Task 2.

1.Exact solution:

y' = -2*y +4*x   y(0)=0
y'+2*y=4*x
y=u*y1
y1'+2*y1=0
y1'/y1=-2
ln|y1|=-2*x
y1=e^(-2*x)
y = u*e^(-2*x)
u'*e^(-2*x)=4*x
u'=(4*x*e^(2*x))'
u=2*x*e^(2*x) - e^(2*x) +c
y = (2*x*e^(2*x) - e^(2*x) +c)*e^(-2*x)
y = 2*x+c*e^(-2*x)-1
y(0)=c*e^(0)-1=0
c=1
y = 2*x+e^(-2*x)-1

2.Structure of the program:

In my program I have functions: `euler`, `impr_euler`,`runge_kutta` where I provide numerical solutions for the given function. They refer to `f`, the original task. I use function `real_f` to find the exact solution. I also have functions `show_exact`, `show_euler`, `show_imp_euler`, `show_runge`, `show_all_methods` to show different graphs for different methods if we need to see them separately or at the same time. And also `show_errors` and `error_steps` for showing errors.

3. Description of each method:

- ```
  def euler(x,y,h):
    for i in range(1,len(x)):
      y[i]=y[i-1]+ h*f(x[i-1],y[i-1])
  ```

This function uses The Euler method to solve f, the original task. For every x from 0 to X we find y, using this formula:
  ```

$$y_{i+1} = y(x_i) + hf(x_i, y(x_i))$$

- ```
  def impr_euler(x,y,h):
    for i in range(1,len(x)):
        k1=f(x[i-1],y[i-1])
        k2=f(x[i-1]+h,y[i-1]+h*k1)
        y[i]=y[i-1]+ h/2*(k1+k2)
  ```

This function uses The Improved Euler method to solve f, the original task. For every x from 0 to X we find y, using this formula:

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f(x_i + h, y_i + hk_{1i}), \\ y_{i+1} &= y_i + \frac{h}{2}(k_{1i} + k_{2i}). \end{aligned}$$

