Bounding a Generalized Stolarsky mean with Hölder means

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Outline

- Generalized Stolarsky means for two positive numbers
- Bounding generalized Stolarsky means by Hölder means

Given two positive numbers a and b, with $a \leq b$, for any $p \in [-\infty, \infty]$, we define their p-Hölder mean as:

$$H_p(a,b) = \left(\frac{1}{2}a^p + \frac{1}{2}b^p\right)^{1/p}. \tag{1}$$

For $p = 0$, we define:

$$H_0(a,b) = \lim_{p \to 0}\left(\frac{1}{2}a^p + \frac{1}{2}b^p\right)^{1/p} = \sqrt{ab}.$$

For $p = \infty$, we define:

$$H_\infty(a,b) = \lim_{p \to \infty}\left(\frac{1}{2}a^p + \frac{1}{2}b^p\right)^{1/p} = \max\{a,b\} = b,$$

and for $p = -\infty$, we define:

$$H_{-\infty}(a,b) = \lim_{p \to -\infty}\left(\frac{1}{2}a^p + \frac{1}{2}b^p\right)^{1/p} = \min\{a,b\} = a.$$

For $0 < a < b$, we define the n-Stolarsky mean of a and b, as:

$$S_n(a,b) = \left(\frac{b^n - a^n}{n(b-a)}\right)^{1/(n-1)}$$

if $n \notin \{0,1\}$.
For $n = 0$, by a limit argument, we can define:

$$S_0(a,b) = \frac{b-a}{\ln(b) - \ln(a)}$$

For $n = 1$, again by a limit argument, we define:

$$S_1(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$$

For $n = \pm\infty$, an argument similar to the one for $H_{\pm\infty}$, we define $S_\infty = \max\{a,b\} = b$ and $S_{-\infty} = \min\{a,b\} = a$.

$H_p(a, b) = E\left([X^p]\right)^{1/p}$, where X is a discrete random variable taking the values $X = a$ and $X = b$, each with probability $1/2$.

$S_n(a, b) = E\left(\left[Y^{n-1}\right]\right)^{1/(n-1)}$, where Y is a continuous random variable uniformly distributed over the interval $[a, b]$.

Question: How can we move in a "continuous" way from the Hölder to the Stolarsky means?

We define the $n-$ generalized Stolarsky mean of a and b as:

$$G_n(a, b) \quad = \quad E\left[Z^{n-1}\right]^{1/n}. \tag{2}$$

That means:

$$G_n(a,b) = \left(\frac{b^n - (b-\lambda)^n + (a+\lambda)^n - a^n}{2\lambda n} \right)^{1/(n-1)}$$

if $n \notin \{0,1\}$.
Another set of limit arguments allows us to define this mean for $n = 0$:

$$G_0(a,b) = \frac{2\lambda}{\ln(b) - \ln(b-\lambda) + \ln(a+\lambda) - \ln(a)}$$

while for $n = 1$:

$$G_1(a,b) = \frac{1}{e} \left(\frac{b^b (a+\lambda)^{a+\lambda}}{(b-\lambda)^{b-\lambda} a^a} \right)^{1/(b-a)}$$

For $n = \pm\infty$, unsurprisingly, we will define $G_\infty = \max\{a,b\} = b$ and $G_{-\infty} = \min\{a,b\} = a$.

Question: Given a number $n \in [-\infty, \infty]$, what are the greatest $p(n)$ and the least $q(n)$ in $[-\infty, \infty]$ such that, for all a and b positive numbers, we have:

$$H_{p(n)}(a,b) \leq S_n(a,b) \leq H_{q(n)}(a,b)? \tag{3}$$

For $n = 0$, $S_0(a, b)$ becomes the logarithmic mean of a and b, and the answer was given for the first time by T.P. Lin in 1974, and later on by C.O. Imoru in 1982.

For $n = 0$, the answer is $p(0) = 0$ and $q(0) = 1/3$, that means, for all $0 < a < b$, we have:

$$\sqrt{ab} \leq \frac{b-a}{\ln(b) - \ln(a)} \leq \left(\frac{a^{1/3} + b^{1/3}}{2}\right)^3. \tag{4}$$

Our General Question: Given $\lambda \in [0, (b-a)/2]$, $n \in [-\infty, \infty]$, what are the greatest $p(n)$ and the least $q(n)$ in $[-\infty, \infty]$ such that, for all a and b positive numbers, we have:

$$H_{p(n)}(a,b) \leq G_n(a,b) \leq H_{q(n)}(a,b)? \tag{5}$$

Particular Question: For $\lambda = (b-a)/3$ and $n = 0$, what are the greatest $p(n)$ and least $q(n)$ in $[-\infty, \infty]$ such that, for all a and b positive numbers, we have:

$$H_p(a,b) \leq \frac{2}{3} \cdot \frac{b-a}{\ln(b) - \ln((a+2b)/3) + \ln((2a+b)/3) - \ln(a)} \leq H_q(a,b)?$$

We can divide all sides of the last inequality by a, and rewrite the inequality as:

$$H_p(1,x) \leq \frac{2}{3} \cdot \frac{x-1}{\ln(x) - \ln(2x+1) + \ln(x+2)} \leq H_q(1,x),$$

for all $x \geq 1$.

