Задача 6.

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Пусть

$$\mathbf{v} = \sum_{i=1}^{n} \xi_i(x) \partial_{x_i}, \quad \mathbf{u} = \sum_{i=1}^{n} \eta_i(x) \partial_{x_i},$$
$$\operatorname{pr}^{(1)} \mathbf{v}(L) = \dot{A}, \quad \operatorname{pr}^{(1)} \mathbf{u}(L) = \dot{B}.$$

Тогда

$$\begin{split} [\mathbf{v}, \mathbf{u}] &= \mathbf{v} \left(\sum_{i} \eta_{i} \partial_{x_{i}} \right) - \mathbf{u} \left(\sum_{j} \xi_{j} \partial_{x_{j}} \right) = \sum_{j} \xi_{j} \partial_{x_{j}} \left(\sum_{i} \eta_{i} \partial_{x_{i}} \right) - \sum_{i} \eta_{i} \partial_{x_{i}} \left(\sum_{j} \xi_{j} \partial_{x_{j}} \right) = \\ &= \sum_{j} \xi_{j} \sum_{i} (\partial_{x_{j}} \eta_{i} \partial_{x_{i}} + \eta_{i} \partial_{x_{j}} \partial_{x_{i}}) - \sum_{i} \eta_{i} \sum_{j} (\partial_{x_{i}} \xi_{j} \partial_{x_{j}} + \xi_{j} \partial_{x_{i}} \partial_{x_{j}}) = \\ &= \sum_{i} \sum_{j} \left(\xi_{j} \frac{\partial \eta_{i}}{\partial x_{j}} - \eta_{j} \frac{\partial \xi_{i}}{\partial x_{j}} \right) \partial_{x_{i}}, \\ &\operatorname{pr^{(1)}}[\mathbf{v}, \mathbf{u}] = [\mathbf{v}, \mathbf{u}] + \sum_{i} \sum_{j} \left(\dot{\xi}_{j} \frac{\partial \eta_{i}}{\partial x_{j}} + \xi_{j} \left(\frac{\partial \eta_{i}}{\partial x_{j}} \right) - \dot{\eta}_{j} \frac{\partial \xi_{i}}{\partial x_{j}} - \eta_{j} \left(\frac{\partial \xi_{i}}{\partial x_{j}} \right) \right) \partial_{\dot{x}_{i}}. \end{split}$$

Заметим, что

$$\left(\frac{\partial \eta_i}{\partial x_j}\right) = \sum_k \frac{\partial^2 \eta_i}{\partial x_j \partial x_k} \dot{x}_k = \frac{\partial}{\partial x_j} \dot{\eta}_i.$$

Отсюда

$$[\operatorname{pr}^{(1)}\mathbf{v}, \operatorname{pr}^{(1)}\mathbf{u}] = \sum_{i} \left(\xi_{i} \sum_{j} \left(\frac{\partial \eta_{j}}{\partial x_{i}} \partial_{x_{j}} + \eta_{j} \partial_{x_{i}} \partial_{x_{j}} + \frac{\partial \dot{\eta}_{j}}{\partial x_{i}} \partial_{\dot{x}_{j}} + \dot{\eta}_{j} \partial_{x_{i}} \partial_{\dot{x}_{j}} \right) + \\ + \dot{\xi}_{i} \sum_{j} \left(\eta_{j} \partial_{\dot{x}_{i}} \partial_{x_{j}} + \frac{\partial \eta_{j}}{\partial x_{i}} \partial_{\dot{x}_{j}} + \dot{\eta}_{j} \partial_{\dot{x}_{i}} \partial_{\dot{x}_{j}} \right) \right) - \sum_{j} (...),$$

и после взаимного уничтожения одинаковых слагаемых получаем

$$\begin{aligned} [\operatorname{pr}^{(1)}\mathbf{v},\operatorname{pr}^{(1)}\mathbf{u}] &= \\ &= \sum_{i} \sum_{j} \left(\xi_{j} \frac{\partial \eta_{i}}{\partial x_{j}} - \eta_{j} \frac{\partial \xi_{i}}{\partial x_{j}} \right) \partial_{x_{i}} + \sum_{i} \sum_{j} \left(\xi_{j} \frac{\partial \dot{\eta}_{i}}{\partial x_{j}} + \dot{\xi}_{j} \frac{\partial \eta_{i}}{\partial x_{j}} - \eta_{j} \frac{\dot{\xi}_{i}}{x_{j}} - \dot{\eta}_{j} \frac{\partial \xi_{i}}{\partial x_{j}} \right) \partial_{\dot{x}_{i}} = \\ &= \operatorname{pr}^{(1)}[\mathbf{u}, \mathbf{v}]. \end{aligned}$$

Нам достаточно доказать

$$[\operatorname{pr}^{(1)}\mathbf{v}, \operatorname{pr}^{(1)}\mathbf{u}](L) = \frac{d}{dt}(\operatorname{pr}^{(1)}\mathbf{v}(B) - \operatorname{pr}^{(1)}\mathbf{u}(A)).$$

Заметим, что

$$\begin{split} \frac{\partial}{\partial x_i} \frac{dA}{dt} &= \frac{\partial^2 A}{\partial t \partial x_i} + \sum_{\beta \geq 0} \sum_{\alpha} \frac{\partial^2 A}{\partial x_i \partial x_{\alpha}^{(\beta)}} x_{\alpha}^{(\beta+1)} = \frac{d}{dt} \frac{\partial A}{\partial x_i}, \\ \frac{\partial}{\partial \dot{x}_i} \frac{dA}{dt} &= \frac{\partial}{\partial \dot{x}_i} \left(\frac{\partial A}{\partial t} + \sum_{\alpha} \frac{\partial A}{\partial x_{\alpha}} \dot{x}_{\alpha} + \sum_{\beta \geq 1} \sum_{\alpha} \frac{\partial A}{\partial x_{\alpha}^{(\beta)}} x_{\alpha}^{(\beta+1)} \right) = \frac{d}{dt} \frac{\partial A}{\partial \dot{x}_i} + \frac{\partial A}{\partial x_i}. \end{split}$$

Значит,

$$\begin{split} \frac{d}{dt}(\mathbf{v}(B) - \mathbf{u}(A)) &= \\ &= \sum_{i} \left(\dot{\xi}_{i} \frac{\partial B}{\partial x_{i}} + \xi_{i} \frac{\partial \dot{B}}{\partial x_{i}} + \dot{\xi}_{i} \left(\frac{\partial \dot{B}}{\partial \dot{x}_{i}} - \frac{\partial B}{\partial x_{i}} \right) - \dot{\eta}_{i} \frac{\partial A}{\partial x_{i}} - \eta_{i} \frac{\partial \dot{A}}{\partial x_{i}} - \dot{\eta}_{i} \left(\frac{\partial \dot{A}}{\partial \dot{x}_{i}} - \frac{\partial A}{\partial x_{i}} \right) \right) = \\ &= \sum_{i} \left(\xi_{i} \sum_{k} \left(\frac{\partial \eta_{k}}{\partial x_{i}} \frac{\partial L}{\partial x_{k}} + \eta_{k} \frac{\partial^{2} L}{\partial x_{i} \partial x_{k}} + \frac{\partial \dot{\eta}_{k}}{\partial x_{i}} \frac{\partial L}{\partial \dot{x}_{k}} + \dot{\eta}_{k} \frac{\partial^{2} L}{\partial \dot{x}_{k} \partial x_{i}} \right) + \\ &+ \dot{\xi}_{i} \sum_{k} \left(\eta_{k} \frac{\partial^{2} L}{\partial x_{k} \partial \dot{x}_{i}} + \frac{\partial \eta_{k}}{\partial x_{i}} \frac{\partial L}{\partial \dot{x}_{k}} + \dot{\eta}_{k} \frac{\partial^{2} L}{\partial \dot{x}_{k} \partial \dot{x}_{i}} \right) - \dots \right), \end{split}$$

и после взаимного уничтожения одинаковых слагаемых получаем

$$\frac{d}{dt}(\mathbf{v}(B) - \mathbf{u}(A)) = \sum_{i} \left(\xi_{i} \sum_{k} \left(\frac{\partial \eta_{k}}{\partial x_{i}} \frac{\partial L}{\partial x_{k}} + \frac{\partial \dot{\eta}_{k}}{\partial x_{i}} \frac{\partial L}{\partial \dot{x}_{k}} \right) - \eta_{i} \sum_{k} \left(\frac{\partial \xi_{k}}{\partial x_{i}} \frac{\partial L}{\partial x_{k}} + \frac{\partial \dot{\xi}_{k}}{\partial x_{i}} \frac{\partial L}{\partial \dot{x}_{k}} \right) \right) = \\
= [\operatorname{pr}^{(1)} \mathbf{v}, \operatorname{pr}^{(1)} \mathbf{u}](L),$$

что и требовалось.