- ```
  def runge_kutta(x,y,h):
      for i in range(1,len(x)):
          k1=f(x[i-1],y[i-1])
          k2=f(x[i-1]+h/2,y[i-1]+(h/2)*k1)
          k3=f(x[i-1]+h/2,y[i-1]+(h/2)*k2)
          k4=f(x[i-1] +h,y[i-1]+h*k3)
          y[i]=y[i-1]+(h/6)*(k1+2*k2+2*k3+k4)
  ```

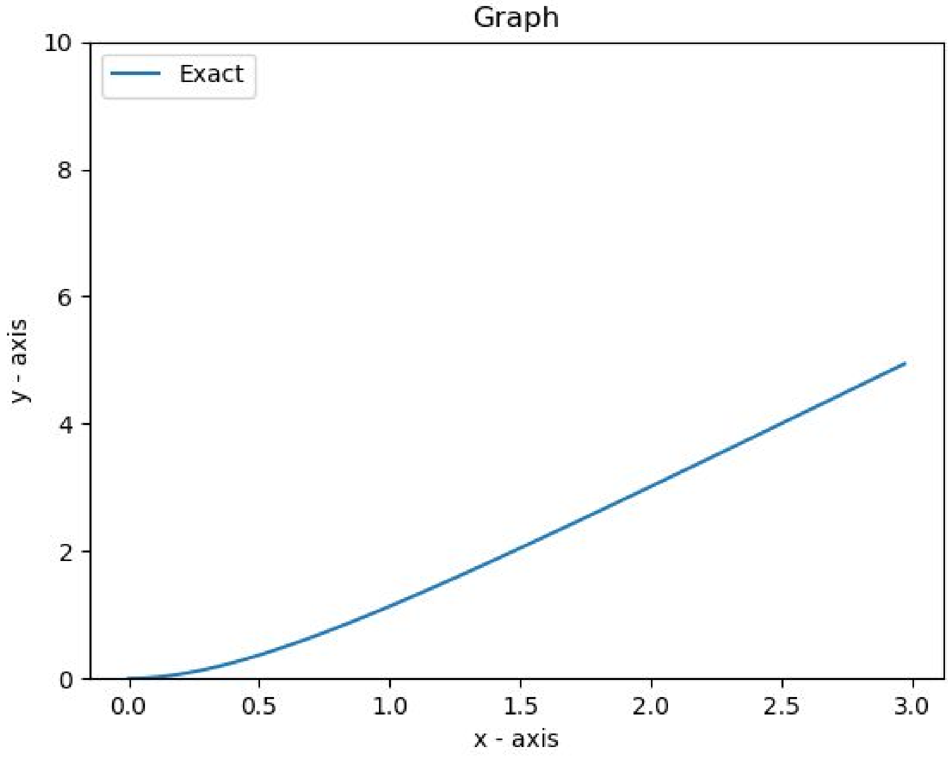
This function uses The Runge-Kutta method to solve f, the original task. For every x from 0 to X we find y, using this formula:

