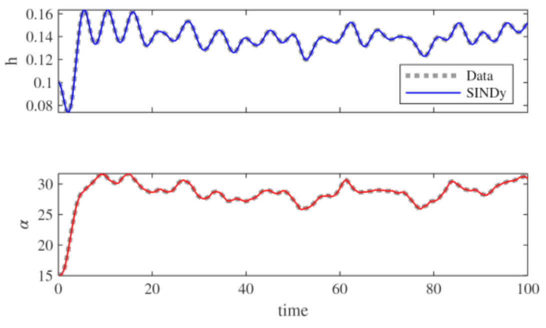


2) Sparse Regression to Solve for Active Terms in the Dynamics

	'$\ddot h$'	'$\ddot\alpha$'
'1'	[0]	[0]
'$\dot h$'	[-0.3912]	[-0.9818]
'$\dot\alpha$'	[0]	[0]
'h'	[-1.4402]	[0]
'α'	[0.3927]	[-0.0182]
'$\dot h h$'	[0]	[0]
'$\dot h\alpha$'	[0]	[0]
...	...	...
'$\dot h\dot h\dot h$'	[0.0652]	[0.1633]
'$\dot h\dot h\alpha$'	[0]	[0]
'$\dot h\dot h h$'	[0]	[0]
'$\dot h h\alpha$'	[-0.1962]	[-0.4913]
...	...	...
'$\dot h\alpha\alpha$'	[0.1959]	[0.4914]
...	...	...
'$hh\alpha$'	[0]	[0]
'$h\alpha\alpha$'	[0]	[0]
'$\alpha\alpha\alpha$'	[-0.0654]	[-0.1639]

