Macroeconomics
Lecture 5 – Permanent Income

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Overview

1. Multi-period consumer problem

2. Permanent Income Hypothesis
   - A simplified model
   - Uncertainty

3. Multi-period small open economy

Table of Contents

Multi-period budget constraints

The consumer lives from $t = 0$ till $t = T$ ($T + 1$ periods)

Initial condition: savings S_{-1} with interest r_{-1}:

$$\begin{cases} C_0 + S_0 = Y_0 + (1 + r_{-1})\, S_{-1} \\ C_1 + S_1 = Y_1 + (1 + r_0)\, S_0 \\ \qquad \cdots \\ C_t + \underline{S_t} = Y_t + (1 + r_{t-1}) S_{t-1} \\ C_{t+1} + S_{t+1} = Y_{t+1} + (1 + r_t)\, \underline{S_t} \\ \qquad \cdots \\ C_T + S_T = Y_T + (1 + r_{T-1}) S_{T-1} \end{cases}$$

$T \to \infty$ can also be considered $\Rightarrow$ **infinite-horizon** model

Inter-temporal budget constraint: step by step

1. Solve for S_0 in period **1** budget constraint:

$$C_1 + S_1 = (1 + r_0)S_0 + Y_1 - C_1 \Leftrightarrow S_0 = \frac{C_1 - Y_1 + S_1}{1 + r_0}$$

2. Plug the obtained expression of S_0 in the period **0** budget constraint. S_0 is eliminated, but now S_1 in period 0 constraint

3. Do step 1 for S_1 in period **2** constraint

4. Plug obtained expression of S_1 back into equation of step 2. S_1 eliminated, but now S_2 in the equation

5. Repeat for $S_2, S_3 \ldots S_T$

$$C_0 + S_0 = Y_0 + (1+r_{-1})S_{-1} \Rightarrow C_0 + \frac{C_1 - Y_1 + S_1}{1+r_0} = Y_0 + (1+r_{-1})S_{-1}$$

$$C_1 + S_1 = Y_1 + (1+r_0)S_0$$

$$(1+r_0)S_0 = C_1 + S_1 - Y_1$$

$$S_0 = \frac{C_1 - Y_1 + S_1}{1+r_0}$$

$$C_2 + S_2 = Y_2 + (1+r_1)S_1$$

$$S_1 = \frac{C_2 - Y_2 + S_2}{1+r_1}$$

$$\Downarrow$$

$$C_0 + \frac{C_1 - Y_1}{1+r_0} + \frac{C_2 - Y_2 + S_2}{(1+r_0)(1+r_1)} =$$

$$= Y_0 + (1+r_{-1})S_{-1}$$

$$C_0 + \frac{C_1}{1+r_0} + \frac{C_2}{(1+r_0)(1+r_1)} = Y_0 + \frac{Y_1}{1+r_0} + \frac{Y_2}{(1+r_0)(1+r_1)} + (1+r_{-1})S_{-1} - \frac{S_2}{(1+r_0)(1+r_1)}$$

$$C_0 + \frac{C_1}{1+r_0} + \frac{C_2}{(1+r_0)(1+r_1)} = Y_0 + \frac{Y_1}{1+r_0} + \frac{Y_2}{(1+r_0)(1+r_1)} + (1+r_{-1})S_{-1} - \frac{S_2}{(1+r_0)(1+r_1)}$$

$$C_0 + \frac{C_1}{1+r_0} + \frac{C_2}{(1+r_0)(1+r_1)} + \frac{C_3}{(1+r_0)(1+r_1)(1+r_2)} + \dots + \frac{C_t}{(1+r_0)\times\dots\times(1+r_{t-1})} \dots$$

$$\dots + \frac{C_T}{(1+r_0)\times\dots\times(1+r_{T-1})} = \sum_{t=0}^{T} \frac{Y_t}{R_{0,t}} + (1+r_{-1})S_{-1} - \frac{S_T}{R_{0,T}}$$

$$R_{0,t} = (1+r_0)\times(1+r_1)\times\dots\times(1+r_{t-1})$$

$$\boxed{R_{0,0} = 1} \qquad R_{0,1} = (1+r_0)$$

$$\sum_{t=0}^{T} \frac{C_t}{R_{0,t}} = \sum_{t=0}^{T} \frac{Y_t}{R_{0,t}} + (1+r_{-1})S_{-1} - \underbrace{\frac{S_T}{R_{0,T}}}_{=0}$$

Inter-temporal budget constraint

Denote cumulative returns as follows:
$R_{0,0} = 1,\ R_{0,1} = (1 + r_0), R_{0,2} = (1 + r_0)(1 + r_1)$ and
$R_{0,t} = (1 + r_0)(1 + r_1)...(1 + r_{t-1})$

Then, the inter-temporal budget constraint with T periods is:

$$\sum_{t=0}^{T} \frac{C_t}{R_{0,t}} = (1 + r_{-1})S_{-1} + \sum_{t=0}^{T} \frac{Y_t}{R_{0,t}} + \underbrace{\frac{S_T}{R_{0,T}}}_{=0}$$

If we assume constant $r_t = r$, then $R_{0,t} = (1 + r)^t$ and we obtain:

$$\sum_{t=0}^{T} \frac{C_t}{(1 + r)^t} = (1 + r)S_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1 + r)^t} + \underbrace{\frac{S_T}{(1 + r)^T}}_{=0}$$

