

UGA M1: Econometrics 1

Probability

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References

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- These lectures are mostly based on Wooldridge, so that will generally be the best reference to read.
- The others are some freely available alternatives that cover similar material.
- ? is most similar to Wooldridge in terms of length and its focus on econometrics.
- ? and ? also cover similar material, but explain things in more depth.

Probability

- Purpose: system for quantifying chance and making predictions about future events
- Interpretations:
 - Relative frequency of an event in many repeated trials
 - Subjective assessment of the likelihood of an event

Basic definitions

- The properties of probability are fairly intuitive, and you can work with probabilities without worrying too much about how probability is formally defined.
- However, if we want to be mathematically rigorous, we should carefully define what we mean by probability.
- This slide formally defines probability. Do not worry if it seems very abstract.

Basic definitions

- **Random experiment:** procedure that has well-defined set of outcomes and “could” be infinitely repeated
- **Sample space:** set of possible outcomes of an experiment,
 $S = \{a_1, a_2, \dots, a_J\}$
- **Event:** any subset of sample space, $A \subseteq S$
- **Probability:** function from all subsets of the sample space, S , to $[0, 1]$ such that
 1. $\mathbb{P}(S) = 1$
 2. $1 \geq \mathbb{P}(A) \geq 0$ for all $A \subseteq S$
 3. If A_1, A_2, \dots are disjoint events then $\mathbb{P}(A_1 \cup A_2 \cup \dots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \dots$
- **Random variable:** function from S to a number

Basic definitions

- An example of a random experiment is rolling a dice. In this case the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
- Some events are: get a 1 = $\{1\}$, get more than 4, $\{5, 6\}$, etc.
- A random variable is a numeric representation of a random experiment.
- For example, our experiment could be flipping a coin. Then $S = \{\text{heads}, \text{tails}\}$.
- A function assigns a number to each element of S , so one example of a random variable is $X = \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$.
- Another is $Y = \begin{cases} 50 & \text{if heads} \\ -257 & \text{if tails} \end{cases}$.
- A third example is $Z = \begin{cases} 1 & \text{if heads} \\ 1 & \text{if tails} \end{cases}$.

Properties of probability

1. $\mathbb{P}(\emptyset) = 0$
2. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
3. $A \subseteq B$ implies $\mathbb{P}(A) \leq \mathbb{P}(B)$
4. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Conditional probability

- **Conditional probability:** $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$
 - Satisfies axioms of unconditional probability (and so also has properties on previous slide)
- **Bayes' Rule** $\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$
- A is **independent** of B iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, denote as $A \perp\!\!\!\perp B$
 - $A \perp\!\!\!\perp B$ implies $\mathbb{P}(A|B) = \mathbb{P}(A)$ and $\mathbb{P}(B|A) = \mathbb{P}(B)$.

Discrete random variables

- Recall: a **random variable** is a function from S to a number
- A random variable is **discrete** if it can only take countably many different values
- E.g. $X \in \{x_1, \dots, x_m\}$
- **Probability mass function (PMF)** of X , $p_i = \mathbb{P}(X = x_i)$ gives the probability that X equals each of its possible values
- **Cumulative distribution function (CDF)**:
$$F(x) = \mathbb{P}(X \leq x) = \sum_{i=1}^m p_i 1\{x_i \leq x\}$$

Continuous random variables

- A random variable is **continuous** if it takes on any single real value with zero probability
- Set of possible values is uncountably infinite (e.g. real line or line segment)
- CDF is continuous and differentiable
- **Probability density function (PDF)**: the derivative of the CDF

$$f(x) = \frac{dF}{dx}(x) \quad \text{and} \quad F(x) = \int_{-\infty}^x f(t)dt$$

Bivariate distributions

■ Discrete:

- Joint PMF $f_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$
- Marginal PMF $f_X(x) = \mathbb{P}(X = x) = \sum_{y \in \text{all values of } Y} \mathbb{P}(X = x, Y = y)$
- Conditional PMF $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

■ Continuous:

- joint CDF $F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^x \int_{-\infty}^y f_{X,Y}(u,t) du dt$
- joint PDF $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}}{\partial x \partial y}(x,y)$
- Marginal CDF for X , $F_X(x) = \mathbb{P}(X \leq x) = F_{X,Y}(x, \infty)$
- Marginal PDF for X , $f_X(x) = \frac{dF_X}{dx}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
- Conditional PDF $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$

Independence

- Random variables X, Y are independent if $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all x, y
- $X \perp\!\!\!\perp Y$ implies $f_{X|Y}(x|y) = f_X(x)$

Expectation

- The **expectation** of X is

$$\mathbb{E}[X] = \begin{cases} \sum_{i=1}^m x_i f_X(x_i) & \text{if } X \text{ discrete} \\ \int_{-\infty}^{\infty} x f_X(x) dx & \text{if } X \text{ continuous} \end{cases}$$

- Is a constant
- a.k.a. expected value, population average, population mean, first moment
- Expectation is linear:

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

- Expectation of function g :

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Variance and other moments

- **k th moment** of X : $\mathbb{E}[X^k]$
- **k th central moment** of X : $\mathbb{E}[(X - \mathbb{E}[X])^k]$
- **Variance** is the 2nd central moment

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2\end{aligned}$$

- $\text{Var}(a + bX) = b^2 \text{Var}(X)$
- **Standard deviation** is $\sqrt{\text{Var}(X)}$

Covariance

- **Covariance** of X and Y :

$$\begin{aligned}\text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[(X - \mathbb{E}[X])Y] = \mathbb{E}[X(Y - \mathbb{E}[Y])]\end{aligned}$$

- $\text{Cov}(X, X) = \text{Var}(X)$
- $\text{Cov}(a_1 + b_1X + c_1Y, a_2 + b_2X + c_2Y) = b_1b_2\text{Var}(X) + c_1c_2\text{Var}(Y) + (b_1c_2 + c_1b_2)\text{Cov}(X, Y)$
- $\text{Var}(a + bX + cY) = b^2\text{Var}(X) + c^2\text{Var}(Y) + 2bc\text{Cov}(X, Y)$
- $\text{Var}\left(\sum_{i=1}^N b_i X_i\right) = \sum_{i=1}^N \left(\sum_{j=1}^N b_i b_j \text{Cov}(X_i, X_j)\right)$

Correlation

- **Correlation** of X and Y is $\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$
- **Cauchy-Schwartz inequality**: $|\text{Cov}(X, Y)| \leq \sqrt{\text{Var}(X)\text{Var}(Y)}$
 - Implies $-1 \leq \text{Corr}(X, Y) \leq 1$
- $\text{Corr}(X, Y) = \pm 1$ iff $Y = a + bX$

Conditional expectation

- **Conditional expectation** of Y given $X = x$:

$$\mathbb{E}[Y|X = x] = \begin{cases} \sum_{i=1}^m y_i f_{Y|X}(y_i|x) & \text{if discrete} \\ \int_{-\infty}^{\infty} y f_{Y|X}(y|x) dy & \text{if continuous} \end{cases}$$

- $\mathbb{E}[Y|X]$ is a function of X
- Has all properties of unconditional expectation
- Properties:
 - $\mathbb{E}[g_1(X) + g_2(X)Y|X] = g_1(X) + g_2(X)\mathbb{E}[Y|X]$
 - **Law of iterated expectations:** $\mathbb{E}_X [\mathbb{E}_{Y|X}[Y|X]] = \mathbb{E}[Y]$
 - $\mathbb{E}[X(Y - \mathbb{E}[Y|X])] = 0$

Conditional variance

- **Conditional variance:** $\text{Var}(Y|X) = \mathbb{E}[(Y - \mathbb{E}[Y|X])^2 | X]$
- Relation to variance:

$$\text{Var}(Y) = \mathbb{E}_X [\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$$

- Analysis of variance (ANOVA)
- $\mathbb{E}_X [\text{Var}(Y|X)]$ is within- X variance
- $\text{Var}(\mathbb{E}[Y|X])$ is between- X variance

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