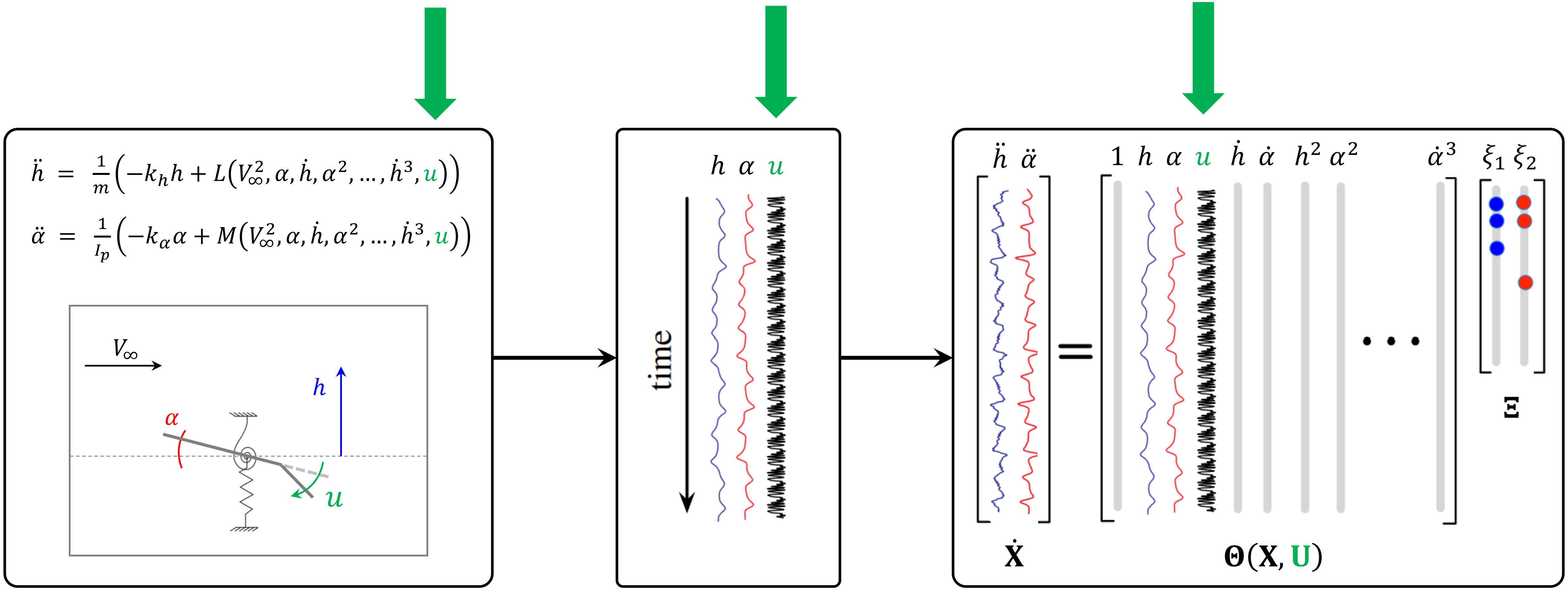
3) Sparse Coefficients of Dynamics

4) SINDy model prediction



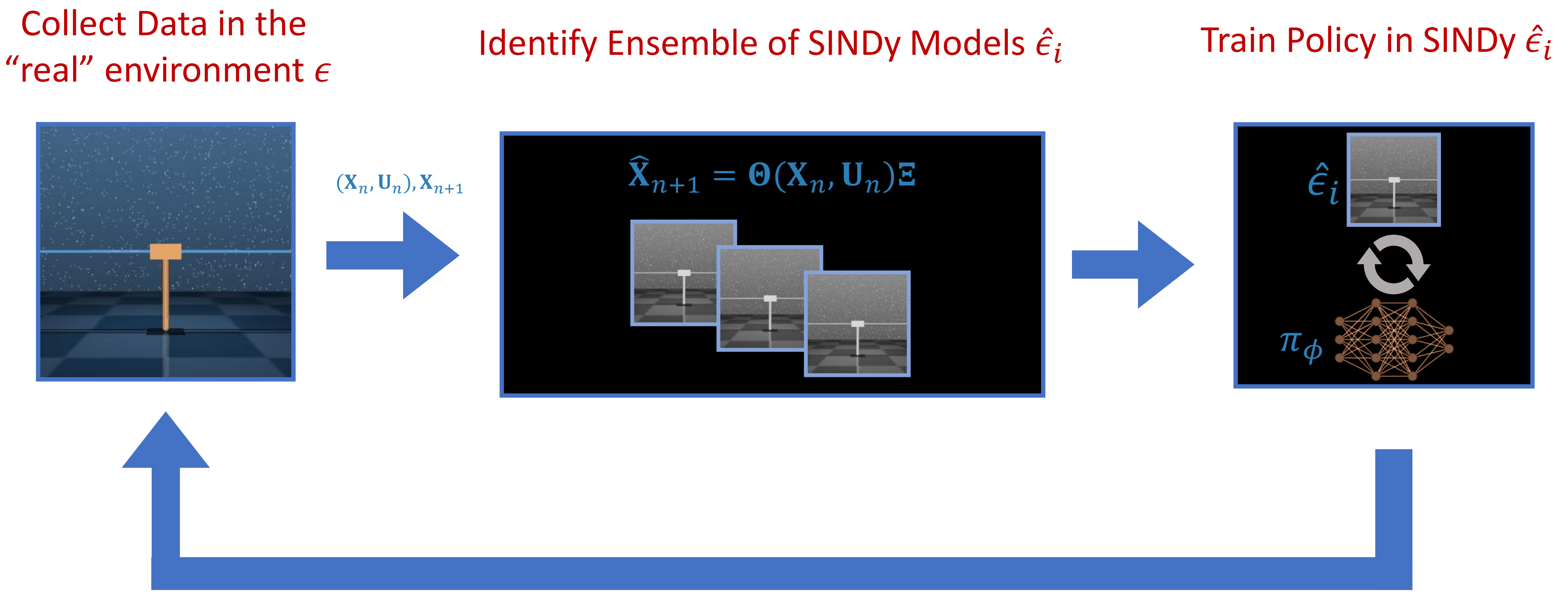
SL Brunton, JL Proctor, JN Kutz. Discovering governing equations from data by sparse identification of nonlinear dynamical systems, PNAS 2016.

SINDy with control



U Fasel, E Kaiser, JN Kutz, BW Brunton, SL Brunton. SINDy with control: A tutorial. IEEE CDC, 2021.

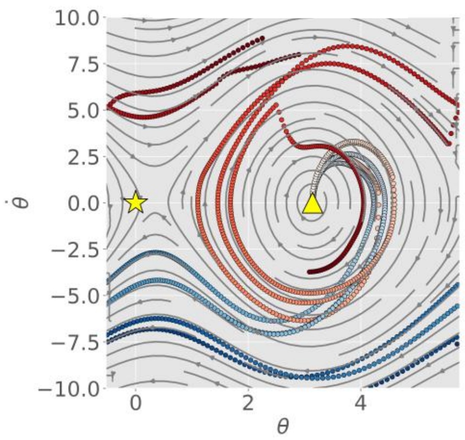
SINDy – Reinforcement Learning: Collaboration University of Washington



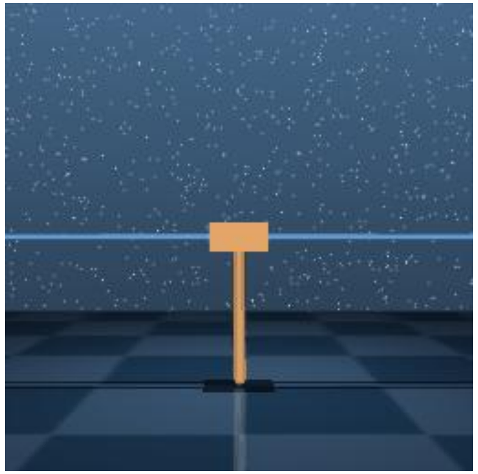
N Zolman, U Fasel, N Kutz, S Brunton. SINDy-RL: Interpretable and Efficient Reinforcement Learning for Fluid Flow Control, arXiv 2403.09110, 2024.

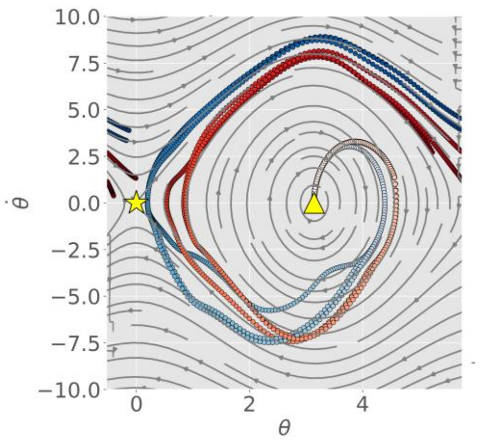
SINDy – Reinforcement Learning: Collaboration University of Washington

