

SINDy – Sparse Identification of Nonlinear Dynamics

Imperial Aeronautics DPSA & EmTech lecture 2025

Urban Fasel

Imperial College
London

SINDy

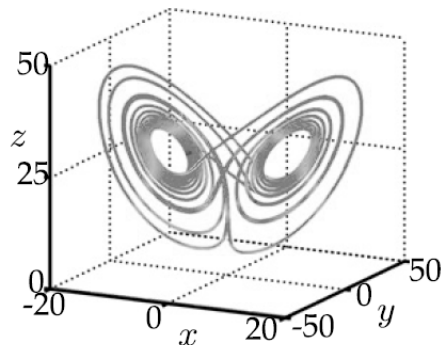
Data



Dynamics (assumptions)



Model structure & coefficients



$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x})$$

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$

Lecture outline

Part 1: SINDy – Sparse Identification of Nonlinear Dynamics

- **Intro:** identifying ODEs
- SINDy – **applications**

Part 2: Coding example

- Matlab / Python **example:** *Lorenz system*
- **PySINDy**

Part 3: SINDy with Control

- My SINDy **research**
- SINDy **Reinforcement Learning**

Identifying ODEs and PDEs from data – SINDy

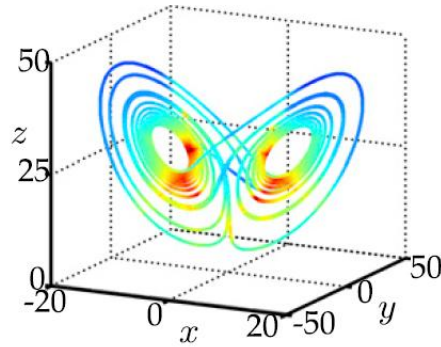
Data



Dynamics (assumptions)



Model structure
& coefficients

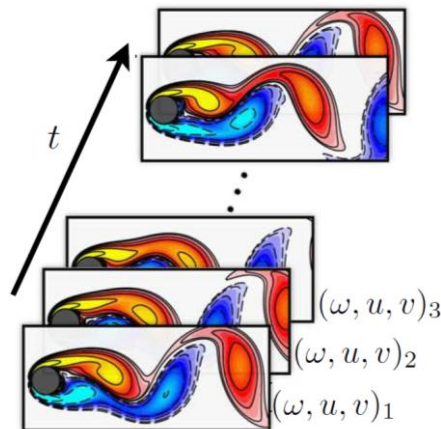


$$\frac{d}{dt}\mathbf{x} = \mathbf{f}(\mathbf{x})$$

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z$$



$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{N}(\mathbf{u})$$

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \frac{1}{Re} \nabla^2 \omega$$

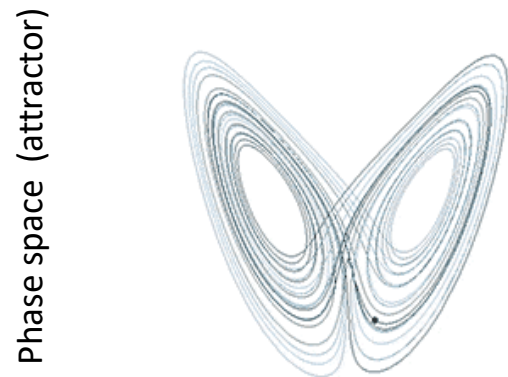
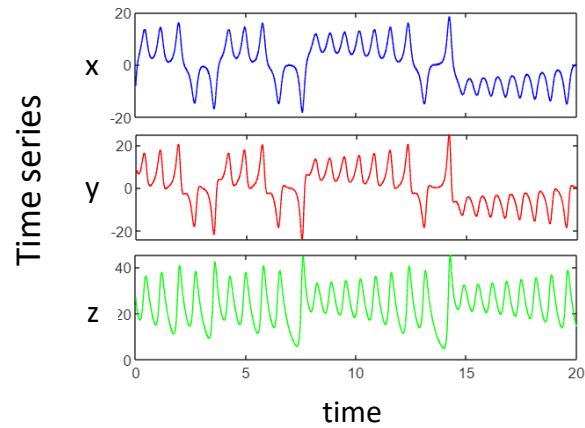
1) True Lorenz System

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = x(\rho - z) - y$$

$$\dot{z} = xy - \beta z.$$

Collect time series data ($\rho = 28$, $\sigma = 10$, $\beta = 8/3$)



Sequential thresholded least squares algorithm



Sparse regression: penalised least squares

$$\rightarrow \hat{\xi}_k = \operatorname{argmin}_{\xi_k} \|\dot{\mathbf{X}}_k - \mathbf{\Theta}(\mathbf{X})\xi_k\|_2^2 + \lambda R(\xi_k)$$

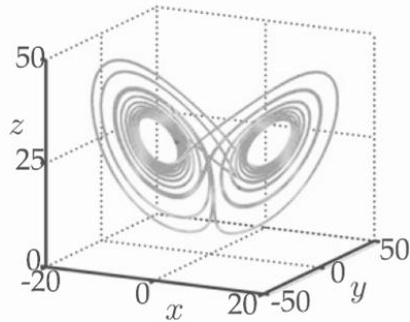
```
function Xi = sparsifyDynamics(Theta,dXdt,lambda,n)
% Compute Sparse regression: sequential least squares
Xi = Theta\dXdt; % Initial guess: Least-squares

% Lambda is our sparsification knob.
for k=1:10
    smallinds = (abs(Xi)<lambda); % Find small coefficients
    Xi(smallinds)=0; % and threshold
    for ind = 1:n % n is state dimension
        biginds = ~smallinds(:,ind);
        % Regress dynamics onto remaining terms to find sparse Xi
        Xi(biginds,ind) = Theta(:,biginds)\dXdt(:,ind);
    end
end
end
```

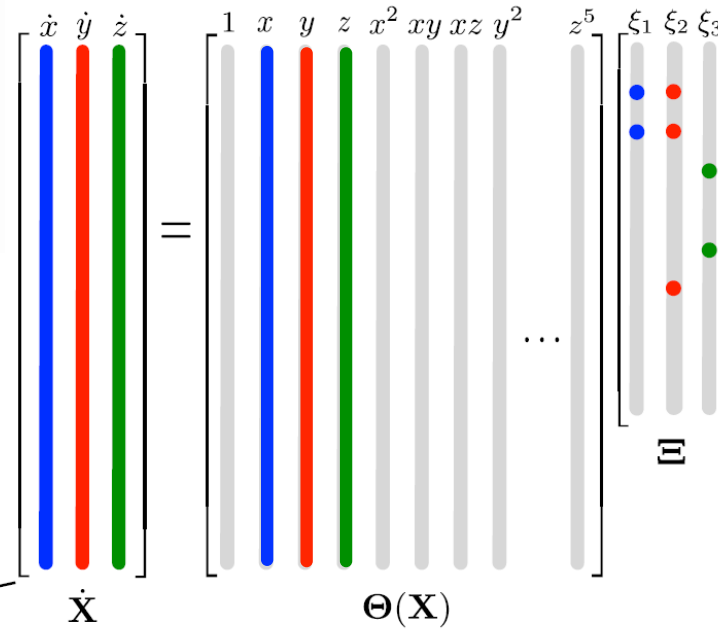
Sparse Regression to Solve for Active Terms in the Dynamics