We can prove that, for all $x \geq 1$, we have:

$$H_0(1, x) = \sqrt{1 \cdot x} \quad \leq \quad \frac{2}{3} \cdot \frac{x - 1}{\ln(x) - \ln(2x + 1) + \ln(x + 2)}.$$

We can also prove, that for all $x \geq 1$, we have:

$$\frac{3}{2} \left[\ln(x) - \ln(2x + 1) + \ln(x + 2)\right] \quad \geq \quad \ln(x), \tag{6}$$

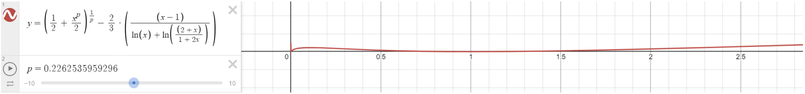
and so, we have:

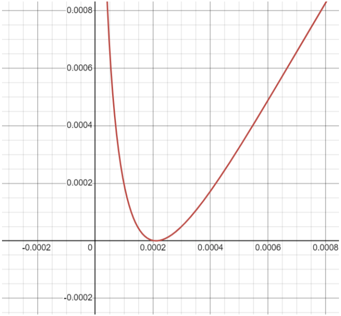
$$\frac{2}{3} \cdot \frac{x - 1}{\ln(x) - \ln(2x + 1) + \ln(x + 2)} \quad \leq \quad \frac{x - 1}{\ln(x) - \ln(1)}$$

$$\text{by Lin inequality} \quad \leq \quad H_{1/3}(1, x).$$

$p = 0$ is optimal, but $q = 1/3$ is not optimal.

Graphical Analysis





$$p = 0.2262535959296\ldots$$

main.py > ...

```python
from scipy.optimize import minimize_scalar
import numpy as np

p = 0

def func(x):
    return pow((1/2+pow(x,p)/2),1/p)-2*((x-1)/(np.log(x)+np.log((2+x)/(1+2*x))))/3

for i in range(1,17):
    p = p + 9*pow(1/10,i)
    while minimize_scalar(func,bounds=(0,0.1),method="bounded").fun > 0:
        p = p - pow(1/10,i)
    print("Decimal #",i,": ", p)

print("Final result:", p)
```

```
@urbandrei ➜/workspaces/HolderStolarsky (main) $ python main.py
Decimal # 1 :  0.20000000000000015
Decimal # 2 :  0.22000000000000008
Decimal # 3 :  0.2260000000000001
Decimal # 4 :  0.22620000000000018
Decimal # 5 :  0.22625000000000015
Decimal # 6 :  0.22625300000000015
Decimal # 7 :  0.22625350000000014
Decimal # 8 :  0.22625359000000014
Decimal # 9 :  0.22625359200000014
Decimal # 10 :  0.2262535926000001
Decimal # 11 :  0.2262535926500001
Decimal # 12 :  0.22625359265000006
Decimal # 13 :  0.22625359265030004
Decimal # 14 :  0.22625359265039005
Decimal # 15 :  0.22625359265039405
Decimal # 16 :  0.22625359265039416
Final result: 0.22625359265039416
@urbandrei ➜/workspaces/HolderStolarsky (main) $
```

$$p = 0.22625359265039416\ldots$$

There is a minimum around $x = 0.02$ we are trying to make identical equal to 0.

To identify the value of p we tried, unsuccessfully for the time being:

- Using differentiation and generating a system of equations.
- Expansion using power series.
- Reverse-search graphical/software result.

For further investigation, I think there might be a way to simplify the system of equations.

THANK YOU.

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We can imagine that we have a rod, with one end at the point a, and the other end at the point b, that is equally charged only at its endpoints with a total charge of 1. This corresponds to the Hölder means.

Then, the charge is dissipating, in a symmetric way, from the endpoints of the road toward the middle point $\frac{a+b}{2}$, until it occupies the whole road in a uniform way. This corresponds to the Stolarsky means.

If the rate of the propagation of the charge is 1, then at time $t = \lambda$, with $\lambda \in \left[0, \dfrac{b-a}{2}\right]$, the subintervals $[a, a+\lambda]$ and $[b, b-\lambda]$ are uniformly charged, while the interval $(a+\lambda, b-\lambda)$ is uncharged. This corresponds to a continuous random variable Z given by the density function:

$$f(x) = \begin{cases} \frac{1}{2\lambda} & \text{if} \quad x \in [a, a+\lambda] \cup [b-\lambda, b] \\ 0 & \text{otherwise} \end{cases}$$

In particular, for $\lambda := \dfrac{b-a}{2}$, and $n = 0$, we have

the logarithmic mean:

$$S_0(a,b) \quad = \quad \frac{b-a}{\ln(b) - \ln(a)}$$

In particular, for $\lambda := \dfrac{b-a}{3}$, and $n = 0$, we have:

$$G_0(a,b) \quad = \quad \frac{2}{3} \cdot \frac{b-a}{\ln(b) - \ln((a+2b)/3) + \ln((2a+b)/3) - \ln(a)}$$