1st Dynamics Fit

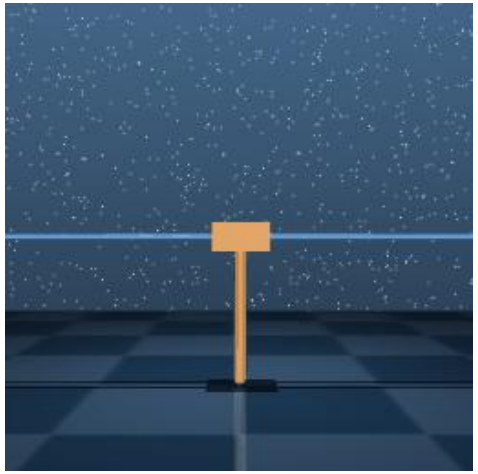


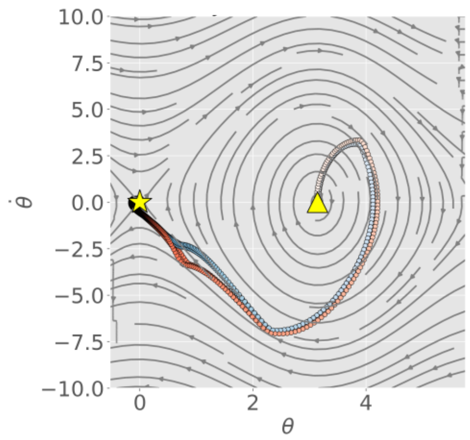
15th Dynamics Fit





25th Dynamics Fit



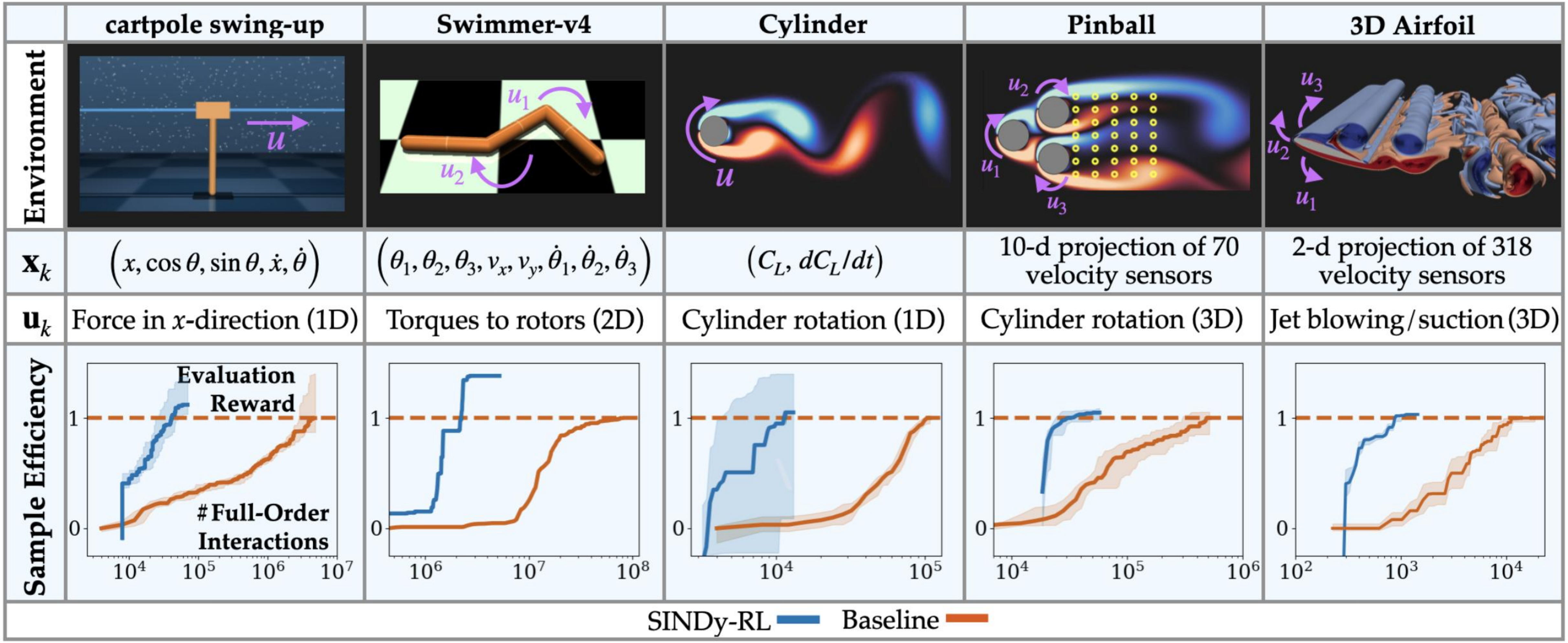


● Ground Truth

● SINDy

N Zolman, U Fasel, N Kutz, S Brunton. SINDy-RL: Interpretable and Efficient Reinforcement Learning for Fluid Flow Control, arXiv 2403.09110, 2024.

SINDy – Reinforcement Learning: Collaboration University of Washington



Training speedup → more than 2 orders of magnitude

N Zolman, U Fasel, N Kutz, S Brunton. SINDy-RL: Interpretable and Efficient Reinforcement Learning for Fluid Flow Control, arXiv 2403.09110, 2024.

SINDy with control 2.0 – Learning a Poincare map → Owen Brook

Data-Driven Stabilisation of Unstable Periodic Orbits of the Three-Body Problem

Owen M. Brook[1], **Jason J. Bramburger**[2], **Davide Amato**[1], **Urban Fasel**[1]
[1] *Department of Aeronautics, Imperial College London, London, UK*
[2] *Department of Mathematics and Statistics, Concordia University, Montréal, QC, Canada*
Corresponding author: Owen Brook (owen.brook20@imperial.ac.uk)

Abstract

Many different models of the physical world exhibit chaotic dynamics, from fluid flows and chemical reactions to celestial mechanics. The study of the three-body problem (3BP) and its many different families of unstable periodic orbits (UPOs) have provided fundamental insight into chaotic dynamics as far back as the 19th century. The 3BP, a conservative system, is inherently challenging to sample due to its volume-preservation property. In this paper we present an interpretable data-driven approach for the state-dependent control of UPOs in the 3BP, through leveraging the inherent sensitivity of chaos and the local manifold structure. We overcome the sampling challenge by utilising prior knowledge of UPOs and a novel augmentation strategy. This enables sample-efficient discovery of a verifiable and accurate local Poincaré map in as few as 55 data points. We suggest that the Poincaré map is best discovered at a surface of section where the norm of the monodromy matrix, i.e. the local sensitivity to small perturbations, is the smallest. To stabilise the UPOs, we apply small velocity impulses once each period, determined by solving a convex system of linear matrix inequalities based on the linearised map. We constrain the norm of the decision variables used to solve this system, resulting in locally optimal velocity impulses directed along the local stable manifold. Critically, this behaviour is achieved in a computationally efficient manner. We demonstrate this sample-efficient and low-energy method across several orbit families in the 3BP, with potential applications ranging from robotics and spacecraft control to fluid dynamics.