1) Lorenz System Data

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z.\end{aligned}$$



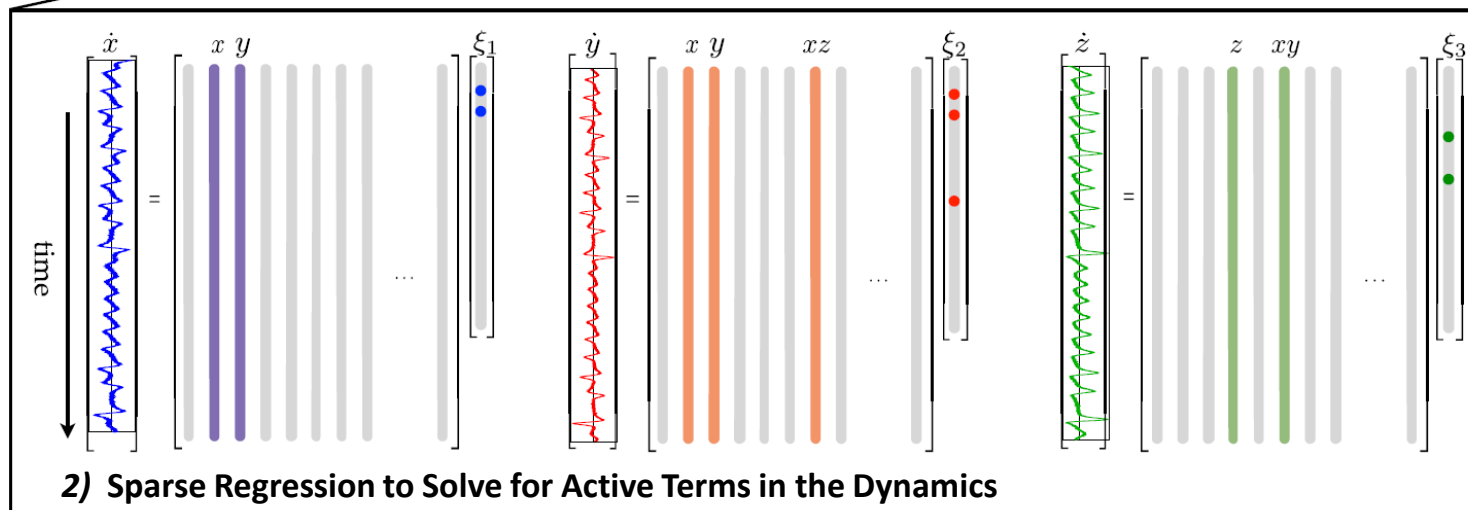
Data In



	'xi_1'	'xi_2'	'xi_3'
'1'	[0]	[0]	[0]
'x'	[-9.9996]	[27.9980]	[0]
'y'	[9.9998]	[-0.9997]	[0]
'z'	[0]	[0]	[-2.6665]
'xx'	[0]	[0]	[0]
'xy'	[0]	[0]	[1.0000]
'xz'	[0]	[-0.9999]	[0]
'yy'	[0]	[0]	[0]
'yz'	[0]	[0]	[0]
...
'yzzzz'	[0]	[0]	[0]
'zzzzz'	[0]	[0]	[0]

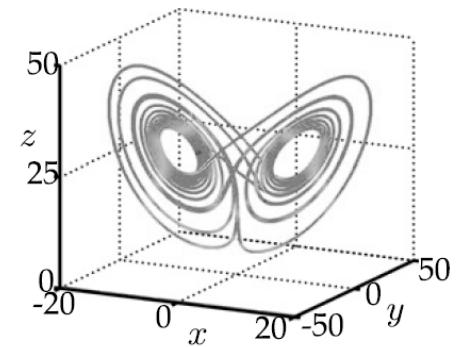
3) Sparse Coefficients of Dynamics

Model Out



2) Sparse Regression to Solve for Active Terms in the Dynamics

4) Identified SINDy model prediction



SINDy – applications

1. Vortex shedding past a cylinder

- Time history of POD coefficients:

- $\dot{x} = \mu x - \omega y + Axz$
- $\dot{y} = \omega x + \mu y + Ayz$
- $\dot{z} = -\lambda(z - x^2 - y^2)$

2. Shock wave dynamics 2D airfoil transonic buffet conditions

- Parametric c_L model for different α

- $c_L(r, \phi) = c_0 + c_1 r + c_2 r \cos(\phi) + c_3 r \sin(\phi) + c_4 r^2 \cos(2\phi) + c_5 r^2 \sin(2\phi)$

3. Cavity flow

- Coefficients of 2 active DMD modes

- $\dot{\alpha}_1 = \lambda_1 \alpha_1 - \mu_1 \alpha_1 |\alpha_1|^2$
- $\dot{\alpha}_5 = \lambda_5 \alpha_5 - \mu_5 \alpha_5 |\alpha_5|^2$

4. Experimental measurements turbulent bluff body wake

- Statistical behavior of the CoP (learning drift and diffusion of SDE)

- $\dot{r} = \lambda r - \mu r^3 + \frac{\sigma^2}{2r} + (\sigma_0 + \sigma_1^2)w(t)$

5. Plasma dynamics (magnetohydrodynamics): 3D spheromak sim

- Dominant POD coefficient dynamics

- $\dot{a}_1 = 0.091a_2 + 0.009a_5$
- $\dot{a}_2 = -0.091a_1 + 0.008a_5 - 0.011a_6$
- ...

6. Experimental weakly turbulent fluid flow in a thin electrolyte layer

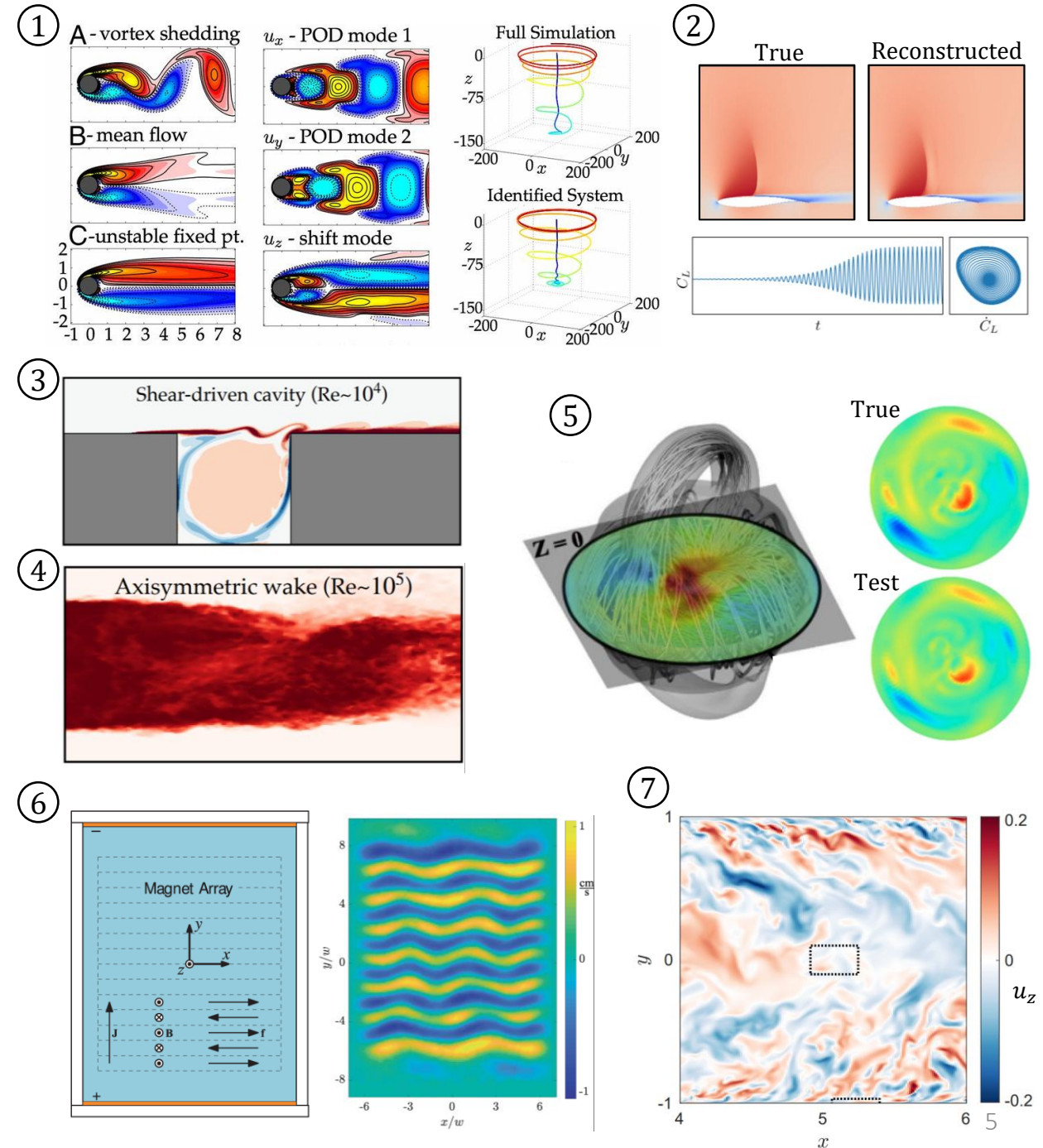
- Measured velocity field, identify PDE: form similar to N-S

- $\partial_t \mathbf{u} = c_1 (\mathbf{u} \cdot \nabla) \mathbf{u} + c_2 \nabla^2 \mathbf{u} + c_3 \mathbf{u} - \rho^{-1} \nabla p + \rho^{-1} \mathbf{f}$

7. Turbulent 3D channel flow ($Re = 1000$) Johns Hopkins database

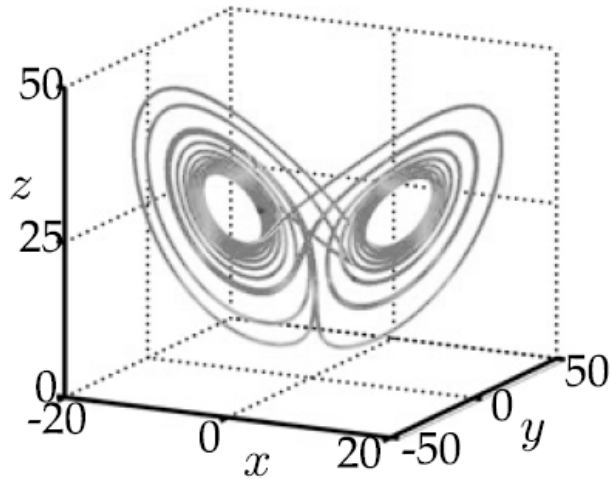
- Identify PDEs: N-S, continuity equation, boundary conditions

- $\partial_t \mathbf{u} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - 0.995 \nabla p + 4.93 \cdot 10^{-5} \nabla^2 \mathbf{u}$

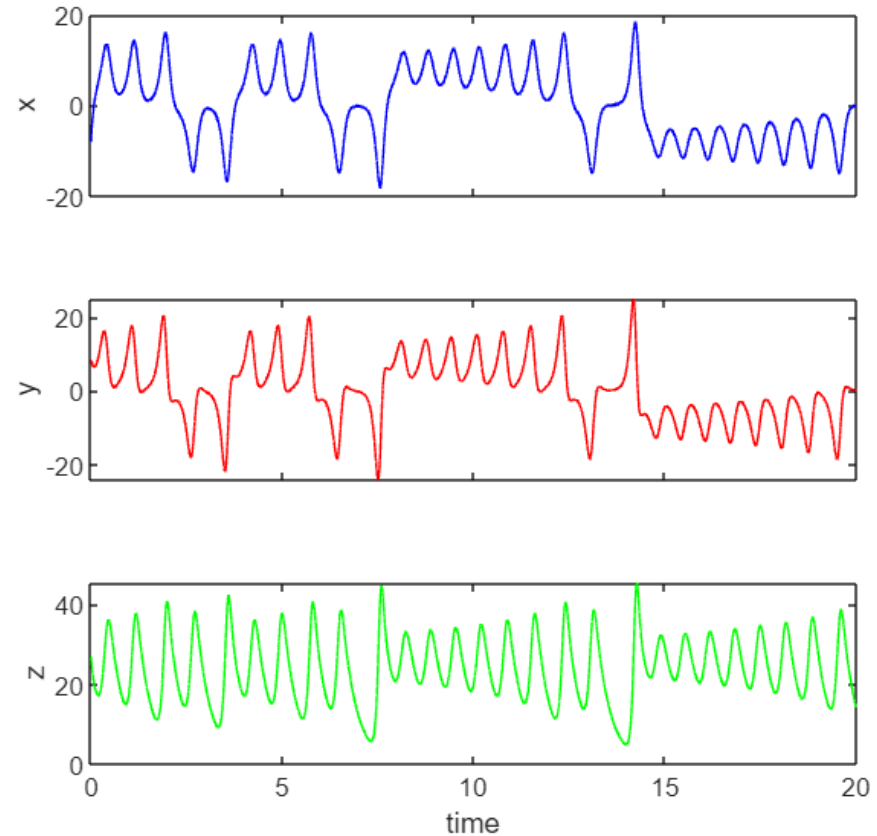


Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z.\end{aligned}$$



Data: time series x, y, z



Additional MATLAB tutorials

Additional Python tutorials

→ <https://github.com/urban-fasel/FiltonWorkshop2024>

→ [https://github.com/urban-fasel/I-X workshop 2025](https://github.com/urban-fasel/I-X_workshop_2025)

Data

→ how much and what quality is needed?

Library

→ how to choose an effective library of candidate terms?

Optimization

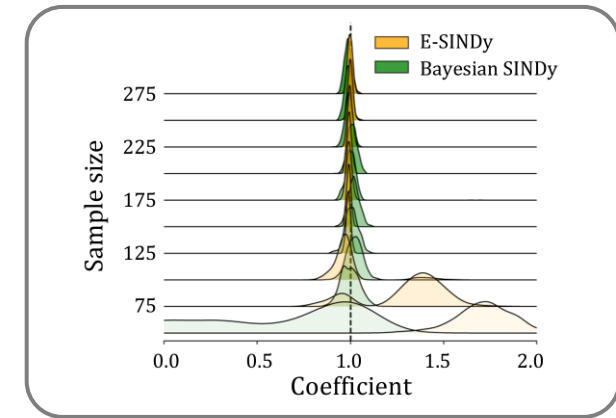
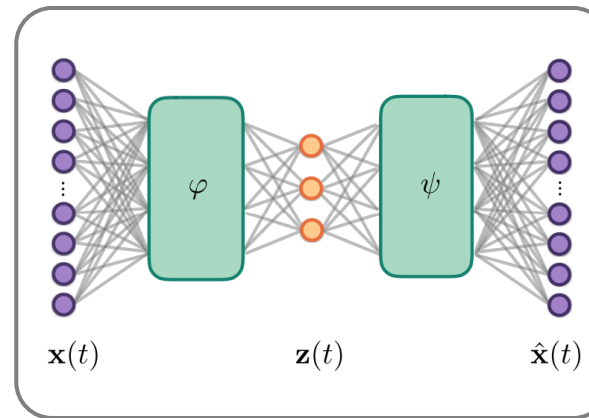
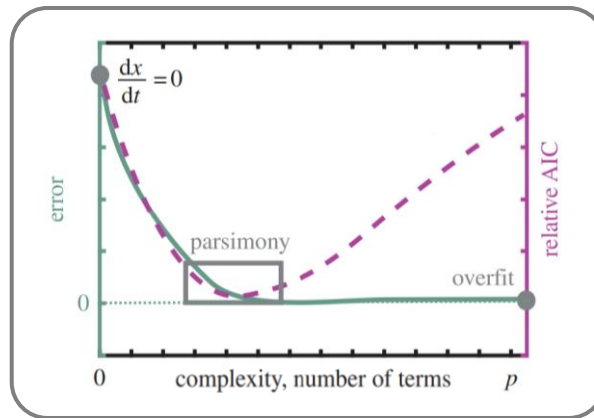
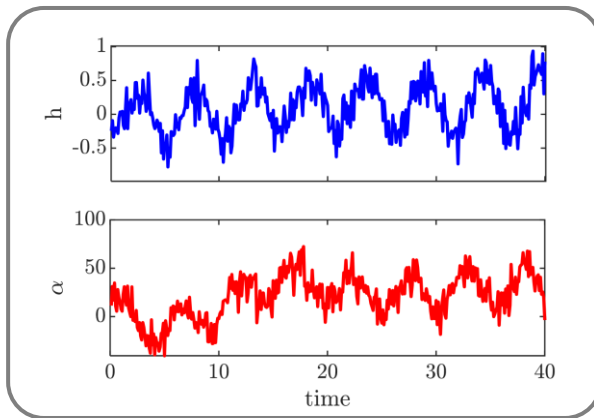
→ what algorithm to use & how to select models?

(UQ

→ how to quantify model coefficient uncertainty?)

(Coordinates

→ do we measure the right variables?)



K Champion, B Lusch, JN Kutz, SL Brunton. Data-driven discovery of coordinates and governing equations, 2019.

H Schaeffer, SG McCalla. Sparse model selection via integral terms, 2017.

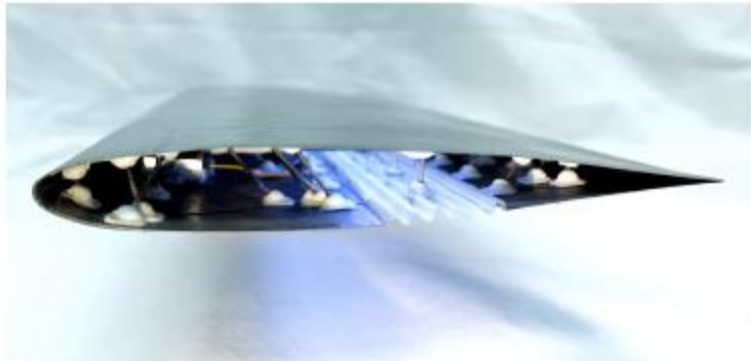
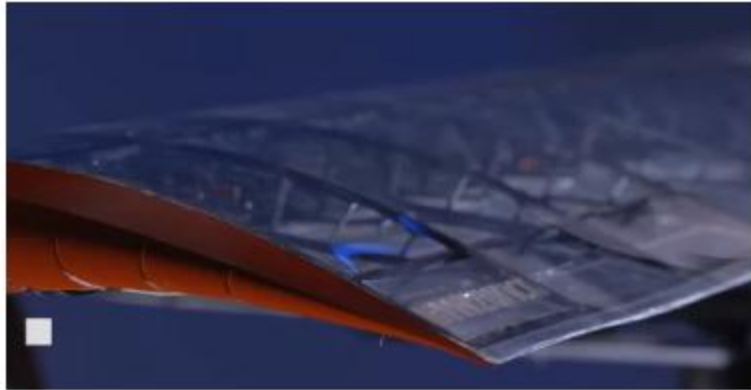
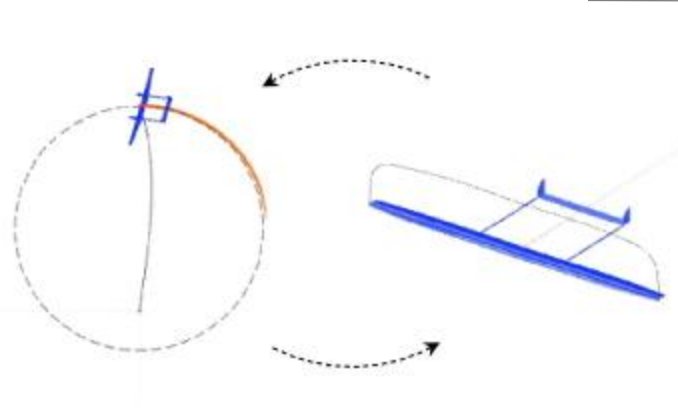
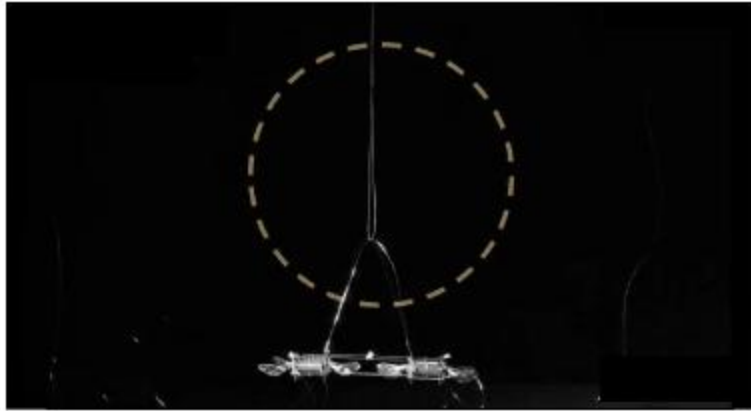
D Messenger, DM Bortz. WEAK SINDy: Galerkin-based data-driven model selection, 2021.

PAK Reinbold, DR Gurevich, RO Grigoriev. Using noisy or incomplete data to discover models of spatiotemporal dynamics, 2020.

NM Mangan, JN Kutz, SL Brunton, JL Proctor. Model selection for dynamical systems via sparse regression and information criteria, 2017.

LM Gao, U Fasel, SL Brunton, JN Kutz. Convergence of uncertainty estimates in Ensemble and Bayesian sparse model discovery, 2023.

Research related to SINDy



(Adaptive) flight systems

1. [Flapping wing MAV](#)
2. [Renewable energy systems](#)
3. [Composite additive manufacturing morphing wing drones](#)
4. [Morphing wings](#)
5. [Airborne wind energy](#)

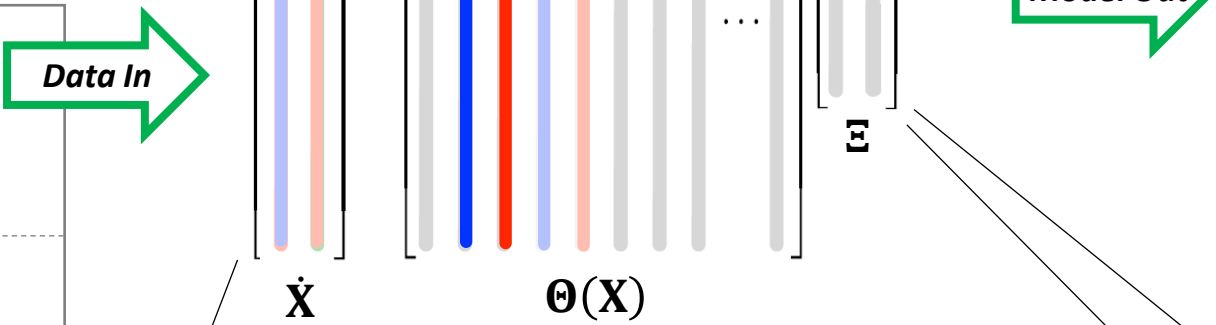
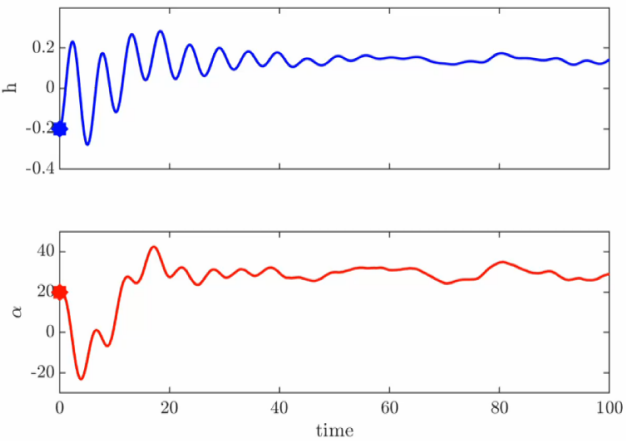
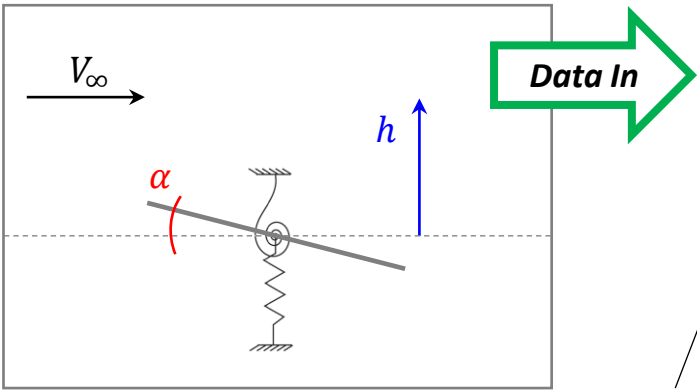
Methods

1. [Co-design optimization](#)
2. **Data driven modeling & control**
 - [DMDc](#), [SINDyC](#), [SINDy-RL](#)
 - [E-SINDy](#), [B-SINDy](#), [SINDy-CP](#)
 - [Poincaré SINDy](#), [Slow Manifolds](#)

SINDy – toy problem in aeroelasticity

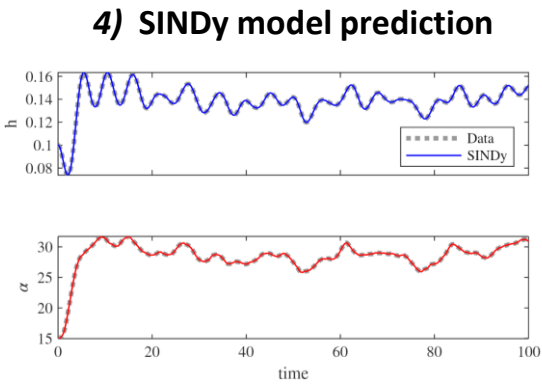
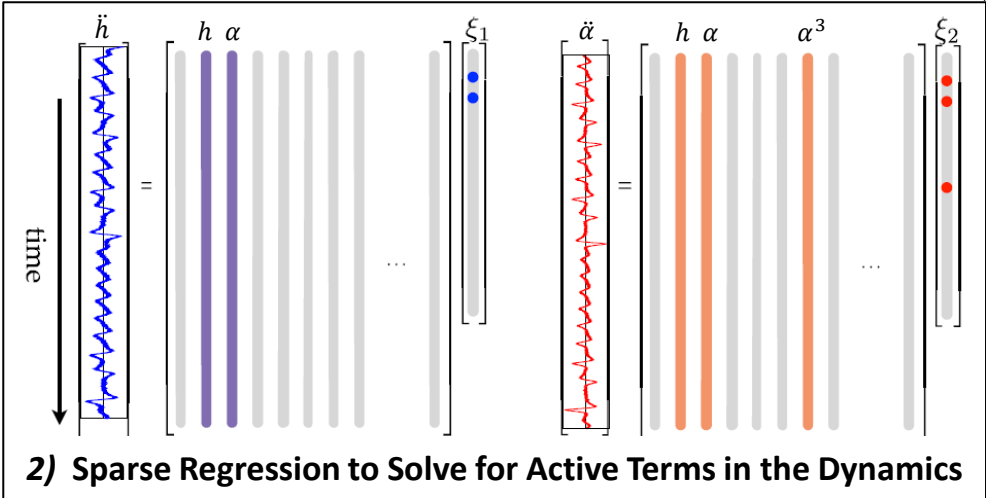
1) 2-DoF Elastically Supported Airfoil

$$\ddot{h} = \frac{1}{m} \left(-k_h h + L(V_\infty^2, \alpha, \dot{h}, \alpha^2, \dots, \dot{h}^3) \right)$$
$$\ddot{\alpha} = \frac{1}{I_p} \left(-k_\alpha \alpha + M(V_\infty^2, \alpha, \dot{h}, \alpha^2, \dots, \dot{h}^3) \right)$$

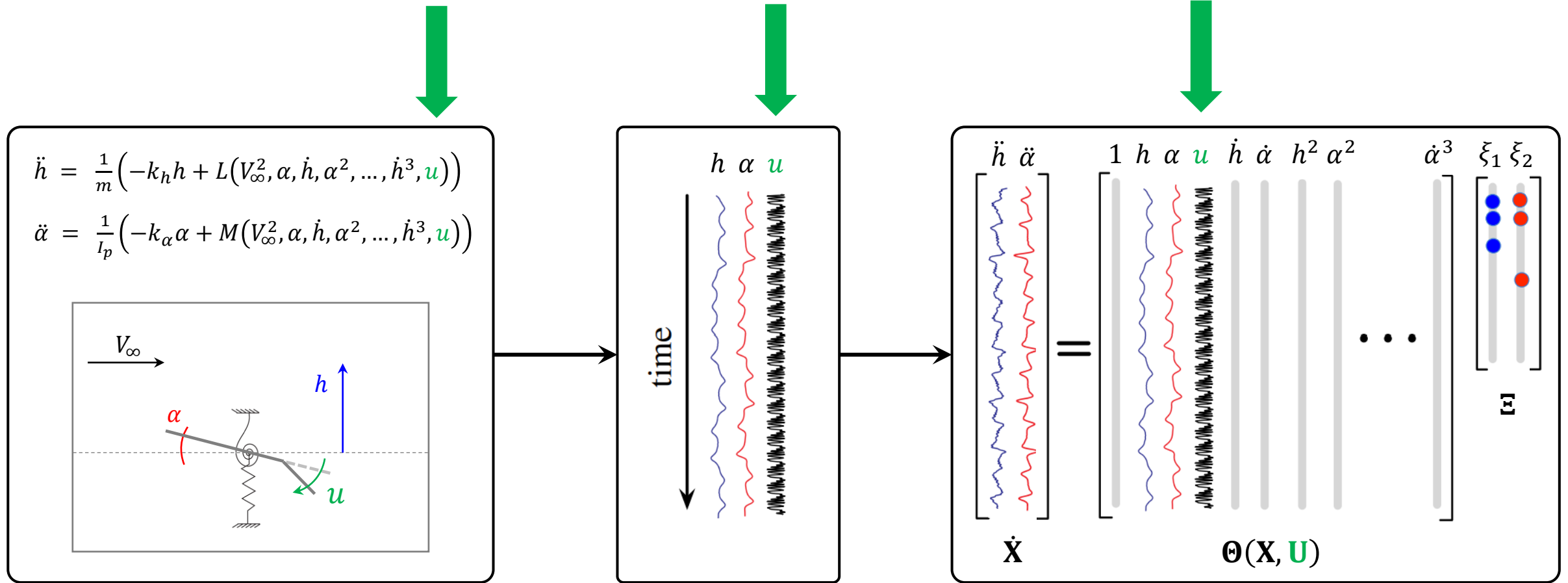


	\dot{h}	$\dot{\alpha}$
'1'	[0]	[0]
' \dot{h} '	[-0.3912]	[-0.9818]
' $\dot{\alpha}$ '	[0]	[0]
'h'	[-1.4402]	[0]
' α '	[0.3927]	[-0.0182]
' $\dot{h}\dot{h}$ '	[0]	[0]
' $\dot{h}\dot{\alpha}$ '	[0]	[0]
...
' $\dot{h}\dot{h}\dot{h}$ '	[0.0652]	[0.1633]
' $\dot{h}\dot{h}\dot{\alpha}$ '	[0]	[0]
' $\dot{h}\dot{\alpha}\dot{h}$ '	[0]	[0]
' $\dot{h}\dot{\alpha}\dot{\alpha}$ '	[-0.1962]	[-0.4913]
...
' $\dot{h}\alpha\alpha$ '	[0.1959]	[0.4914]
...
' $h\dot{h}\alpha$ '	[0]	[0]
' $h\alpha\alpha$ '	[0]	[0]
' $\alpha\alpha\alpha$ '	[-0.0654]	[-0.1639]

3) Sparse Coefficients of Dynamics



SINDy with control

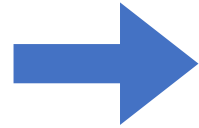


SINDy – Reinforcement Learning: Collaboration University of Washington

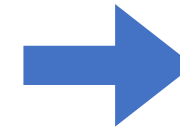
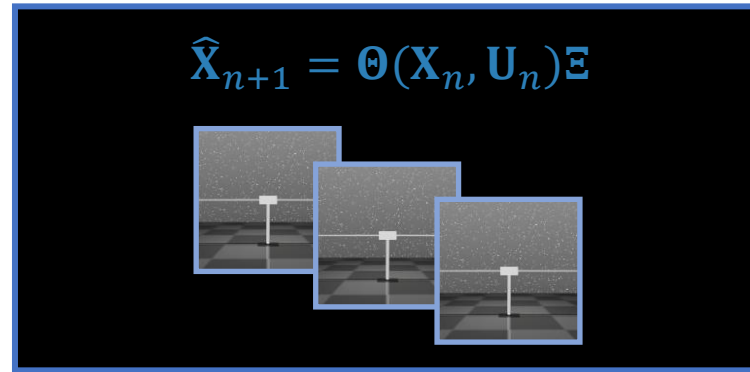
Collect Data in the
“real” environment ϵ



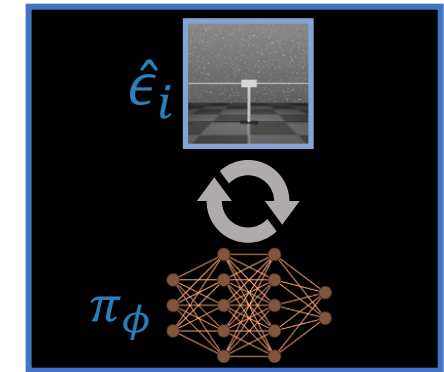
$(\mathbf{X}_n, \mathbf{U}_n), \mathbf{X}_{n+1}$



Identify Ensemble of SINDy Models $\hat{\epsilon}_i$



Train Policy in SINDy $\hat{\epsilon}_i$



SINDy – Reinforcement Learning: Collaboration University of Washington

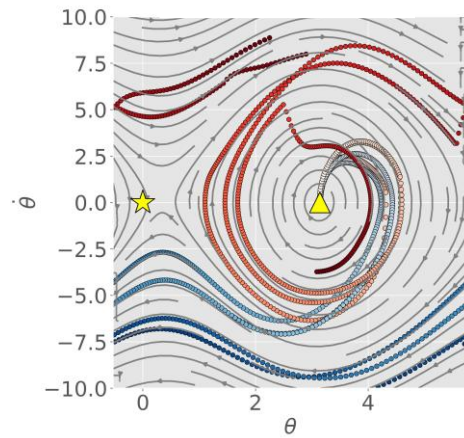
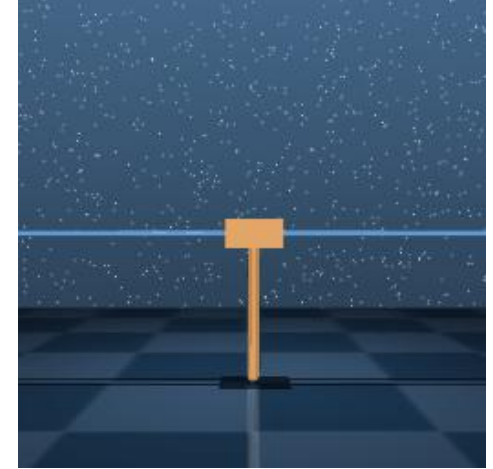
1st Dynamics Fit



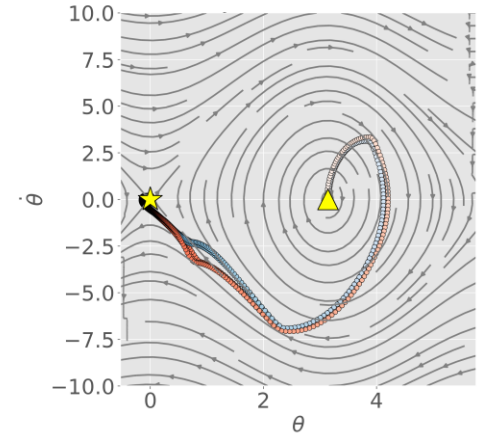
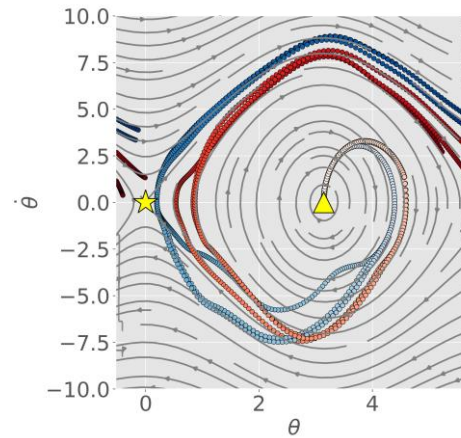
15th Dynamics Fit




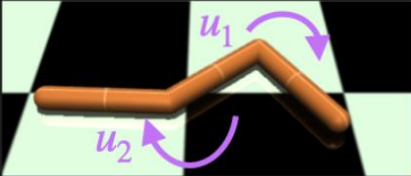
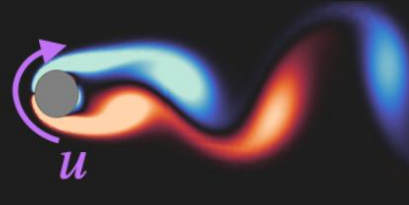
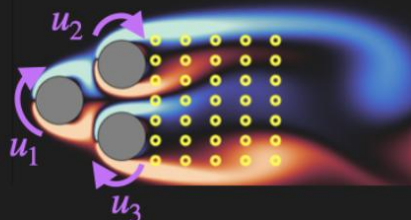
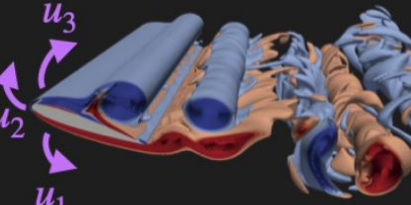
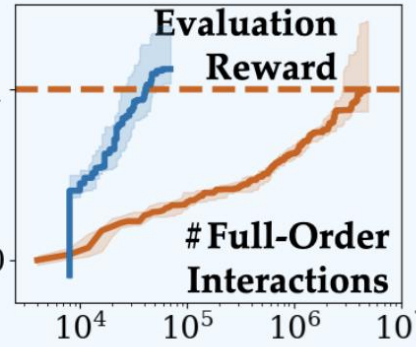
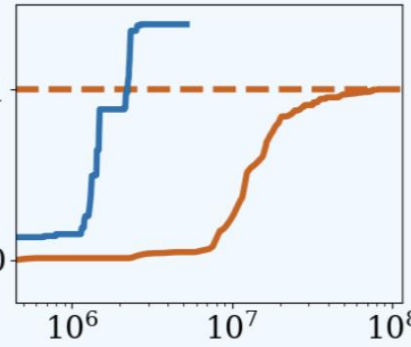
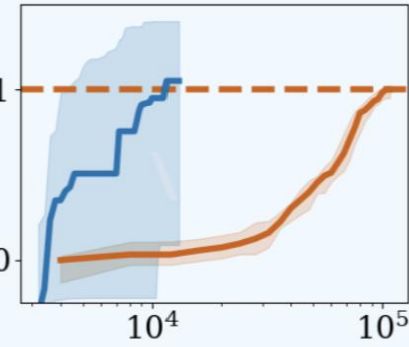
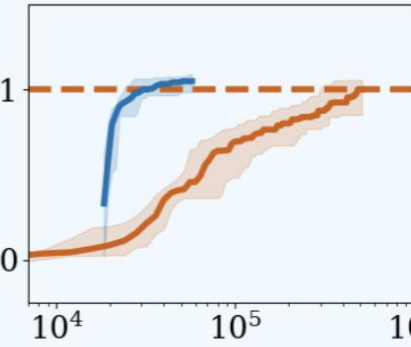
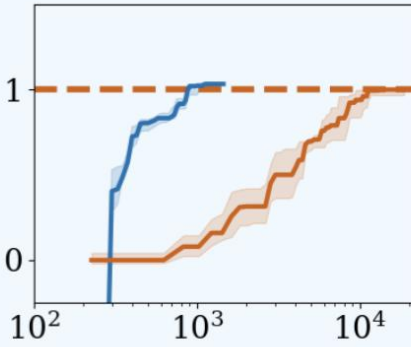
25th Dynamics Fit



● Ground Truth
● SINDy



SINDy – Reinforcement Learning: Collaboration University of Washington

	cartpole swing-up	Swimmer-v4	Cylinder	Pinball	3D Airfoil
Environment					
\mathbf{x}_k	$(x, \cos \theta, \sin \theta, \dot{x}, \dot{\theta})$	$(\theta_1, \theta_2, \theta_3, v_x, v_y, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$	$(C_L, dC_L/dt)$	10-d projection of 70 velocity sensors	2-d projection of 318 velocity sensors
\mathbf{u}_k	Force in x-direction (1D)	Torques to rotors (2D)	Cylinder rotation (1D)	Cylinder rotation (3D)	Jet blowing/suction (3D)
Sample Efficiency					
SINDy-RL — Baseline					

Training speedup → more than 2 orders of magnitude

SINDy with control 2.0 – Learning a Poincare map → Owen Brook

DATA-DRIVEN STABILISATION OF UNSTABLE PERIODIC ORBITS OF THE THREE-BODY PROBLEM

Owen M. Brook¹, Jason J. Bramburger², Davide Amato¹, Urban Fasel¹

¹ Department of Aeronautics, Imperial College London, London, UK

² Department of Mathematics and Statistics, Concordia University, Montréal, QC, Canada

Corresponding author: Owen Brook (owen.brook20@imperial.ac.uk)

Abstract

Many different models of the physical world exhibit chaotic dynamics, from fluid flows and chemical reactions to celestial mechanics. The study of the three-body problem (3BP) and its many different families of unstable periodic orbits (UPOs) have provided fundamental insight into chaotic dynamics as far back as the 19th century. The 3BP, a conservative system, is inherently challenging to sample due to its volume-preservation property. In this paper we present an interpretable data-driven approach for the state-dependent control of UPOs in the 3BP, through leveraging the inherent sensitivity of chaos and the local manifold structure. We overcome the sampling challenge by utilising prior knowledge of UPOs and a novel augmentation strategy. This enables sample-efficient discovery of a verifiable and accurate local Poincaré map in as few as 55 data points. We suggest that the Poincaré map is best discovered at a surface of section where the norm of the monodromy matrix, i.e. the local sensitivity to small perturbations, is the smallest. To stabilise the UPOs, we apply small velocity impulses once each period, determined by solving a convex system of linear matrix inequalities based on the linearised map. We constrain the norm of the decision variables used to solve this system, resulting in locally optimal velocity impulses directed along the local stable manifold. Critically, this behaviour is achieved in a computationally efficient manner. We demonstrate this sample-efficient and low-energy method across several orbit families in the 3BP, with potential applications ranging from robotics and spacecraft control to fluid dynamics.

1 Introduction

Chaotic systems are characterised by their sensitivity to small perturbations, where two trajectories starting arbitrarily close together will diverge exponentially in time. For a long time, this meant that these systems were largely avoided and efforts were generally directed to suppress or evade chaos [1]. However, chaotic

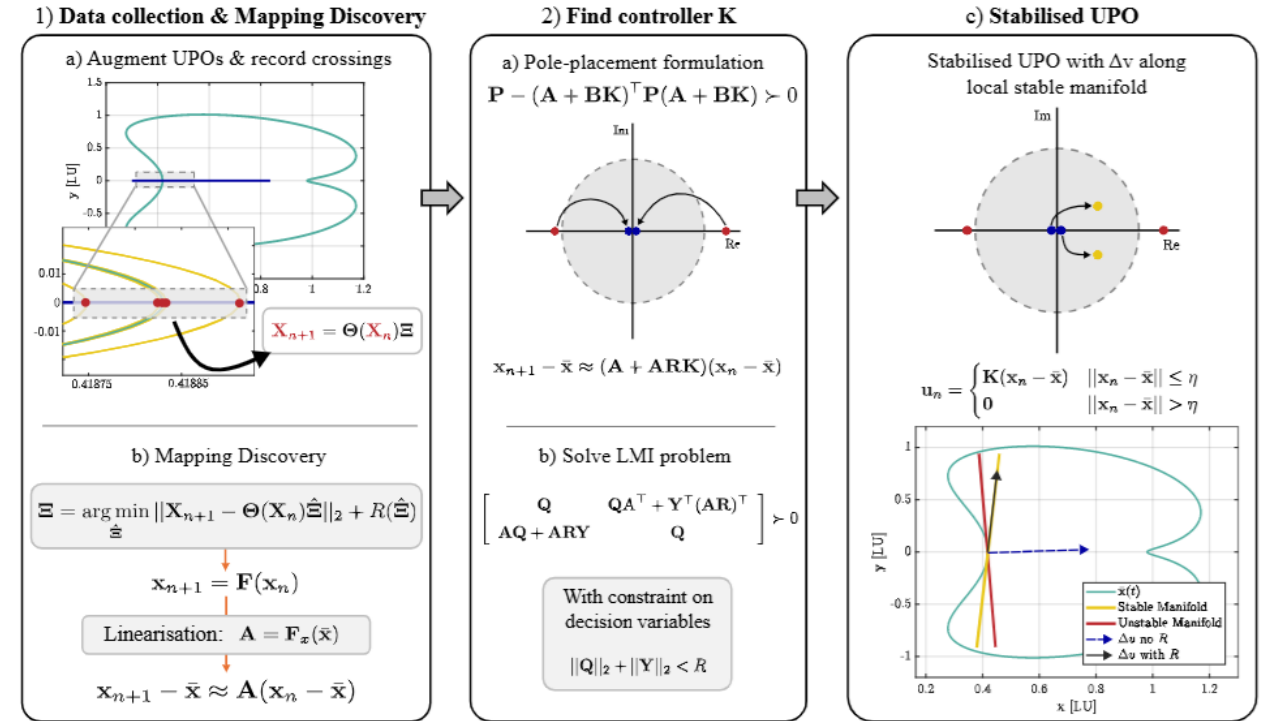


Figure 1: Our method for stabilising a Lyapunov UPO: 1) Initial conditions of known UPOs are selected and small velocity perturbations added. The states (x, \dot{x}, \dot{y}) are recorded at successive crossings of the surface of section $\Sigma = \{(x, \dot{x}, \dot{y}) | y = 0, x < 0.84\}$. A Poincaré map is discovered using SINDy and linearised at the desired UPO \bar{x} . 2) A controller K is found through solving a pole-placement problem using linear matrix inequalities with a constraint on the decision variables. 3) Constraint causes the magnitude of the eigenvalues to increase and aligns the impulse Δv along the local stable manifold, a locally-optimal solution.

arXiv paper: <https://arxiv.org/abs/2507.08630>

Lecture outline

Part 1: SINDy – Sparse Identification of Nonlinear Dynamics

- **Intro:** identifying ODEs
- SINDy – **applications**

Part 2: Coding example

- Matlab / Python **example:** *Lorenz system*
- **PySINDy**

Part 3: SINDy with Control

- My SINDy **research**
- SINDy **Reinforcement Learning**