$$\begin{aligned} k_{1i} &= f(x_i, y_i), \\ k_{2i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{1i}\right), \\ k_{3i} &= f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_{2i}\right), \\ k_{4i} &= f(x_i + h, y_i + hk_{3i}). \end{aligned}$$

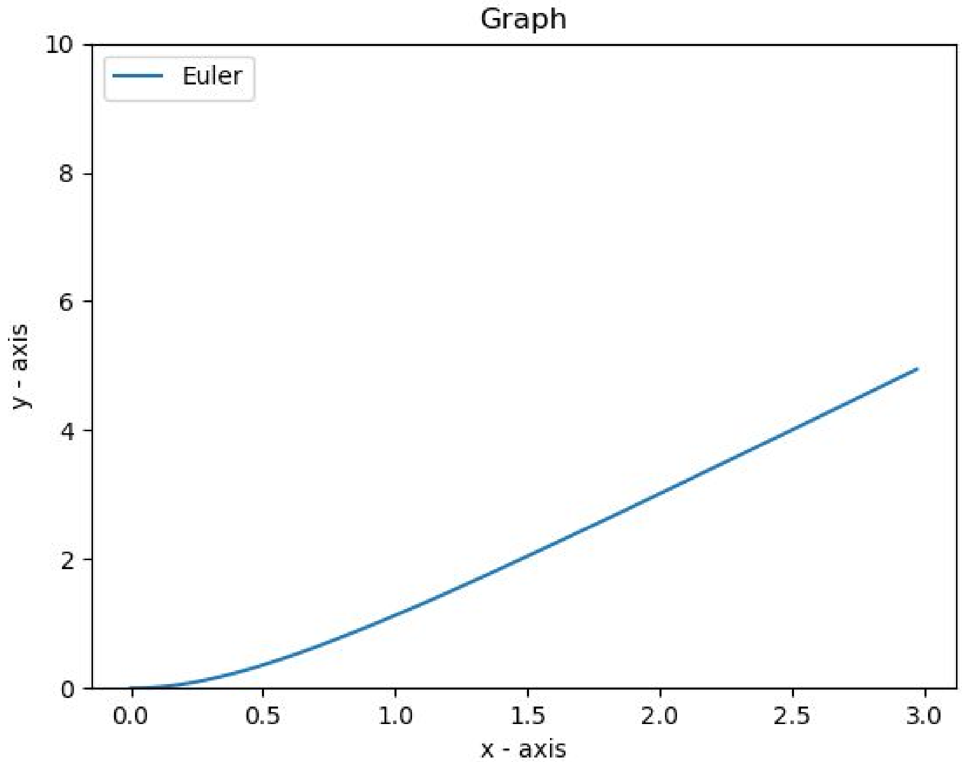
$$y_{i+1} = y_i + \frac{h}{6}(k_{1i} + 2k_{2i} + 2k_{3i} + k_{4i}).$$

4. Graphs

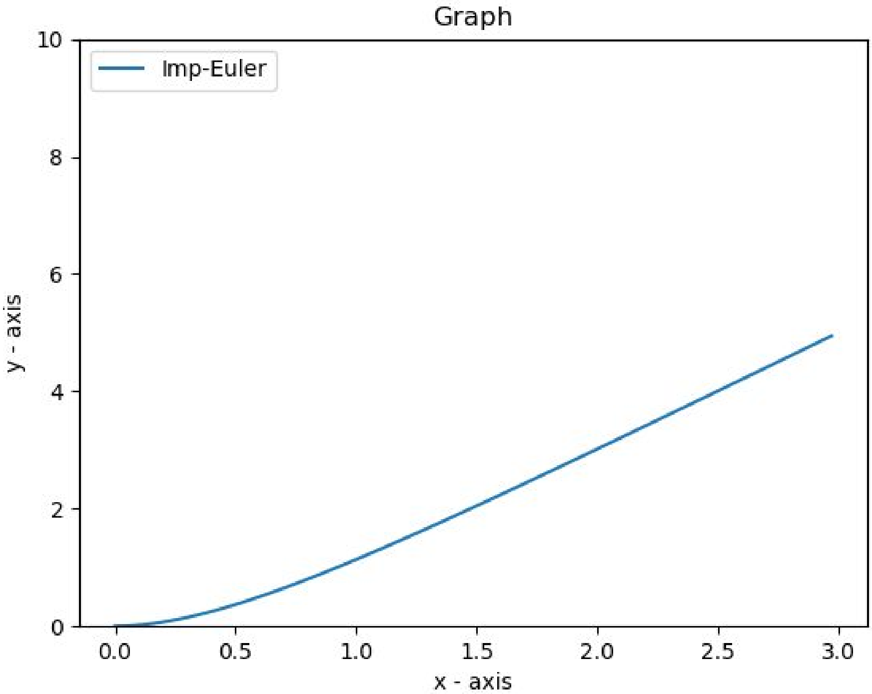
Graph for the exact solution (how other solutions are supposed to look like):



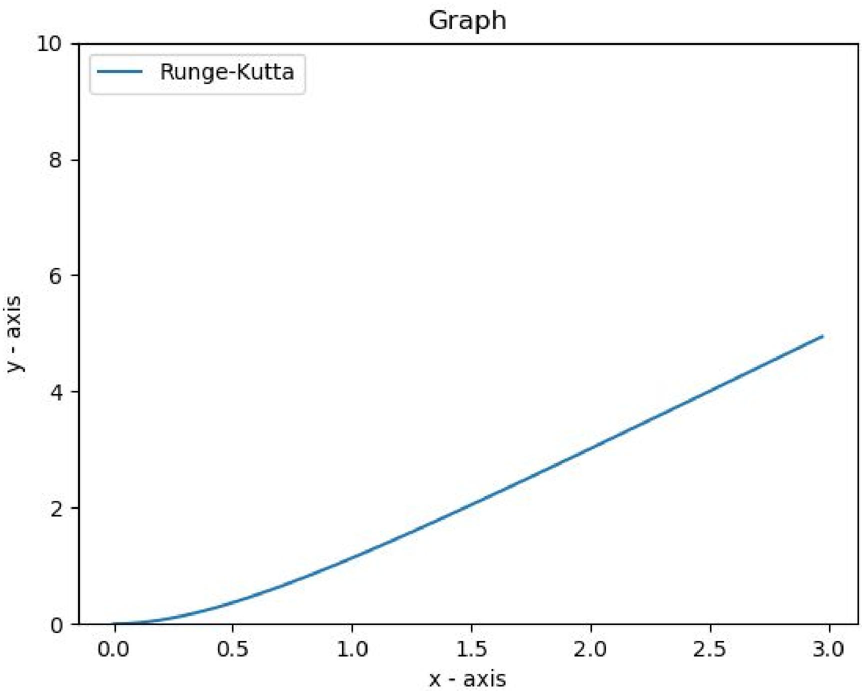
Graph for the Euler method:



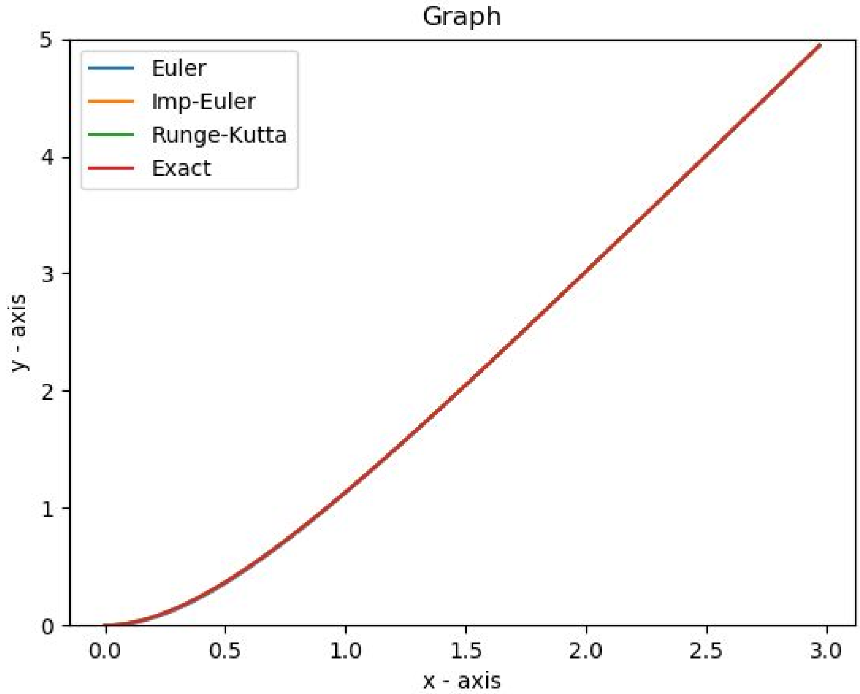
Graph for the Improved Euler method:



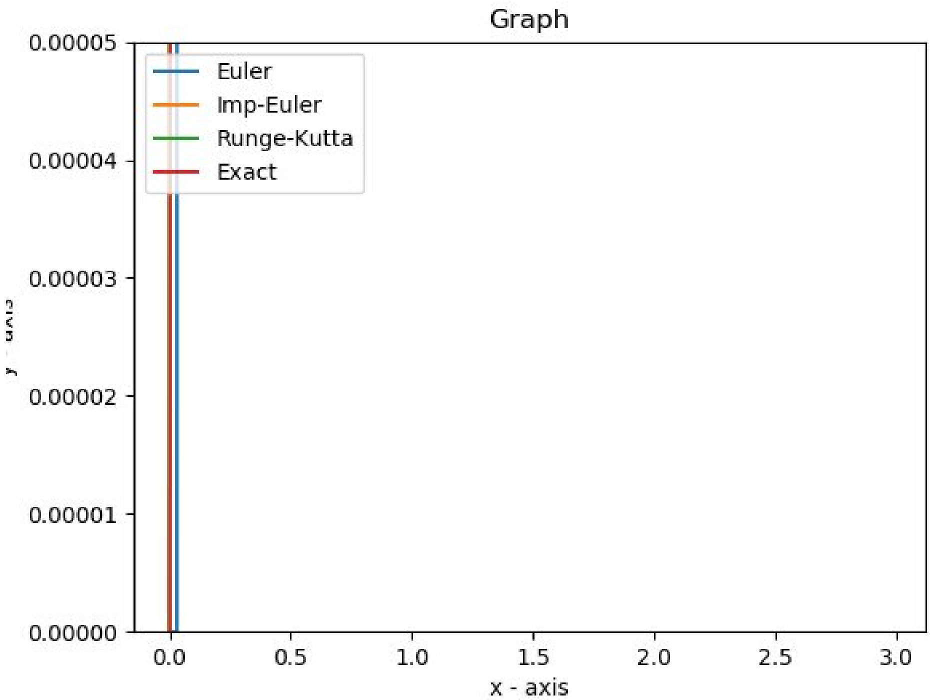
Graph for the The Runge-Kutta method:



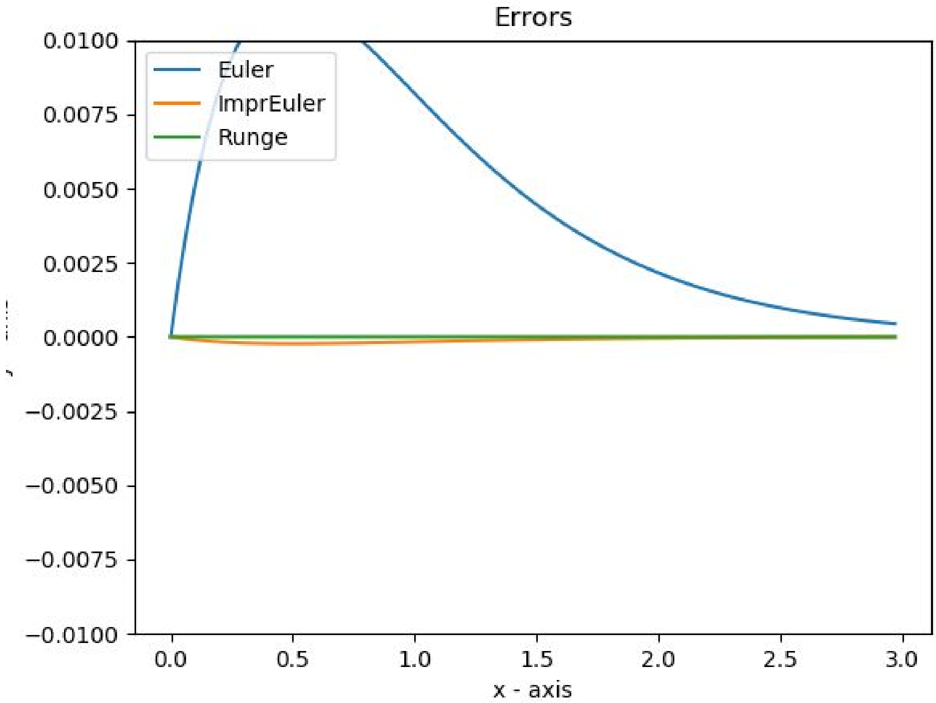
Since they look similar, I would like to show how they look when i put them next to each other:



But the difference between graphs is more obvious if I change limits



And the Error graph



5. Global error.

For finding the best method, we can take a look at the global error: