

Lecture Notes on Holographic Renormalization

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Abstract

We review the formalism of holographic renormalization. We start by discussing mathematical results on asymptotically anti-de Sitter spacetimes. We then outline the general method of holographic renormalization. The method is illustrated by working all details in a simple example: a massive scalar field on anti-de Sitter spacetime. The discussion includes the derivation of the on-shell renormalized action, of holographic Ward identities, anomalies and RG equations, and the computation of renormalized one-, two- and four-point functions. We then discuss the application of the method to holographic RG flows. We also show that the results of the near-boundary analysis of asymptotically AdS spacetimes can be analytically continued to apply to asymptotically de Sitter spacetimes. In particular, it is shown that the Brown-York stress energy tensor of de Sitter spacetime is equal, up to a dimension dependent sign, to the Brown-York stress energy tensor of an associated AdS spacetime.

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1 Introduction

The AdS/CFT correspondence [1, 2, 3, 4, 5] offers the best understood example of a gravity/gauge theory duality. According to this duality string theory on an asymptotically anti-de Sitter spacetime (AAdS) times a compact manifold M is exactly equivalent to the quantum field theory (QFT) “living” on the boundary of AAdS. This is a strong/weak coupling duality: the strong coupling regime of the quantum field theory corresponds to the weak coupling regime of the string theory and vice versa. The exact equivalence between the two formulations means that, at least in principle, one can obtain complete information on one side of the duality by performing computations on the other side. In these lecture notes we discuss how to obtain renormalized QFT correlation functions by performing computations on the gravity side of the correspondence.

Despite much effort, string theory on AdS spacetimes is still poorly understood. At low energies, however, the theory is well approximated by supergravity. The relevant description is in terms of a $d+1$ dimensional supergravity theory with an AAdS ground state coupled to the Kaluza-Klein (KK) modes that result from the reduction of the $10d$ (or $11d$) supergravity on the compact manifold M . On the gauge theory side the low energy limit corresponds to considering the large 't Hooft limit, $\lambda = g_{YM}^2 N \gg 1$. Suppressing loop contributions on the gravitational side corresponds to considering the *QFT* in the large N limit. In this limit the gravitational computations involve classical solutions of the supergravity theory coupled to the infinite set of the KK modes. In some cases one can further simplify things by consistently truncate the KK modes. In these cases it is sufficient to work with a lower dimensional gauged supergravity. The existence of a consistent truncation implies that there is a subset of gauge invariant operators on the gauge theory side which are closed under OPE's. To compute correlation functions of these operators it is sufficient to study classical solutions of the $d+1$ dimensional gauged supergravity.

In quantum field theory, correlation functions suffer from UV divergences, and one needs to renormalize the theory to make sense of them. A general phenomenon of the gravity/gauge theory correspondence is the so-called UV/IR connection [6], i.e. UV divergences in the field theory are related to IR divergences on the gravitational side, and vice versa. On the gravitational side, long distance (IR) is the same as near

the boundary. The purpose of these lecture notes is explain how to deal with these IR divergences, i.e. how to “holographically renormalize”.

In the QFT the cancellation of the UV divergences does not depend on the IR physics. This implies that the holographic renormalization should only depend on the near-boundary analysis. Furthermore, in QFT a major role in the renormalization program is played by symmetries and the corresponding Ward identities. If the UV subtractions respect some symmetry then the corresponding Ward identity holds, otherwise it is anomalous. The corresponding statement on the gravitational side is that the near-boundary analysis that determines the IR divergences should be sufficient to establish the holographic Ward identities and anomalies. On the other hand, correlation functions capture the dynamics of the theory so the near-boundary analysis should not be sufficient to determine them. To determine correlation functions one needs exact solutions of the bulk field equations. The subtractions necessary to render the correlations functions finite should be consistent with each other. This is guaranteed if they are made by means of counterterms. In early studies of correlation functions in AdS/CFT the divergences were simply dropped from correlators. In the holographic renormalization program we shall implement all subtractions by means of covariant counterterms.

The regularization and renormalization method discussed in these notes was first introduced in [7] and it was promoted to a systematic method in [8]. It was applied to RG flows in [9, 10]. Counterterms for AdS gravity were also introduced in [11], see also [12, 13, 14, 15]. The holographic renormalization method is described in detail in [10]. The discussion in these notes should be viewed as complementary to the discussion there. In general we refrain from discussing in detail material that are sufficiently discussed elsewhere. The emphasis here is in the general features of the method illustrated by the simplest possible example.

These lecture notes are organized as follows. In the next section we state the main result, namely the reformulation of the computation of correlation function in terms of renormalized 1-point functions in the presence of sources. In section 3 we introduce asymptotically AdS spacetimes and we discuss in detail the results of Fefferman and Graham [16] on hyperbolic manifolds. In section 4 we present the method of holographic renormalization. In particular, we discuss how to obtain

asymptotic solutions of the field equations, regularize and renormalize the on-shell action, compute exact 1-point functions, derive Ward identities, compute correlation functions and derive RG equations. All of these are illustrated by a simple example, a massive scalar in AdS, in section 5. In section 6 we discuss the application of the method to holographic RG flows. We conclude by discussing open problems and future directions.

In the appendix we show that all *local* results derived via the near-boundary analysis can be straightforwardly “analytically continued” to asymptotically dS spacetimes. In particular, the Brown-York stress energy tensor [17] associated with an asymptotically dS spacetime [18] is always equal, up to a sign, to the stress energy of the AdS space related to the dS space by a specific analytic continuation. The sign is plus in $D = 4k + 3$ (bulk) dimensions and minus in $D = 4k + 1$ dimensions. In particular, in three dimensions the two stress energy tensors are the same and in five dimensions they differ by a sign.

2 Statement of results

According to the gravity/gauge theory correspondence, for every bulk field Φ there is a corresponding gauge invariant boundary operator O_Φ . In particular, bulk gauge fields correspond to boundary symmetry currents. As we will review later, AAdS spaces have a boundary at spatial infinity, and one needs to impose appropriate boundary conditions there. The partition function of the bulk theory is then a functional of the fields parametrizing the boundary values of the bulk fields. According to the prescription proposed in [2, 3], the boundary values of the fields are identified with sources that couple to the dual operator, and the on-shell bulk partition function with the generating functional of QFT correlation functions,

$$Z_{\text{SUGRA}}[\phi_{(0)}] = \int_{\Phi \sim \phi_{(0)}} D\Phi \exp(-S[\Phi]) = \langle \exp \left(- \int_{\partial AAdS} \phi_{(0)} O \right) \rangle_{QFT}. \quad (2.1)$$

where the expectation value on the right hand side is over the QFT path integral, and $\partial AAdS$ denotes the boundary of the asymptotically AdS space. We will work exclusively in the leading saddle point approximation where this relation becomes,

$$S_{\text{onshell}}[\phi_{(0)}] = -W_{QFT}[\phi_{(0)}] \quad (2.2)$$

where $S_{\text{onshell}}[\phi_{(0)}]$ is the on-shell supergravity action and $W_{\text{QFT}}[\phi_{(0)}]$ is the generating function of QFT connected graphs. Correlation functions of the operator O are now computed by functional differentiation with respect to the source,

$$\begin{aligned}\langle O(x) \rangle &= \left. \frac{\delta S_{\text{onshell}}}{\delta \phi_{(0)}(x)} \right|_{\phi_{(0)}=0} \\ \langle O(x)O(x_2) \rangle &= \left. -\frac{\delta^2 S_{\text{onshell}}}{\delta \phi_{(0)}(x)\delta \phi_{(0)}(x_2)} \right|_{\phi_{(0)}=0} \\ \langle O(x_1)\cdots O(x_n) \rangle &= (-1)^{n+1} \left. \frac{\delta^3 S_{\text{onshell}}}{\delta \phi_{(0)}(x_1)\cdots \delta \phi_{(0)}(x_n)} \right|_{\phi_{(0)}=0}\end{aligned}\quad (2.3)$$

etc.

As we mentioned, QFT correlation functions diverge and the right hand side of (2.2) is not well-defined without renormalization. Similarly, the left hand side is also divergent due to the infinite volume of the spacetime and one needs to appropriately renormalize. In section 4 we will lay out the renormalization method but we start by stating the final result.

Given a classical action $S[\Phi, A_\mu, G_{\mu\nu}, \dots]$ that depends on a number of fields $\Phi, A_\mu, G_{\mu\nu}$, etc there exist *exact* renormalized 1-point functions, one for each bulk field,

$$\begin{aligned}\Phi &\rightarrow \langle O(x) \rangle_s = \frac{1}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{\text{ren}}}{\delta \phi_{(0)}(x)} \sim \phi_{(2\Delta-d)}(x) \\ A_\mu &\rightarrow \langle J_i(x) \rangle_s = \frac{1}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{\text{ren}}}{\delta A_{i(0)}(x)} \sim A_{mi}(x) \\ G_{\mu\nu} &\rightarrow \langle T_{ij}(x) \rangle_s = \frac{2}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{\text{ren}}}{\delta A_{i(0)}(x)} \sim g_{(d)ij}(x)\end{aligned}\quad (2.4)$$

where S_{ren} is the renormalized on-shell action (to be discussed shortly), J_i is the boundary symmetry current that couples to the bulk gauge field A_μ , T_{ij} is the boundary stress energy tensor that couples to the boundary metric $g_{(0)ij}$ etc. The fields on the right hand side, $\phi_{(2\Delta-d)}, A_{mi}, g_{(d)ij}$ appear in the asymptotic expansion of the solutions of the bulk field equation. The exact definition is given in the section 4. The asymptotic analysis of the field equations does not determine these coefficient but given an exact solution of the field equations it is straightforward to extract $\phi_{(2\Delta-d)}, A_{mi}, g_{(d)ij}$. These functions are in general non-local functions of the sources $\phi_{(0)}, A_{0i}, g_{(0)ij}$. Notice that the relations (2.4) hold for *any* solution of the bulk field

equations, and as such it is a property of the theory rather than of each specific solution.