1 Introduction

Chaotic systems are characterised by their sensitivity to small perturbations, where two trajectories starting arbitrarily close together will diverge exponentially in time. For a long time, this meant that these systems were largely avoided and efforts were generally directed to suppress or evade chaos [1]. However, chaotic

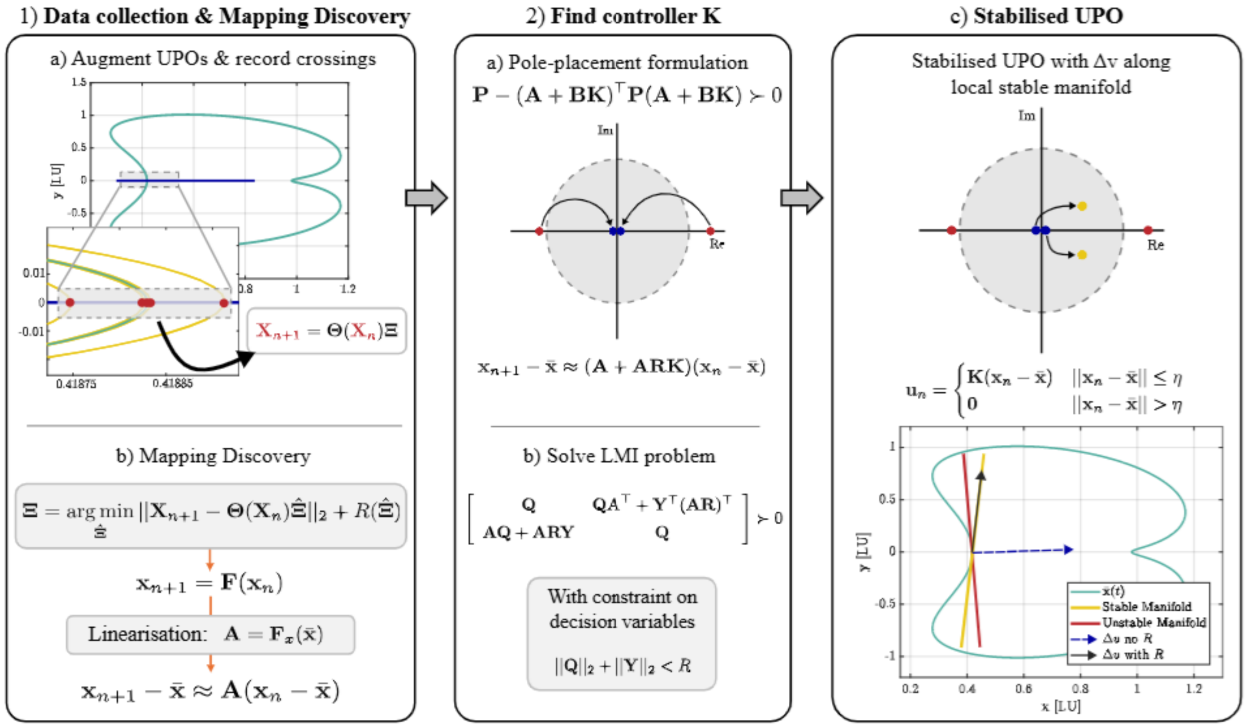


Figure 1: Our method for stabilising a Lyapunov UPO: 1) Initial conditions of known UPOs are selected and small velocity perturbations added. The states $(x, \dot{x}, \dot{y})$ are recorded at successive crossings of the surface of section $\Sigma = \{(x, \dot{x}, \dot{y}) | y = 0, x < 0.84\}$. A Poincaré map is discovered using SINDy and linearised at the desired UPO $\bar{\mathbf{x}}$. 2) A controller $\mathbf{K}$ is found through solving a pole-placement problem using linear matrix inequalities with a constraint on the decision variables. 3) Constraint causes the magnitude of the eigenvalues to increase and aligns the impulse Δv along the local stable manifold, a locally-optimal solution.

arXiv paper: https://arxiv.org/abs/2507.08630

Owen M. Brook, Jason J. Bramburger, Davide Amato, Urban Fasel. Data-Driven Stabilisation of Unstable Periodic Orbits of the Three-Body Problem, arXiv 2507.08630, 2024.

Lecture outline

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- **Intro**: identifying ODEs
- SINDy – **applications**

Part 2: Coding example

- Matlab / Python **example**: *Lorenz system*
- **PySINDy**

Part 3: SINDy with Control

- My SINDy **research**
- SINDy **Reinforcement Learning**