The subscript s in (2.4) denotes that the expectation values are in the presence of sources. This means that if we want to compute higher point functions we only need to differentiate (2.4) and then set the sources to zero,

$$\langle O(x_1) \cdots O(x_n) \rangle \sim \frac{\delta \phi_{(2\Delta-d)}(x_1)}{\delta \phi_{(0)}(x_2) \cdots \delta \phi_{(0)}(x_n)} \Big|_{\phi_{(0)}=0} \quad (2.5)$$

This relation replaces the relations (2.3). If one knew $\phi_{(2\Delta-d)}$ exactly as a function of sources, one would have solved the theory since using (2.5) one can determine all n -point functions. We will see later how to determine $\phi_{(2\Delta-d)}$ using (bulk) perturbation theory.

The exact one-point functions allow one to establish the holographic Ward identities in full generality. For example, in the case where the only source turned on is the boundary metric, i.e. we only consider the Einstein-Hilbert term in the bulk action, one can show that the bulk field equations imply the correct diffeomorphism and conformal Ward identities [8],

$$\begin{aligned} \nabla^i \langle T_{ij} \rangle_s &= 0 \\ \langle T_i^i \rangle_s &= \mathcal{A} \end{aligned} \quad (2.6)$$

where \mathcal{A} is the holographic Weyl anomaly [7].

3 Asymptotically anti-de Sitter spacetimes

In this section we discuss in detail asymptotically AdS spacetimes. Recall that the AdS spacetime is a maximally symmetric solution of Einstein's equations with negative cosmological constant,

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = \Lambda G_{\mu\nu} \quad (3.1)$$

The AdS space is conformally flat. This implies that the Weyl tensor vanishes,

$$W_{\mu\nu\kappa\lambda} = 0. \quad (3.2)$$

This equation combined with Einstein equations implies that the curvature tensor of the AdS_{d+1} spacetime is given by²

$$R_{\mu\nu\kappa\lambda} = \frac{1}{l^2}(G_{\kappa\mu}G_{\nu\lambda} - G_{\mu\lambda}G_{\nu\kappa}) \quad (3.3)$$

where l^2 is the AdS radius ($\Lambda = -d(d-1)/2l^2$).

The metric (in convenient coordinates) is given by

$$ds^2 = \frac{l^2}{\cos^2 \theta}(-dt^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2) \quad (3.4)$$

where $0 \leq \theta < \pi/2$. Notice that the metric has a second order pole at $\theta = \pi/2$. This is where the boundary of AdS is located. Because of the second order pole, the bulk metric does not yield a metric at the boundary. It yields a conformal structure instead. Let us consider a function $r(x)$ that is positive in the interior of AdS, but has a first order pole at the boundary. Such a function is called a “defining function”. We now multiply the AdS metric by r^2 and evaluate it at the boundary,

$$g_{(0)} = r^2 G|_{\pi/2} \quad (3.5)$$

For instance, one could choose $r = \cos \theta$. This metric is finite but it is only defined up to conformal transformations. Indeed, if r is a good defining function, then so is re^w , where w is a function with no zeros or poles at the boundary. So the AdS metric yields a conformal structure at the boundary, i.e. a metric up to conformal transformations.

We will now define asymptotically AdS spaces by generalizing the above considerations. First, let us define *conformally compact manifolds* following [19]. Let X be the interior of a manifold-with-boundary \bar{X} , and let $M = \partial X$ be its boundary. We will call a metric G conformally compact if it has a second order pole at M but there exists a defining function (i.e. $r(M) = 0$, $dr(M) \neq 0$ and $r(X) > 0$) such that

$$g = r^2 G \quad (3.6)$$

smoothly extends to \bar{X} , $g|_M = g_{(0)}$, and is non-degenerate. As in the case of AdS space, this procedure defines a conformal structure on M . There is another quantity

²Our curvature convention is $R_{\mu\nu\kappa}{}^\lambda = \partial_\mu \Gamma_{\nu\kappa}^\lambda + \Gamma_{\mu\rho}^\lambda \Gamma_{\nu\kappa}^\rho - \mu \leftrightarrow \nu$, $R_{\mu\nu} = R_{\mu\lambda\nu}{}^\lambda$, $R = G^{\mu\nu} R_{\mu\nu}$.

that smoothly extends to \bar{X} and its restriction to M is independent of choices (i.e. it depends only on the conformal structure),

$$|dr|_g^2 = g^{\mu\nu} \partial_\mu r \partial_\nu r \quad (3.7)$$

This can be shown by using the definition (3.6).

One can calculate the curvature of the bulk metric, G ,

$$R_{\kappa\lambda\mu\nu}[G] = |dr|_g^2 (G_{\kappa\mu} G_{\nu\lambda} - G_{\mu\lambda} G_{\nu\kappa}) + \mathcal{O}(r^{-3}) \quad (3.8)$$

Notice that the leading term is of order r^{-4} . So conformally compact manifolds have a curvature tensor that near to the boundary (i.e. $r = 0$) looks like the curvature tensor of AdS space (3.3). Notice that up to this point we did not impose that the metric G is Einstein, i.e. satisfies (3.1). A short computation shows that Einstein's equations imply,

$$|dr|_g^2|_M = \frac{1}{l^2} \quad (3.9)$$

In this case the Riemann tensor near the boundary is exactly the same as that of AdS space. We are thus lead to the following definition

Definition: *An Asymptotically AdS metric is a conformally compact Einstein metric.*

We set $l = 1$ from now on. Notice that this definition does not impose any condition on the topology of the boundary.

A question of interest for us is whether given a conformal structure at infinity one can determine an Einstein bulk metric with the prescribed boundary conditions. This has been answered in [16] (see [20] for a review). The first step is to prove the following theorem

Theorem: There is always a preferred defining function such that

$$|dr|_g^2 = 1 \quad (3.10)$$

in a neighborhood of the boundary M .

The idea of the proof is as follow. Let r_0 be a defining function such that (3.10) does not hold, and consider another defining function $e^w r_0$. Then equation (3.10) becomes a differential equation for w that always has a solution in the neighborhood of M (recall that $|dr|_g^2|_M = 1$).

We now consider Gaussian coordinates emanating from the boundary. We take the inward (radial) coordinate to be the affine parameter of the geodesics with tangent

∇r . Clearly these are good coordinates as long as we do not meet any caustics. In particular, we can take the defining function as the radial coordinate. Then the bulk metric in the neighborhood of the boundary takes the form,

$$ds^2 = \frac{1}{r^2}(dr^2 + g_{ij}(x, r)dx^i dx^j) \quad (3.11)$$

By construction, $g_{ij}(x, r)$ has a smooth limit as $r \rightarrow 0$, so it can be written as

$$g_{ij}(x, r) = g_{(0)ij} + r g_{(1)ij} + r^2 g_{(2)ij} + \dots \quad (3.12)$$

One may now determine the coefficients $g_{(k)ij}$, $k > 0$ from Einstein's equations. Explicit computation shows that in pure gravity all coefficients multiplying odd powers of r vanish up to the order r^d . To simplify the computation of the even coefficients we introduce the new coordinate $\rho = r^2$ [7]. This is the coordinate used throughout this paper. In these coordinates the metric is given by

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j, \\ g(x, \rho) = g_{(0)} + \dots + \rho^{d/2} g_{(d)} + h_{(d)} \rho^{d/2} \log \rho + \dots \quad (3.13)$$

The analysis of the Einstein equations is exactly analogous to the analysis of the scalar field equation that we will discuss in detail in section 5. Details can be found in [8]. Here we briefly summarize the main points. Einstein's equations can be solved order by order in the ρ variable. The resulting equations are *algebraic* so the solution is insensitive to the sign of the cosmological constant and the signature of spacetime. In appendix A we use this fact to derive the corresponding asymptotic expansion for asymptotically de Sitter spacetimes. However, when the cosmological constant is equal to zero the corresponding equations are differential [21] and impose restrictions on $g_{(0)}$ as well. This means that, in general, the various coefficients in the asymptotic expansion of the metric that contribute to divergences in the on-shell actions are non-local with respect to each other. This implies that in the case of asymptotically flat spacetimes there is no universal set of local counterterms that can remove the divergences from the on-shell action for any solution. This is one of the main reasons the program of holographic renormalization does not extend in any straightforward way to the case of asymptotically flat spacetimes.

In the case at hand, the equations uniquely determine the coefficients $g_{(2)}, \dots, g_{(d-2)}$, $h_{(d)}$ and the trace and covariant divergence of $g_{(d)}$. The coefficient $h_{(d)}$ is present only

when d is even, and it is equal to the metric variation of the holographic conformal anomaly. The explicit expressions for $g_{(2)}, \dots, g_{(d-2)}, h_{(d)}$ and the trace and covariant divergence of $g_{(d)}$ can be found in appendix A of [8]. $g_{(d)}$ is directly related to the 1-point function of the dual stress energy tensor. In general, the solution obtained by this procedure is only valid near the boundary. More powerful techniques are needed in order to obtain solutions that extend to the deep interior. The three dimensional case is special in that one can exactly solve the equations to all orders [22]. Even in this case, however, the coordinate patch (3.13) does not in general cover the entire spacetime, see [23, 24] for related work. A review of the purely gravitational case can be found in [25].

These results were extended in the case of matter coupled to gravity in [8]. In this case the bulk equation reads

$$R_{\mu\nu} - \frac{1}{2}RG_{\mu\nu} = T_{\mu\nu} \quad (3.14)$$

where $T_{\mu\nu} = \Lambda G_{\mu\nu} +$ matter contribution. The equations in this case have a near-boundary solution provided the matter contribution to $T_{\mu\nu}$ is softer than the cosmological constant contribution. In these cases the matter fields are dual to marginal or relevant operators. If the matter stress energy tensor diverges faster than the cosmological constant term, the matter fields correspond to irrelevant operators. In this case in order to obtain a near-boundary solution the sources should be considered infinitesimal.

4 Holographic Renormalization Method

In this section we outline all steps involved in the method of holographic renormalization.

4.1 Asymptotic solution

In the first step we obtain the most general solution of the bulk field equations with prescribed, but arbitrary, Dirichlet boundary condition. Let us suppress all spacetime and internal indices and denote collectively bulk fields by $\mathcal{F}(x, \rho)$. Near the boundary

each **h field** has an asymptotic expansion of the form

$$\mathcal{F}(x, \rho) = \rho^m \left(f_{(0)}(x) + f_{(2)}(x)\rho + \cdots + \rho^n (f_{(2n)}(x) + \log \rho \tilde{f}_{(2n)}(x)) + \dots \right) \quad (4.1)$$

where ρ is the radial coordinate of AdS. We use coordinates where the AAdS metric takes the asymptotic form,

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j, \\ g_{ij}(x, \rho) &= g_{(0)ij}(x) + g_{(2)ij}(x)\rho + \cdots \end{aligned} \quad (4.2)$$

This is the coordinate system that has been discussed in section 3. The case $g_{(0)ij} = \delta_{ij}$, $g_{(2k)ij} = 0$, $k > 0$, yields the AdS metric (setting $\rho = z^2$ one gets the AdS metric in Poincaré coordinates).³

The field equations are second order differential equations in ρ , so there are two independent solutions. Their asymptotic behaviors are ρ^m and ρ^{m+n} , respectively. In almost all examples discussed in the literature, n and $2m$ are non-negative integers and the expansion involves integral powers of ρ (but see [26] for a counterexample). None of these features is essential to the method. The form of the subleading terms in the asymptotic expansion is determined by the bulk field equations. Notice also that if n is not an integer the logarithmic term $\tilde{f}_{(2n)}$ in (4.1) would be absent. We assume below that n is an integer, since this is the case in all examples in the literature, but one can easily generalize.

The boundary field $f_{(0)}$ that multiplies the leading behavior, ρ^m , is interpreted as the source for the dual operator. In the near-boundary analysis one solves the field equations iteratively by treating the ρ -variable as a small parameter. This yields *algebraic* equations for $f_{(2k)}$, $k < n$, that uniquely determine $f_{(2k)}$ in terms of $f_{(0)}(x)$ and derivatives up to order $2k$. These equations leave $f_{(2n)}(x)$ undetermined⁴. This was to be expected: the coefficient $f_{(2n)}(x)$ is the Dirichlet boundary condition for a solution which is linearly independent from the one that starts as ρ^m . The undetermined function $f_{(2n)}$ is related to the exact 1-point function of the corresponding

³Throughout these notes we work with Euclidean signature. Most of the results, however, are independent of the signature of spacetime.

⁴In the case \mathcal{F} is the metric $G_{\mu\nu}$ or a bulk gauge field A_μ the bulk field equations partly determine the corresponding $f_{(2n)}(x)$. For instance, as we discussed in the previous section, the bulk field equations determine the divergence and the trace of $g_{(d)ij}(x)$, but leave undetermined the remaining components.

operator. The logarithmic term in (4.1) is necessary in order to obtain a solution. It is related to conformal anomalies of the dual theory, and it is also fixed in terms of $f_{(0)}(x)$.

To summarize, the asymptotic analysis of the bulk field equations yields:

- $f_{(0)}(x)$ is the field theory source,
- $f_{(2)}(x), \dots, f_{(2n-2)}$, and $\tilde{f}_{(2n)}$ are uniquely determined by the bulk field equations and are local functions of $f_{(0)}$,
- $\tilde{f}_{(2n)}$ is related to conformal anomalies,
- $f_{(2n)}(x)$ is undetermined by the near-boundary analysis.

4.2 Regularization

Having obtained the most general asymptotic solution of the field equations, we now proceed to compute the on-shell value of the action. To regularize the on-shell action we restrict the range of the ρ integration, $\rho \geq \epsilon$, and we evaluate the boundary terms at $\rho = \epsilon$, where ϵ is a small parameter. A finite number of terms which diverge as $\epsilon \rightarrow 0$ can be isolated, so that the on-shell action takes the form

$$S_{\text{reg}}[f_{(0)}; \epsilon] = \int_{\rho=\epsilon} d^4x \sqrt{g_{(0)}} [\epsilon^{-\nu} a_{(0)} + \epsilon^{-(\nu+1)} a_{(2)} + \dots - \log \epsilon a_{(2\nu)} + \mathcal{O}(\epsilon^0)] \quad (4.3)$$

where ν is a positive number that only depends on the scale dimension of the dual operator and $a_{(2k)}$ are local functions of the source(s) $f_{(0)}$. The logarithmic divergence directly gives the conformal anomaly, as discussed in [7]. The divergences do not depend on $\tilde{f}_{(2n)}$, i.e. the coefficients that the near-boundary analysis does not determine.

4.3 Counterterms

The counterterm action is defined as

$$S_{\text{ct}}[\mathcal{F}(x, \epsilon); \epsilon] = -\text{divergent terms of } S_{\text{reg}}[f_{(0)}; \epsilon] \quad (4.4)$$

where divergent terms are expressed in terms of the fields $\mathcal{F}(x, \epsilon)$ ‘living’ at the regulated surface $\rho = \epsilon$ and the induced metric there, $\gamma_{ij} = g_{ij}(x, \epsilon)/\epsilon$. This is

required for covariance and entails an “inversion” of the expansions (4.1) up to the required order. In other words, in order to determine S_{ct} we first invert the series (4.1) to obtain $f_{(0)} = f_{(0)}(\mathcal{F}(x, \epsilon), \epsilon)$, and then substitute in the coefficients $a_{(2k)}(f_{(0)}(x)) = a_{(2k)}(\mathcal{F}(x, \epsilon), \epsilon)$, and finally insert those in (4.3).

4.4 Renormalized on-shell action

To obtain the renormalized action we first define a subtracted action at the cutoff

$$S_{\text{sub}}[\mathcal{F}(x, \epsilon); \epsilon] = S_{\text{reg}}[f_{(0)}; \epsilon] + S_{\text{ct}}[\mathcal{F}(x, \epsilon); \epsilon]. \quad (4.5)$$

$-\text{div} \int_{\text{reg}} (f_{(0)}, \xi)$

The subtracted action has a finite limit as $\epsilon \rightarrow 0$, and the renormalized action is a functional of the sources defined by this limit, i.e.

$$S_{\text{ren}}[f_{(0)}] = \lim_{\epsilon \rightarrow 0} S_{\text{sub}}[\mathcal{F}; \epsilon] \quad (4.6)$$

The distinction between S_{sub} and S_{ren} is needed because the variations required to obtain correlation functions are performed before the limit $\epsilon \rightarrow 0$ is taken.

4.5 Exact 1-point functions

The 1-point function of the operator O_F in the presence of sources is defined as

$$\langle O_F \rangle_s = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta f_{(0)}} \quad (4.7)$$

It can be computed by rewriting it in terms of the fields living at the regulated boundary,

$$\langle O_F \rangle_s = \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon^{d/2-m}} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub}}}{\delta F(x, \epsilon)} \right) \quad (4.8)$$

By construction (4.8) has a limit as $\epsilon \rightarrow 0$, but it is a good check on all previous steps to explicitly verify that the divergent terms indeed cancel. Explicit evaluation of the limit yields

$$\langle O_F \rangle_s \sim f_{(2n)} + C(f_{(0)}) \quad (4.9)$$

where $C(f_{(0)})$ is a function that depends locally on the sources, so it yields contact terms to higher point functions. The exact form of $C(f_{(0)})$ depends on the theory under consideration and in general is scheme dependent. The coefficient in front of $f_{(2n)}$ also depends on the theory under consideration (but it is scheme independent).

4.6 Ward identities

Having obtained explicit formulas for the holographic 1-point functions it is straightforward to verify whether the expected Ward identities hold. For instance, the 1-point function of the boundary stress energy tensor is given by

$$\langle T_{ij} \rangle_s \sim g_{(d)ij} + C(g_{(0)ij}), \quad (4.10)$$

Sources

see [8, 25] for the exact formulas. As discussed in the previous section, the bulk field equations only determine the divergence and trace of $g_{(d)ij}$. This information, however, is enough in order to compute the divergence and the trace of T_{ij} . Once Ward identities are established at the level of 1-point functions (in the presence of sources), they hold in general, since n -point functions can be obtained by further differentiation of 1-point functions with respect to the sources.

4.7 RG transformations

The energy scale on the boundary theory is associated with the radial coordinate of the bulk spacetime. RG transformations can be studied by using bulk diffeomorphisms that induce a Weyl transformation on the boundary metric. Such transformations have been studied in [27]. Here we will consider the simplest of such transformations,

$$\rho = \rho' \mu^2, \quad x^i = x^{i'} \mu. \quad (4.11)$$

This transformation is an isometry of AdS. Since we know how bulk fields transform under bulk diffeomorphisms, we can readily compute how the $f_{(2n)}$ transforms under (4.11), and therefore find what is the RG transformation of n -point functions.

4.8 n -point functions

To compute n -point functions we need exact (as opposed to asymptotic) solutions of the bulk field equations with prescribed but arbitrary boundary conditions. Given such an exact solution one can read-off $f_{(2n)}$ as a function of $f_{(0)}$ by considering the asymptotics of the solution. Then n -point functions can be computed using (2.5).

Given that the bulk equations are coupled non-linear equations, the general Dirichlet problem is in general not tractable. We proceed by linearizing the bulk field

Weyl scale
Σ

equations. Solving the equations for linearized fluctuations, i.e. determining the bulk-to-boundary propagator, allows one to determine the linear in $f_{(0)}$ term of $f_{(2n)}$. This is sufficient in order to obtain 2-point functions. Even in the absence of exact solutions, higher point functions can be determined perturbatively: one solves the bulk field equations perturbatively and thus determines the terms of $f_{(2n)}$ that are quadratic or higher orders in $f_{(0)}$.

5 Example: Massive scalar

In this section we illustrate the method by working through all steps in the simplest possible example: a free massive scalar field in AdS spacetime. The action is given by

$$S = \frac{1}{2} \int d^{d+1}x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) \quad (5.1)$$

The spacetime metric is give by⁵

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} dx^i dx^i. \quad (5.2)$$

The bulk field equation is equal to

$$(-\square_G + m^2)\Phi = -\frac{1}{\sqrt{G}} \partial_\mu (\sqrt{G} G^{\mu\nu} \partial_\nu \Phi) + m^2 \Phi = 0 \quad (5.3)$$

5.1 Asymptotic Solution

We want to obtain asymptotic solutions of (5.3). The scalar field, however, couples to the Einstein equation through its stress energy tensor. This means that in general we need to solve the coupled system of gravity-scalar field equations. In favorable circumstances the equations decouple near the boundary and one can study (5.3) in a fixed gravitational background. This issue is discussed at length in [10] and we refer there for more details. The current example is such a favorable case (but the example in section 5.9 is not), so we proceed by solving (5.3) in the gravitational background given in (5.2).

We look for a solution of the form

$$\Phi(x, \rho) = \rho^{(d-\Delta)/2} \phi(x, \rho),$$

⁵We denote by μ, ν , etc. $(d+1)$ -dimensional indices and by i, j , etc. d -dimensional indices.

$$\phi(x, \rho) = \phi_{(0)} + \rho\phi_{(2)} + \rho^2\phi_{(4)} + \cdots \quad (5.4)$$

Inserting this in (5.3) yields,

$$0 = [(m^2 - \Delta(\Delta - d))\phi(x, \rho) - \rho(\Box_0\phi(x, \rho) + 2(d - 2\Delta + 2)\partial_\rho\phi(x, \rho) + 4\rho\partial_\rho^2\phi(x, \rho))] \quad (5.5)$$

where $\Box_0 = \delta^{ij}\partial_i\partial_j$. The easiest way to solve (5.5) is to successively differentiate with respect to ρ and then set $\rho = 0$. Setting $\rho = 0$ in (5.5) implies,

$$(m^2 - \Delta(\Delta - d)) = 0 \quad (5.6)$$

which is the well-known relation between the mass and the conformal weight Δ of the dual operator. With (5.6) satisfied, (5.5) reduces to

$$\Box_0\phi(x, \rho) + 2(d - 2\Delta + 2)\partial_\rho\phi(x, \rho) + 4\rho\partial_\rho^2\phi(x, \rho) = 0 \quad (5.7)$$

Setting $\rho = 0$ we get

$$\phi_{(2)}(x) = \frac{1}{2(2\Delta - d - 2)}\Box_0\phi_{(0)} \quad (5.8)$$

Notice that we solved an *algebraic* equation in order to determine $\phi_{(2)}$. In particular, the solution remains valid if we change the signature of spacetime (the only different in that case is in the meaning of \Box_0), or we analytically continue the bulk metric from AdS to dS. This is a generic feature of the asymptotic solutions we discuss.

Now differentiate (5.7) with respect to ρ and set $\rho = 0$. The result is

$$\phi_{(4)}(x) = \frac{1}{4(2\Delta - d - 4)}\Box_0\phi_{(2)} \quad (5.9)$$

Continuing this way one obtains all coefficients in the expansion (5.4),

$$\phi_{(2n)} = \frac{1}{2n(2\Delta - d - 2n)}\Box_0\phi_{(2n-2)} \quad (5.10)$$

This procedure stops, however, when $2\Delta - d - 2n = 0$. In this case we need to introduce a logarithmic term at order $\rho^{\Delta/2}$ in (5.4) to obtain a solution. To be concrete consider the case $2\Delta - d - 2 = 0$, i.e. $\Delta = d/2 + 1$. The new asymptotic expansion is given by

$$\phi(x, \rho) = \phi_{(0)} + \rho(\phi_{(2)} + \log \rho \psi_{(2)}) + \cdots \quad (5.11)$$

Inserting this expression in (5.7) we now get that

$$\psi_{(2)} = -\frac{1}{4}\square_0\phi_{(0)} \quad (5.12)$$

and we find that $\phi_{(2)}$ is *not* determined by the field equations.

In the general case $\Delta = d/2 + k$, with k an integer, a similar computation yields,

$$\psi_{(2\Delta-d)} = -\frac{1}{2^{2k}\Gamma(k)\Gamma(k+1)}(\square_0)^k\phi_{(0)} \quad (5.13)$$

and again $\phi_{(2\Delta-d)}$ is *not* determined by the bulk field equations.

5.2 Regularization

We now evaluate the regularized action on the asymptotic solution just found,

$$\begin{aligned} S_{\text{reg}} &= \frac{1}{2} \int_{\rho \geq \epsilon} d^{d+1}x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) \\ &= \frac{1}{2} \int_{\rho \geq \epsilon} d^{d+1}x \sqrt{G} \Phi (-\square_G + m^2) \Phi - \frac{1}{2} \int_{\rho=\epsilon} d^d x G^{\rho\rho} \Phi \partial_\rho \Phi \end{aligned} \quad (5.14)$$

where we use the convention to have $\rho = \epsilon$ at the lower end of radial integration. Since the bulk field equations are satisfied, the bulk term vanishes and by inserting the explicit asymptotic solution we obtain,

$$\begin{aligned} S_{\text{reg}} &= - \int_{\rho=\epsilon} d^d x \epsilon^{-\Delta+\frac{d}{2}} \left(\frac{1}{2} (d-\Delta) \phi(x, \epsilon)^2 + \epsilon \phi(x, \epsilon) \partial_\epsilon \phi(x, \epsilon) \right), \\ &= \int_{\rho=\epsilon} d^d x \left(\epsilon^{-\Delta+\frac{d}{2}} a_{(0)} + \epsilon^{-\Delta+\frac{d}{2}+1} a_{(2)} + \dots - \log \epsilon a_{(2\Delta-d)} \right) \end{aligned} \quad (5.15)$$

where

$$\begin{aligned} a_{(0)} &= -\frac{1}{2}(d-\Delta)\phi_{(0)}^2, & a_{(2)} &= -(d-\Delta+1)\phi_{(0)}\phi_{(2)} = -\frac{d-\Delta+1}{2(2\Delta-d-2)}\phi_{(0)}\square_0\phi_{(0)}, \\ a_{(2\Delta-d)} &= -\frac{d}{2^{2k+1}\Gamma(k)\Gamma(k+1)}\phi_{(0)}(\square_0)^k \phi_{(0)} \text{ source} \end{aligned} \quad (5.16)$$

As promised, the coefficients $a_{(2\nu)}$ of the divergent terms are local functions of the source $\phi_{(0)}$.

5.3 Counterterms

To obtain the counterterms we need to **invert the series (5.4)**. This is needed because it is $\Phi(x, \epsilon)$ rather than $\phi_{(0)}$ that transforms as a scalar under bulk diffeomorphisms

at $\rho = \epsilon$. To second order we obtain,

$$\begin{aligned}\phi_{(0)} &= \epsilon^{-(d-\Delta)/2} \left(\Phi(x, \epsilon) - \frac{1}{2(2\Delta - d - 2)} \square_\gamma \Phi(x, \epsilon) \right) \\ \phi_{(2)} &= \epsilon^{-(d-\Delta)/2-1} \frac{1}{2(2\Delta - d - 2)} \square_\gamma \Phi(x, \epsilon)\end{aligned}\tag{5.17}$$

where \square_γ is the Laplacian of the induced metric $\gamma_{ij} = \frac{1}{\epsilon} \delta_{ij}$ at $\rho = \epsilon$. These results are sufficient in order to rewrite $a_{(0)}$ and $a_{(2)}$ in terms of $\Phi(x, \epsilon)$. The counterterm action is then given by (4.4)

$$S_{\text{ct}} = \int \sqrt{\gamma} \left(\frac{d-\Delta}{2} \Phi^2 + \frac{1}{2(2\Delta - d - 2)} \Phi \square_\gamma \Phi \right) + \dots\tag{5.18}$$

-div S / log epsilon

where the dots indicate higher derivative terms. Notice that when $\Delta = d/2 + 1$ the coefficient of the $\Phi \square_\gamma \Phi$ is replaced by $-\frac{1}{4} \log \epsilon$. Similarly, when $\Delta = d/2 + k$ there is a k -derivative logarithmic counterterm.

5.4 Renormalized on-shell action

The renormalized action in the minimal subtraction scheme is given by (4.5). We still have the freedom to add finite counterterms. This corresponds to the scheme dependence in the field theory. For instance, in order to have a manifestly supersymmetric scheme where $S_{\text{ren}} = 0$ when evaluated on the background, it may be necessary to add finite counterterms. This phenomenon was observed in [9].

5.5 Exact 1-pt function

Equation (4.8) adapted to our case gives

$$\langle O_\Phi \rangle_s = \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon^{\Delta/2}} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub}}}{\delta \Phi(x, \epsilon)} \right)\tag{5.19}$$

For concreteness we will discuss the $\Delta = d/2 + 1$ case. Now,

$$\begin{aligned}\delta S_{\text{sub}} &= \delta S_{\text{reg}} + \delta S_{\text{ct}} \\ &= \int_{\rho \geq \epsilon} d^{d+1} x \delta \Phi (-\square_G + m^2) \Phi \\ &\quad + \int_{\rho = \epsilon} d^d x \delta \Phi \left(-2\epsilon \partial_\epsilon \Phi + (d - \Delta) \Phi - \frac{1}{2} \log \epsilon \square_\gamma \Phi \right)\end{aligned}\tag{5.20}$$

Using the fact that the bulk field equations hold we obtain,

$$\frac{\delta S_{\text{sub}}}{\delta \Phi} = -2\epsilon\partial_\epsilon\Phi + (d - \Delta)\Phi - \frac{1}{2}\log\epsilon\Box_\gamma\Phi \quad (5.21)$$

Inserting this in (5.19) and substituting for Φ the explicit asymptotic solution we found, we find that the divergent terms cancel, as they should, and the finite part is equal to

$$\langle O_\Phi \rangle_s = -2(\phi_{(2)} + \psi_{(2)}) \quad (5.22)$$

As promised, the 1-point function depends on the part of the asymptotic solution that is not determined by the near-boundary analysis. $\psi_2(x)$ is a local function of the sources, see (5.12). We called such contributions $C(\phi_{(0)})$ in (4.9). Actually this term is scheme dependent. Indeed by adding the finite counterterm

$$S_{\text{ct,fin}} = -\frac{1}{4}\int d^d x \phi_{(0)}\Box_0\phi_{(0)} = -\frac{1}{2}\int d^d x \sqrt{\gamma}\mathcal{A} \quad (5.23)$$

in the action we can remove completely the factor of $\psi_{(2)}$ from the the 1-point function. Notice that \mathcal{A} in (5.23) is the matter conformal anomaly [28].

Finally, let us mention that for general Δ the result is [8]

$$\langle O_\Phi \rangle_s = -(2\Delta - d)\phi_{(2\Delta-d)} + C(\phi_{(0)}) \quad (5.24)$$

5.6 RG transformations

To determine the RG transformations of the correlation functions we need to determine how the coefficients in the asymptotic solution transform under (4.11). Since $\Phi(x, \rho)$ is scalar, we have

$$\Phi'(x', \rho') = \Phi(x, \rho) \quad (5.25)$$

This equation implies

$$\phi'_{(0)}(x') = \mu^{d-\Delta}\phi_{(0)}(x'\mu) \quad (5.26)$$

$$\phi'_{(2)}(x') = \mu^{d-\Delta+2}\phi_{(2)}(x'\mu) \quad (5.27)$$

$$\dots \quad \dots$$

$$\psi'_{(2\Delta-d)}(x') = \mu^\Delta\psi_{(2\Delta-d)}(x'\mu) \quad (5.28)$$

$$\phi'_{(2\Delta-d)}(x') = \mu^\Delta(\phi_{(2\Delta-d)}(x'\mu) + \log\mu^2\psi_{(2\Delta-d)}(x'\mu)) \quad (5.29)$$

Notice that (5.26) implies

$$\mu \frac{\partial}{\partial \mu} \phi_{(0)}(x' \mu) = -(d - \Delta) \phi_{(0)}(x' \mu) \quad (5.30)$$

which is the correct RG transformation rule for a source of an operator of dimension Δ .

Now using (5.29) we can obtain the transformed 1-point function,

$$\langle O(x') \rangle'_s = \mu^\Delta \left(\langle O(x' \mu) \rangle_s - (2\Delta - d) \log \mu^2 \psi_{(2\Delta-d)}(x' \mu) \right) \quad (5.31)$$

Notice the new term can be obtained by addition of the following finite counterterm,

$$S_{\text{ct,fin}}(\mu) = \int d^d x \sqrt{\gamma} \frac{1}{2} \log \mu^2 \mathcal{A} \quad \text{cf anomaly} \quad (5.32)$$

where \mathcal{A} is the matter conformal anomaly. This result is as expected: we are computing conformal field theory correlation functions. The correlation function should thus have a trivial scale dependence, up to the effects of conformal anomalies. Indeed, we see from (5.31) that the transformation of the 1-point function of the operator of dimension Δ has the expected scaling term and an additional term that is related to the conformal anomaly. In other words, all non-trivial scale dependence is driven by the conformal anomaly.

5.7 Correlation functions

The considerations so far involved only the near-boundary analysis. We have derived holographic 1-point functions, but these involve coefficients in the asymptotic expansion of the bulk fields that the near-boundary analysis does not determine. We will now see how these are determined by obtaining the exact solution of the bulk field equations. In the case at hand, the field equation is linear in Φ and can be solved exactly. In more general circumstances the field equations are non-linear and cannot be solved in full generality. One may, however, linearize around the background and solve the linearized fluctuation equations. This is sufficient to obtain 2-point functions since we only need to know $\phi_{(2\Delta-d)}$ to linear order in the source in order to obtain them. We will discuss higher point functions in the next section.

For concreteness we work in $d = 4$ and we consider the case $\Delta = d/2 + 1 = 3$. Let $\rho = z^2$ and $\Phi = z^{d/2} \chi$, and we also Fourier transform in the x coordinates. The bulk

field equation (5.3) becomes

$$z^2 \partial_z^2 \chi + z \partial_z \chi - (k^2 z^2 + 1) \chi = 0 \quad (5.33)$$

This is the modified Bessel differential equation. The solution that is regular in the interior is

$$\begin{aligned} \chi &= K_1(kz) \\ &= \frac{1}{kz} + \left(\frac{1}{4}(-1 + 2\gamma) - \frac{1}{2}(-\log 2 + \log kz) \right) kz + \dots \end{aligned} \quad (5.34)$$

where in the second line we give the asymptotic expansion near $z = 0$, and $k = |k|$.

Converting back to the ρ coordinate we get

$$\Phi(k, \rho) = \rho^{(d-\Delta)/2} \phi_{(0)}(k) \left(1 + \rho \left(\left(\frac{1}{4}(-1 + 2\gamma) + \frac{1}{2} \log \frac{k}{2} \right) k^2 + \frac{1}{4} k^2 \log \rho \right) \right) + \dots \quad (5.35)$$

where $\phi_{(0)}(k)$ represents the overall normalization of χ . We now read off the various coefficients

$$\psi_2(k) = \frac{1}{4} k^2 \phi_{(0)}(k) \rightarrow \psi_2(x) = -\frac{1}{4} \square \phi_{(0)}(x) \quad (5.36)$$

$$\phi_{(2)}(k) = -2\phi_{(0)} \left(\frac{1}{4}(-1 + 2\gamma) + \frac{1}{2} \log \frac{k}{2} \right) k^2 \quad (5.37)$$

Notice that the exact solution correctly reproduces the value of $\psi_{(2)}$ we determined by the near boundary analysis (5.12). Furthermore, the exact solution determines $\phi_{(2)}$. Notice that $\phi_{(2)}$ is related non-locally to the source $\phi_{(0)}$ (their relation involves an infinite number of derivatives).

Inserting in (5.22) we get

$$\langle O_\Phi(k) \rangle_s = -2\phi_{(0)}(k) \left[\left(\frac{1}{4}(-1 + 2\gamma) - \frac{1}{2} \log 2 + \frac{1}{4} k^2 \right) + \frac{1}{4} k^2 \log k^2 \right] \quad (5.38)$$

The terms in parenthesis lead to contact terms in the 2-point function and can be omitted. We now use (2.5) to obtain the 2-point function,

$$\begin{aligned} \langle O_\Phi(k) O_\Phi(-k) \rangle &= -\frac{\delta \phi_{(2)}(k)}{\delta \phi_{(0)}(-k)} \\ &= \frac{1}{2} k^2 \log \frac{k^2}{\mu^2} \end{aligned} \quad (5.39)$$

where we have also introduced the scale μ (this scale is introduced by adding a local counterterm proportional to the anomaly).

Fourier transforming (5.39) we get (see appendix A.2 of [29]).

$$\langle O_\Phi(x)O_\Phi(0) \rangle = \frac{4}{\pi^4} \left(-\frac{1}{32} \square \square \frac{1}{x^2} \log x^2 M^2 \right) \quad (5.40)$$

where $M = \gamma\mu/2$ and γ is the Euler constant. The expression in brackets in the right hand side is the renormalized version of $1/x^6$ (see (A.1) of [29]), i.e. it is equal to $1/x^6$ for $x \neq 0$, but it is non-singular (as a distribution) at $x = 0$, so

$$\langle O_\Phi(x)O_\Phi(0) \rangle = \frac{4}{\pi^4} \mathcal{R} \frac{1}{x^6} \quad (5.41)$$

where we used $\mathcal{R} \frac{1}{x^6}$ to denote the renormalized version of $1/x^6$. This is manifestly the correct two-point function for the an operator of dimension 3. Notice that, as promised, we got the renormalized 2-point function. The normalization of (5.41) also agrees with the normalization derived in [30]. For $\Delta = d/2 + k$, where k is an integer, the normalization is given by

$$c_\Delta = (2\Delta - d) \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})} \quad (5.42)$$

5.8 RG equation

Let us now consider the RG equation satisfied by the two point function we just derived,

$$M \frac{\partial}{\partial M} \langle O_\Phi(x)O_\Phi(0) \rangle = \frac{4}{\pi^4} \left(-\frac{1}{32} \square \square \frac{2}{x^2} \right) = \square \delta^{(4)}(x) \quad (5.43)$$

This is the expected equation. The scale dependence of the correlator originates only from the conformal anomaly. As derived in [28] (following [31]) the trace of the stress energy tensor is related to the scale dependence of the correlator by

$$\int d^4x \langle T_i^i \rangle = \sum_{k=1}^{\infty} \frac{1}{k!} \int d^4x_1 \dots d^4x_k M \frac{\partial}{\partial M} \langle O(x_1) \dots O(x_k) \rangle \quad (5.44)$$

Thus local terms in the scale derivative of correlation functions lead to conformal anomalies. In non-conformal theories, the M derivative of the correlation function is non-local implying that there is non-trivial β -function. In such cases the left hand side of (5.44) contains a β -function term as well. We refer to [32] for related work.

5.9 n -point functions

⁶The results presented in this section were obtained in collaboration with Dan Freedman and Umut Gürsoy.

We discuss in this section the perturbative computation of n -point functions. We consider the case $d = 4$ and $\Delta = d/2 + 1 = 3$ and we illustrate the method by computing a four-point function. Our starting point is the following bulk action

$$S = \int d^5x \sqrt{G} \left(\frac{1}{2} G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \frac{1}{2} m^2 \Phi^2 - \frac{1}{3} \Phi^4 \right). \quad (5.45)$$

where $m^2 = -3$ in this case. The reason we choose this specific action is that the near-boundary analysis relevant for this action was performed in [10] in connection with the computation of 2-point functions in the GPPZ flow [33]. We are interested here in computing the 4-point function of the operator dual to the scalar in the AdS vacuum rather than the domain-wall vacuum corresponding to the GPPZ flow, but as emphasized earlier, the results of the near-boundary analysis are valid for *any* solution of the bulk field equation, so we can freely borrow the results of [10]⁷. The 1-point function is given in (5.39) of [10],

$$\langle O_\Phi \rangle_s = -2(\phi_{(2)} + \psi_{(2)}) + \frac{2}{9} \phi_{(0)}^3 \quad (5.46)$$

The last two terms lead to contact terms in correlation functions. $\psi_{(2)}$ was discussed in the previous section. The last term gives an ultra local contribution to the 4-point function, i.e. it contributes only when all four operators are at the same point. In the remainder we discuss the contribution at separated points.

The field equation is given by

$$(-\square_G + m^2)\Phi = g\Phi^3 \quad (5.47)$$

where $g = 4/3$. The idea is now to solve this equation perturbatively in g (one can justify this by introducing a coupling constant in the Φ^4 term in (5.45)). Let

$$\Phi = \Phi_0 + g\Phi_1 + \dots \quad (5.48)$$

then

$$(-\square_G + m^2)\Phi_0 = 0 \quad (5.49)$$

$$(-\square_G + m^2)\Phi_1 = \Phi_0^3 \quad (5.50)$$

⁷The analysis involves the steps we discussed in the previous section. In this case, however, one has to consider the coupled system of gravity-scalar equations rather than study the scalar field equation in a fixed gravitational background as was done in the previous section. We refer to section 5.2 of [10] for details.

etc. Equation (5.49) was solved in the previous section. It will be convenient to quote the result in a way that is valid for other values of Δ as well,

$$\Phi_0(z, \vec{x}) = \int d^4y K_\Delta(z, \vec{x} - \vec{y}) \phi_{(0)}(y) \quad (5.51)$$

where the bulk-to-boundary propagator is given by

$$K_\Delta(z, \vec{x} - \vec{y}) = C_\Delta \left(\frac{z}{z^2 + (\vec{x} - \vec{y})^2} \right)^\Delta \quad (5.52)$$

and

$$C_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2} \Gamma(\Delta - \frac{d}{2})} \quad (5.53)$$

Note that $z^2 = \rho$, as in the previous section.

Equation (5.50) is solved by

$$\Phi_1(x) = \int d^{d+1}y G_\Delta(x, y) \Phi_0(y)^3, \quad (5.54)$$

where $G_\Delta(x, x')$ is the bulk-to-bulk propagator,

$$(-\square_G + m^2)G_\Delta(x, x') = \delta(x, x'), \quad \delta(x, x') = \frac{1}{\sqrt{G}}\delta(x - x') \quad (5.55)$$

The explicit solution is given by (see for instance [5])

$$\begin{aligned} G_\Delta(x, x') &= \frac{2^{-\Delta} C_\Delta}{2\Delta - d} \xi^\Delta F\left(\frac{\Delta}{2}, \frac{\Delta}{2} + \frac{1}{2}; \Delta - \frac{d}{2} + 1; \xi^2\right), \\ \xi &= \frac{2zz'}{z^2 + z'^2 + (\vec{x} - \vec{x}')^2} \end{aligned} \quad (5.56)$$

where $x = (z, \vec{x})$ and $x' = (z', \vec{x}')$, and F is a hypergeometric function.

Having obtained a solution of the bulk field equation to order g , the next task is to obtain the contribution of Φ_1 to $\phi_{(2)}$. From (5.54) follows that we need the near-boundary expansion of the bulk-to-bulk propagator. The latter is given by

$$G_\Delta(x, x') = z^\Delta \frac{1}{(2\Delta - d)} K_\Delta(z', \vec{x} - \vec{x}') + \mathcal{O}(z^{\Delta+2}) \quad (5.57)$$

This follows trivially from (5.56) upon using $F(\Delta/2, (\Delta + 1)/2; \Delta - d/2 + 1; 0) = 1$. Since $\rho = z^2$, this exactly has the correct ρ dependence to contribute to $\phi_{(2)}$ (or $\phi_{(2\Delta-d)}$ in the general case).

We are now ready to compute the 4-point function. By definition,

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = 2 \frac{\delta^3 \phi_{(2)}(x_1)}{\delta \phi_{(0)}(x_2) \delta \phi_{(0)}(x_3) \delta \phi_{(0)}(x_4)} \quad (5.58)$$

where we used (5.46). Using (5.54), (5.57) and (5.51) we finally obtain,

$$\langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle = 3!g \int d^{d+1}x \sqrt{G} \prod_{k=1}^4 K_{\Delta}(z, (\vec{x}_k - \vec{x})) \quad (5.59)$$

This is the correctly normalized 4-point function [30]. The discussion generalizes to different interaction terms. The crucial ingredient is the relation (5.57).

6 RG flows

In the previous section we illustrated in detail how to compute **holographically renormalized correlation functions** of conformal field theories. The method can be used to obtain correlation functions for all quantum field theories that can be obtained via a deformation or a vev from a CFT that has a holographic dual.

cft $\xrightarrow{\text{def}}$ qft

6.1 The vacuum

We have seen in section 5 that the asymptotic expansion of a scalar field that is dual to a dimension Δ operator is of the form,

$$\Phi = \rho^{(d-\Delta)/2} \underbrace{\phi_{(0)}}_{\text{source}} + \dots + \rho^{\Delta/2} \underbrace{\phi_{(2\Delta-d)}}_{\text{1 pt. fnc}} + \dots \quad (6.1)$$

$\langle O \rangle \sim \phi_0$

*op dim Δ
dual
 Φ*

and that $\phi_{(0)}$ has the interpretation of a source and $\phi_{(2\Delta-d)}$ of a 1-point function. It follows from this that if we consider a supergravity solution where the metric is asymptotically AdS and there is a non-trivial scalar turned on then we either have an operator deformation of the CFT or the CFT is in a different non-conformal vacuum.

*gnd
 \downarrow*

- **Operator deformation.** In this case the near-boundary expansion of Φ is $\Phi \sim \rho^{(d-\Delta)/2} \phi_0$, and this corresponds to the addition of the term $\phi_0 O$ in the Lagrangian of the boundary theory.

- **VEV deformation.** In this case the near-boundary expansion of Φ is $\Phi \sim \rho^{\Delta/2} \phi_0$, and the boundary Lagrangian is still the same, but the vev of the dual operator is non-zero, $\langle O \rangle \sim \phi_0$, and the vacuum spontaneously breaks conformal invariance.

We now for concreteness restrict ourselves to $d = 4$. The most general form of a bulk solution that preserves Poincaré invariance in four dimensions is

$$\begin{aligned} ds^2 &= e^{2A(r)} \delta_{ij} dx^i dx^j + dr^2 \\ \Phi &= \Phi(r) \end{aligned} \quad (6.2)$$

The action that governs the dynamics of this system is

$$S = \int d^5x \sqrt{G} \left(\frac{1}{4} R + \frac{1}{2} G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) \right). \quad (6.3)$$

In supergravity theories, the truncation of the theory to **single scalar** usually leads to a potential $V(\Phi)$ that is related to a superpotential $W(\Phi)$ as

$$V(\Phi) = \frac{1}{2} (\partial_\Phi W)^2 - \frac{4}{3} W^2. \quad (6.4)$$

This form of the potential, however, also follows from more general arguments using gravitational stability. A generalized positive energy argument [34], was used in [35], to show that the potential must have the form (6.4) when there is a single scalar field⁸. The argument in [35] implies that when the *AdS* critical point is stable, there is a “superpotential” W such that the critical point of $V(\Phi)$ associated with the *AdS* geometry is also a critical point of W .⁹ In the *AdS/CFT* correspondence, positivity of energy about a given *AdS* critical point is mapped into unitarity of the corresponding CFT. It follows that in all cases the dual CFT is unitary the potential can be written as in (6.4) [36].

When (6.4) holds a simple *BPS* analysis [36, 37] of the domain wall action yields the **flow equations**

$$\frac{dA(r)}{dr} = -\frac{2}{3} W(\Phi), \quad \frac{d\Phi(r)}{dr} = \partial_\Phi W(\Phi) \quad (6.5)$$

These equations have also been obtained from several other standpoints, such as fermion transformation rules in the (truncated) supergravity theory [38], and the Hamilton-Jacobi framework [39, 40].

6.2 Correlation functions

Solutions to the first order equations (6.5) provide the vacuum of the dual quantum field theory. Many such classical solutions are known (see, for instance, [38, 41, 42, 33,

⁸For several scalars the obvious generalization of the form (6.4) implies stability, but the converse is not necessarily true.

⁹If one relaxes this requirement the potential can always be written in the form (6.4); one just views (6.4) as a differential equation for W [37]. In this case, however, as the original critical point may not be a critical point of W , the results of [35] about gravitational stability do not necessarily apply.

43, 44, 45, 26]). The computation of correlation function along RG flows is analogous to the case of correlators of a CFT we just discussed. The details have been spelled out in [9] and [10] to which we refer for a more complete discussion. Here we will only discuss a few selected topics.

The first step in computing correlation functions is to perform the **near-boundary analysis**. At the end of this analysis one ends up with a number of **exact 1-point functions and a number of Ward identities** that relates them. As emphasized on many occasions, the results of this step only depend on the action one starts from and can be applied to any particular solution.

Let us consider the case of a scalar coupled to gravity. This system is relevant for the study of correlation functions of the stress energy tensor and a scalar operator. The near-boundary analysis in this case leads to the following diffeomorphism and **Weyl Ward identities**,

$$\nabla^i \langle T_{ij} \rangle_s = -\langle O \rangle_s \nabla_j \phi_{(0)} \quad (6.6)$$

$$\langle T^i_i \rangle_s = (\Delta - 4) \phi_{(0)} \langle O \rangle_s + \mathcal{A} \quad (6.7)$$

where \mathcal{A} is the holographic conformal anomaly. These results hold provided the bulk action is covariant and admits an AdS critical point. It is a good check on computations of the near-boundary analysis to verify that the actual 1-point functions do satisfy these Ward identities. It is also easy to establish that these are the expected field theory Ward identities. Indeed by definition,

$$\delta S_{\text{ren}} = \int d^4x \sqrt{g_{(0)}} \left[\frac{1}{2} \langle T_{ij} \rangle \delta g_{(0)}^{ij} + \langle O \rangle \delta \phi_{(0)} \right] \quad (6.8)$$

Invariance of (6.8) under diffeomorphisms

$$\delta g_{(0)}^{ij} = -(\nabla^i \xi^j + \nabla^j \xi^i), \quad \delta \phi_{(0)} = \xi^i \nabla_i \phi_{(0)} \quad (6.9)$$

yields (6.6) and invariance, up to an anomaly, under Weyl transformations

$$\delta g_{(0)}^{ij} = -2\sigma g_{(0)}^{ij}, \quad \delta \phi_{(0)} = (\Delta - 4)\sigma \phi_{(0)} \quad (6.10)$$

yields (6.7).

Let us now use these results in the context of RG flows.

- *Operator deformation*

In this case there is a non-zero background value of $\phi_0 = \varphi_0$, so we obtain from (6.7),

$$\langle T_i^i \rangle = (\Delta - 4)\varphi_0 \langle O \rangle + \mathcal{A} \quad (6.11)$$

which leads to the identification of the β function

$$\beta = (\Delta - 4)\varphi_0 \quad (6.12)$$

Notice that this is the β -function for the coupling constant associated with the operator we added in the Lagrangian, not the β -function for the gauge coupling constant. Thus we get the correct RG equation,

$$T_i^i = \beta O + \mathcal{A}, \quad (6.13)$$

see the discussion in section 5.8.

- *VEV deformation*

In this case it is $\langle O \rangle$ which has a background value, $\langle O \rangle_B$. In this case combining (6.6) and (6.7) and going to momentum space we derive for the connected correlator¹⁰

$$\langle T_{ij}(p) O(-p) \rangle = -\frac{\Delta - 4}{3} \langle O \rangle_B \pi_{ij} \quad (6.14)$$

where $\pi_{ij} = \delta_{ij} - p_i p_j / p^2$. We thus find that the two-point function exhibits a massless pole. This is the expected dilaton pole due to the Goldstone boson of spontaneously broken conformal symmetry.

It remains to actually compute the correlators. As long as the derivation is consistent with the near-boundary analysis, the correlation functions are guaranteed to exhibit the physics of operator deformation or spontaneous symmetry breaking we just discussed.

To obtain 2-point function we need to linearize around the domain-wall background and then solve the resulting fluctuation equations. The treatment of fluctuations is universal for the gravity-scalar-vector sector [46, 47, 48]. What is important

¹⁰Notice that the stress energy tensor as defined by (6.8) includes the term $g_{(0)ij} \phi_{(0)} O$. This originate from the term $\int \sqrt{g_{(0)}} \phi_{(0)} O$. The standard field theory stress energy tensor does not include this term and it is related to T_{ij} by $\langle T_{ij} \rangle_{QFT} = \langle T_{ij} \rangle + \varphi_{(0)} \langle O \rangle g_{(0)ij}$, where $\varphi_{(0)}$ is the fluctuation part of the source $\phi_{(0)}$. When the background vev $\langle O \rangle$ does not vanish one must correct for this effect. This has been taken into account in (6.14).

for the computation of the correlation function is that one should consider linear combinations of fluctuations that only depend on the conformal structure at infinity, i.e. the fluctuations should not change if we change the representative of the conformal structure [9]. To illustrate this point we discuss the fluctuation equations of the gravity-scalar sectors.

We look for fluctuations around the solution $(A(r), \varphi_B(r))$ of (6.5). The background plus **linear fluctuations** is described by

$$\begin{aligned} ds^2 &= e^{2A(r)}[\delta_{ij} + h_{ij}(x, r)]dx^i dx^j + (1 + h_{rr})dr^2 \\ \Phi &= \varphi_B(r) + \tilde{\varphi}(x, r) \end{aligned} \quad (6.15)$$

where h_{ij} , h_{rr} and $\tilde{\varphi}$ are considered infinitesimal. This choice does not completely fix the bulk diffeomorphisms. One can perform the one-parameter family of ‘**gauge transformations**’

$$r = r' + \epsilon^r(r', x'), \quad x^i = x'^i + \epsilon^i(r', x') \quad (6.16)$$

with

$$\epsilon^i = \delta^{ij} \int_r^\infty dr' e^{-2A(r)} \partial_j \epsilon^r \quad (6.17)$$

where we only display the fluctuation-independent part of ϵ^i . These diffeos are related to those which induce the Weyl transformation (6.10) of the sources. The gauge choice is also left invariant by the linearization of the $4d$ diffeomorphisms in (6.9).

We decompose the metric fluctuation as

$$h_{ij}(x, r) = h_{ij}^T(x, r) + \delta_{ij} \frac{1}{4} h(x, r) - \partial_i \partial_j H(x, r) + \nabla_{(i} h_{j)}^L \quad (6.18)$$

$4d$ diffeos can be used to set $h_i^L = 0$ and we choose to do so. The equation for the transverse traceless modes decouples from the equations for $(\tilde{\varphi}, h, H, h_{rr})$. The coupled graviton-scalar field equations in the axial gauge where $h_{rr} = 0$ were derived in [46], and we now include h_{rr} . The fluctuation equations are [9]

$$[\partial_r^2 + 4A' \partial_r + e^{-2A} \square] f(x, r) = 0, \quad h_{ij}^T = h_{(0)ij}^T f(x, r) \quad (6.19)$$

$$h' = -\frac{16}{3} \varphi' \tilde{\varphi} + 4A' h_{rr} \quad (6.20)$$

$$H'' + 4A' H' - \frac{1}{2} e^{-2A} h - h_{rr} e^{-2A} = 0 \quad (6.21)$$

$$2A' H' = \frac{1}{2} e^{-2A} h + \frac{8}{3} \frac{1}{p^2} W_\varphi (\tilde{\varphi}' - W_{\varphi\varphi} \tilde{\varphi} - \frac{1}{2} W_\varphi h_{rr}) \quad (6.22)$$

where $h_{(0)ij}^T$ is transverse, traceless, and independent of r .

The fluctuations $(\tilde{\varphi}, h, H, h_{rr})$ transform under the ‘gauge transformations’ in (6.6). This implies that they depend not only on the conformal structure at infinity, but also on the specific representative chosen. To remedy for this we look for gauge invariant combinations. Such combinations are the following [9],

$$R \equiv h_{rr} - 2\partial_r \left(\frac{\tilde{\varphi}}{W_\varphi} \right), \quad h + \frac{16}{3} \frac{W}{W_\varphi} \tilde{\varphi}, \quad H' - \frac{2}{W_\varphi} e^{-2A} \tilde{\varphi} \quad (6.23)$$

In terms of these variables the equations simplify,

$$h + \frac{16}{3} \frac{W}{W_\varphi} \tilde{\varphi} = -\frac{16}{3} \frac{e^{2A}}{p^2} \left(R(WW_{\varphi\varphi} - \frac{4}{3}W^2 - \frac{1}{2}W_\varphi^2) + \frac{1}{2}R'W \right) \quad (6.24)$$

$$H' - \frac{2}{W_\varphi} e^{-2A} \tilde{\varphi} = \frac{1}{p^2} \left(2R(W_{\varphi\varphi} - \frac{4}{3}W) + R' \right) \quad (6.25)$$

Equation (6.20) takes the form

$$\left(h + \frac{16}{3} \frac{W}{W_\varphi} \tilde{\varphi} \right)' = -\frac{8}{3}WR \quad (6.26)$$

Differentiating (6.24) leads to the second order differential equation

$$R'' + (2W_{\varphi\varphi} - 4W)R' - (4W_\varphi^2 - 2W_\varphi W_{\varphi\varphi\varphi} - \frac{32}{9}W^2 + \frac{8}{3}WW_{\varphi\varphi} + p^2 e^{-2A})R = 0 \quad (6.27)$$

We thus find that in order to obtain **2-point functions we need to solve the second order ODE (6.27)**. In favorable circumstances this ODE reduces to a hypergeometric equation whose solutions and their asymptotics are known. In such cases one can explicitly work out the 2-point functions. This was explicitly done for two RG flows, one involving an operator deformation and the other a vev in [9]. Equivalent results were obtained simultaneously in [49] and earlier progress was reported in [50]. We refer to these works for the details.

7 Conclusions

In these lectures notes we presented a systematic method for computing renormalized correlation functions in the gravity/gauge theory correspondence. The method is

complete and can be applied in all cases where the spacetime is asymptotically AdS.¹¹ This means that the dual theory should flow in the UV to a fixed point. Even though this appears to be very general there is a class of gravity/gauge theory dualities where the spacetime is not asymptotically AdS.

The first such example is the case of the gravity/gauge theory duality involving non-conformal branes [52, 53]. In this case the duality involves spacetimes that are conformal to AdS. It should be possible to generalize the method by working in the dual frame, i.e. the frame where the spacetime is exactly AdS. It would be interesting to work out the details.

Another case of interest is the dualities of the Klebanov-Strassler type [54]. A computation of a two-point function for this geometry has been presented in [55]. The results of that paper indicate that a generalization of the method to these geometries should be possible. The main difference of this case with the cases discussed here is that the spacetime geometry contains logarithms at leading order. This means that the form of the asymptotics in the near-boundary analysis should be modified accordingly.

It would also be interesting to develop the method from the ten dimensional point of view. Our discussion was entirely in terms of $d + 1$ -dimensional fields. There is no problem of principle with this since we can always KK reduce the $10d$ theory to $(d + 1)$ -dimensions. The $10d$ perspective, however, can have many advantages. To name a few: geometries that are singular from the $5d$ point of view may be non-singular from the $10d$ point of view; each of the KK towers is represented by a single field, and more importantly some of the gravity/gauge theory dualities, such as the Polchinski-Strassler duality [56], are formulated more naturally from the $10d$ point of view. On the other hand, the $10d$ geometries of the form $AdS \times M$ have degenerate boundaries, the asymptotic expansions are not universal, and the subtractions are made by counterterms that look non-covariant from the $10d$ point of view [57, 58, 59]. A higher dimensional point of view is also relevant for extending the program to cover

¹¹The method has been developed in the Lagrangian formalism. One can also recast holographic RG flows in the Hamilton-Jacobi formalism [39, 40]. By exploring the relation between the Lagrangian and Hamilton-Jacobi formalisms, one can transcribe all steps of the holographic renormalization method to the Hamilton-Jacobi (HJ) method. The issue of renormalization in this setting has been addressed in [51].

dualities such as [60].

In general, fully understanding how to deal with degenerate boundaries would most likely lead to developing tools that are useful in extending the program of holographic renormalization to more general geometries.

Another future direction is to apply the holographic renormalization technique to the DBI action. This would be relevant for computing correlation functions in the defect RG flows [61, 62, 63] that arise in the context of the AdS/dCFT duality [64]. Relevant work can be found in [65]. Finally, it would also be interesting to investigate the generalization of the method to the plane wave geometry.

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A Analytic continuation to De Sitter

Recently there has been interest in a possible holographic interpretation of dS gravity [66, 67]. Looking for supporting evidence to a possible dS/CFT correspondence, many authors investigated the asymptotic symmetries of asymptotically de Sitter spacetime and the associated conserved charges [67, 68, 69, 70, 71, 18]. We show in this appendix that the near-boundary analysis of asymptotically AdS spacetimes can be analytically continued to asymptotically dS spacetimes. It follows that counterterms and exact 1-point functions in the presence of sources have a straightforward continuation. A particular case is the holographic stress energy tensor. Thus, although the results on the asymptotic symmetries are consistent with a possible dS/CFT correspondence, they do not necessarily constitute a piece of evidence for such a correspondence since they follow from corresponding AdS results and their holographic interpretation may be due to the AdS/CFT duality. We emphasize that this discussion is valid for local properties only. The global properties of de Sitter spacetime differ significantly from

the global properties of AdS spacetimes. To compute correlation functions using the methods described in this review, one has to address global issues as well. Exact solutions for massive scalar fields on de Sitter spacetimes have been recently discussed in this context in [72, 74].

Recall that Einstein's equations in the two cases read (our curvature conventions are given in footnote 2),

$$R_{\mu\nu} = \frac{d}{l_{AdS}^2} G_{\mu\nu} \quad \text{AdS} \quad (\text{A.1})$$

$$R_{\mu\nu} = -\frac{d}{l_{dS}^2} G_{\mu\nu} \quad \text{dS} \quad (\text{A.2})$$

So the one equation is mapped to the other by

$$l_{AdS}^2 \leftrightarrow -l_{dS}^2 \quad (\text{A.3})$$

Let us now consider the metrics. The Euclidean AdS metric in Poincaré coordinates,

$$ds_{AdS}^2 = \frac{l_{AdS}^2}{r^2} (dr^2 + dx^i dx^i), \quad (\text{A.4})$$

is mapped to the “big bang” metric,

$$ds_{dS}^2 = \frac{l_{dS}^2}{t^2} (-dt^2 + dx^i dx^i), \quad (\text{A.5})$$

by using (A.3) and in addition take

$$r^2 \rightarrow -t^2. \quad (\text{A.6})$$

Notice that the de Sitter spacetime has two boundaries, one at past infinity, and another at future infinity. The metric (A.5) covers only half of de Sitter spacetime, and only one of the two boundaries is covered by this coordinate patch.

Inspection of the arguments presented in section 3 shows that near each of its boundaries, an asymptotically dS metric can be brought to the form,

$$\begin{aligned} ds_{dS}^2 &= l_{ds}^2 \left(-\frac{d\tilde{\rho}^2}{4\tilde{\rho}^2} + \frac{1}{\tilde{\rho}} \tilde{g}_{ij}(x, \tilde{\rho}) dx^i dx^j \right) \\ \tilde{g}_{ij}(x, \tilde{\rho}) &= \tilde{g}_{(0)ij} + \tilde{\rho} \tilde{g}_{(2)ij} + \cdots \end{aligned} \quad (\text{A.7})$$

where $\tilde{\rho}$ is related to t by $\tilde{\rho} = t^2$. As we discussed in section 3, the near-boundary equations for the coefficients $g_{(k)ij}$ are algebraic. It follows that one can immediately write the solutions for $\tilde{g}_{(k)ij}$ starting from the solutions $g_{(k)ij}$ relevant for the

asymptotically AdS space,

$$\begin{aligned}\tilde{g}_{(4k)ij} &= g_{(4k)ij} \\ \tilde{g}_{(4k+2)ij} &= -g_{(4k+2)ij},\end{aligned}\tag{A.8}$$

where $k = 0, 1, \dots$. The minus sign originates from (A.6), which implies $\rho \rightarrow -\tilde{\rho}$.

Notice that there are two asymptotic expansions, one for each boundary. These expansions should be matched in the region of overlap of the two coordinate patches they cover. Given that the field equations are second order in $\tilde{\rho}$, if we fix $\tilde{g}_{(0)}$ at both boundaries, we effectively uniquely fix the solution throughout spacetime. The coefficients $\tilde{g}_{(d)ij}(x)$ that the near-boundary analysis does not determine are now determined by matching the two asymptotic expansions. Recall that correlation functions in the AdS/CFT correspondence are determined by obtaining global solutions, and reading off the coefficient $\tilde{g}_{(d)ij}$. The above argument suggests that the $\tilde{g}_{(d)ij}(x)$ is a local function of the sources, and thus the corresponding correlation functions consist of only contact terms. It would be interesting to make this argument more precise and to also investigate different boundary conditions. For a closely related discussion we refer to [73]. For now we follow the practice in most of current literature and focus on one of the two boundaries of dS.

Given the asymptotic solution one can proceed to renormalize the on-shell action. We will present in parallel both the AdS and dS case. The action is given by

$$S = \frac{1}{16\pi G} \left[\int_M d^{d+1}x \sqrt{|G|} (R + 2\Lambda) - \int_{\partial M} d^d x \sqrt{|\gamma|} 2K \right], \tag{A.9}$$

where K is the trace of the second fundamental form and γ is the induced metric on the boundary.

The (regularized) on-shell value of the bulk term in (A.9) can be computed using the following results. From Einstein equations we get,

$$\begin{aligned}\text{dS :} \quad R + 2\Lambda &= -\frac{2d}{l_{dS}^2} \\ \text{AdS :} \quad R + 2\Lambda &= \frac{2d}{l_{AdS}^2}\end{aligned}\tag{A.10}$$

Furthermore we need the trace of the second fundamental form

$$\begin{aligned}\text{dS :} \quad K &= -\frac{1}{l_{dS}^2} (d - \rho \text{Tr } g^{-1} \partial_\rho g) \\ \text{AdS :} \quad K &= \frac{1}{l_{AdS}^2} (d - \tilde{\rho} \text{Tr } \tilde{g}^{-1} \partial_{\tilde{\rho}} \tilde{g})\end{aligned}\tag{A.11}$$

where the relative minus sign in the dS case compared to AdS case originates from the fact that the normal to the boundary vector is spacelike in the one case and timelike in the other.

Inserting these values in (A.9) we get

$$\begin{aligned} S_{AdS} &= \frac{l_{AdS}^{d-1}}{16\pi G} \left(\int d^d x \left[\int_{\epsilon} \left(d\rho \frac{d\sqrt{\det g(x, \rho)}}{\rho^{d/2+1}} \right) - 2 \frac{\sqrt{\det g(x, \epsilon)}}{\epsilon^{d/2}} (d - \epsilon \text{Tr } g^{-1} \partial_{\epsilon} g) \right] \right) \\ S_{dS} &= -\frac{l_{dS}^{d-1}}{16\pi G} \left(\int d^d x \left[\int_{\tilde{\epsilon}} \left(d\tilde{\rho} \frac{d\sqrt{\det \tilde{g}(x, \tilde{\rho})}}{\tilde{\rho}^{d/2+1}} \right) - 2 \frac{\sqrt{\det \tilde{g}(x, \tilde{\epsilon})}}{\tilde{\epsilon}^{d/2}} (d - \tilde{\epsilon} \text{Tr } \tilde{g}^{-1} \partial_{\tilde{\epsilon}} \tilde{g}) \right] \right) \end{aligned}$$

Thus the regularized dS on-shell action can be obtained from the AdS one by simply multiplying by minus ones and taking $g \rightarrow \tilde{g}$. It follows that the corresponding counterterms required to render the action finite can also be obtained from the AdS ones by $g \rightarrow \tilde{g}$ and by multiplying by minus one.

We now want to compare the Brown-York stress energy tensor [17] in the AdS and dS cases. By definition,

$$T_{ij} = \frac{2}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}}{\delta g_{(0)}^{ij}} \quad (\text{A.12})$$

where S_{ren} is defined as usual (4.5). The computation of (A.12) was carried out in full generality in [8]. Since the renormalized actions of dS and AdS differ only by sign, the results carry over to the dS case as well. One simply has to multiply by an overall minus and change $g_{(k)}$ to $\tilde{g}_{(k)}$. Let us discuss in some detail the three and five dimensional cases. All other cases are similar.

- $d+1=3$ case

The relevant equation is (3.10) of [8]. We get

$$T_{ij}^{dS} = -\frac{l_{dS}}{8\pi G} (\tilde{g}_{(2)ij} - \tilde{g}_{(0)ij} \text{Tr } \tilde{g}_{(2)}) = \frac{l_{dS}}{8\pi G} (g_{(2)ij} - g_{(0)ij} \text{Tr } g_{(2)}) \quad (\text{A.13})$$

where we used (A.8). Thus,

$$T_{ij}^{dS} = T_{ij}^{AdS} \quad (\text{A.14})$$

and

$$T_i^i = -\frac{1}{24} \left(\frac{3l_{dS}}{2G} \right) R. \quad (\text{A.15})$$

Thus the central charge $c = 3l_{dS}/2G$ comes out positive.

- $d+1=5$ case

In this case the relevant equation is (3.15) of [8], and we get

$$T_{ij} = -\frac{4l_{dS}^3}{16\pi G_N} [\tilde{g}_{(4)ij} - \frac{1}{8}\tilde{g}_{(0)ij}[(\text{Tr } \tilde{g}_{(2)})^2 - \text{Tr } \tilde{g}_{(2)}^2] - \frac{1}{2}(\tilde{g}_{(2)}^2)_{ij} + \frac{1}{4}\tilde{g}_{(2)ij}\text{Tr } \tilde{g}_{(2)}] \quad (\text{A.16})$$

This expression involves $\tilde{g}_{(4)}$ or $\tilde{g}_{(2)}^2$. It follows from (A.8) that the dS stress energy tensor is minus the AdS one,

$$T_{ij}^{dS} = -T_{ij}^{AdS} \quad (\text{A.17})$$

In asymptotically AdS spacetimes positive energy theorems [75] guarantee that the mass of global AdS spacetime is the lowest possible mass. In the AdS/CFT correspondence, this translates to the statement the ground state energy is the lowest energy of the system. Provided there are no subtleties in converting the local statement (A.17) to the statement about the corresponding charges, the relation (A.17) implies that the dS “mass” as defined in [18] is the maximum possible mass among all asymptotically de Sitter spacetimes¹². This was conjectured to be true in [18].

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¹²One also needs to establish that the analytic continuation maps the “allowed” asymptotically dS solutions (i.e. without cosmological singularities) to well-behaved asymptotically AdS solutions (i.e. no naked singularities).

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