For $T \to \infty$, sufficient to assume that S does not grow/fall at a rate faster than r: $\lim_{T\to\infty} \frac{S_T}{(1+r)^T} = 0$ – **transversality condition**

Lifetime utility

Each next period's instantaneous utility gets multiplied by $\beta < 1$ one more time: **geometric discounting**

$$\begin{aligned} U(C_0, C_1, \ldots C_T) &= u(C_0) + \beta u(C_1) + \ldots + \beta^t u(C_t) + \ldots + \beta^T u(C_T) \\ &= \sum_{t=0}^{T} \beta^t u(C_t) \end{aligned}$$

if initial period t : $\sum_{s=t}^{T} \beta^{s-t} u(C_s)$

A dynamic Lagrangian

It is (paradoxically) more convenient to write a Lagrangian with $T+1$ period constraints instead of the inter-temporal one $\Rightarrow$ $T+1$ Lagrange multipliers $\lambda_0, ..., \lambda_t, ..., \lambda_T$:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t [u(C_t) + \lambda_t (Y_t + (1 + r_{t-1})S_{t-1} - C_t - S_t)]$$

- discounting with β^t applies to **both** period t utility and period t Lagrange multiplier (it doesn't mean anything, done for convenience)

- to obtain the Euler equation, derivative w.r.t. C_t and either C_{t+1} **or** S_t (S_t is harder, but useful for future models)

$$\frac{\partial \mathcal{L}}{\partial C_t} = \boxed{\beta^t \left(u'(C_t) + \lambda_t \cdot (-1) \right) = 0} \Rightarrow \quad u'(C_t) - \lambda_t = 0$$

$$\Leftrightarrow \lambda_t = u'(C_t)$$

Euler equation as derivative with respect to S_t

Write periods t and $t+1$ of the Lagrangian explicitly to see that S_t appears twice:

$$\frac{\beta^{t+1}}{\beta^t} = \beta$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\begin{aligned}
\mathcal{L} = & u(C_0) + \lambda_0(Y_0 + (1 + r_{-1})S_{-1} - C_0 - S_0) \\
& + \dots \\
& + \beta^t[u(C_t) + \underline{\lambda_t}(Y_t + (1 + r_{t-1})S_{t-1} - C_t - S_t)] \\
& + \beta^{t+1}[u(C_{t+1}) + \underline{\lambda_{t+1}}(Y_{t+1} + (1 + r_t)S_t - C_{t+1} - S_{t+1})] \\
& + \dots \\
& + \beta^T[u(C_T) + \lambda_T(Y_T + (1 + r_{T-1})S_{T-1} - C_T - S_T)]
\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial S} = \boxed{\beta^t \lambda_t (-1) + \beta^{t+1} \lambda_{t+1}(1 + r_t) = 0}$$

$$\frac{\partial \mathcal{L}}{\partial S_t} = \boxed{\beta^t \lambda_t (-1) + \beta^{t+1} \lambda_{t+1} (1+r_t) = 0}$$

divide by β^t: $-\lambda_t + \beta \lambda_{t+1} (1+r_t) = 0 \Leftrightarrow$

divide by $\beta\lambda_{t+1}$: $\begin{cases} \dfrac{\lambda_t}{\beta \lambda_{t+1}} = 1+r_t \\ \lambda_t = u'(C_t) \end{cases}$

$$\Leftrightarrow \frac{u'(C_t)}{\beta u'(C_{t+1})} = 1+r_t$$ – Euler equation

Table of Contents

Table of Contents

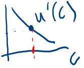
Permanent income hypothesis: simplifying assumptions

To simplify the study of temporary and permanent income shocks, make two assumptions:

1. Constant real interest $r_t = r$
2. $\beta = \frac{1}{1+r}$

Then, a constant consumption level $C_t = \bar{C}$ follows from Euler equation:

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r \Leftrightarrow u'(C_t) = \beta(1+r)u'(C_{t+1})$$



$$\Leftrightarrow u'(C_t) = \frac{1}{1+r}(1+r)u'(C_{t+1})$$

$$\Leftrightarrow u'(C_t) = u'(C_{t+1}) \Leftrightarrow C_t = C_{t+1} = \bar{C}$$

This is an extreme case of **consumption smoothing**

Permanent consumption: calculation

Substitute constant consumption level in intertemporal budget constraint, then use sum of geometric series:

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} = (1+r)S_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}$$

$\bar{C}$ $\quad \beta = \frac{1}{1+r}$

$$\bar{C} \sum_{t=0}^{T} \frac{1}{(1+r)^t} = (1+r)S_{-1} + \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}$$

$$\bar{C} \sum_{t=0}^{T} \beta^t = (1+r)S_{-1} + \sum_{t=0}^{T} \beta^t Y_t$$

$$\bar{C} \frac{1-\beta^{T+1}}{1-\beta} = (1+r)S_{-1} + \sum_{t=0}^{T} \beta^t Y_t$$

$$\boxed{\bar{C} = \frac{1-\beta}{1-\beta^{T+1}} \left((1+r)S_{-1} + \sum_{t=0}^{T} \beta^t Y_t \right)}$$

$$\frac{1}{(1+r)^t} = \left(\frac{1}{1+r}\right)^t = \beta^t$$

$$\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \quad \beta^0$$

$$\sum_{t=T+1}^{\infty} \beta^t = \frac{\beta^{T+1}}{1-\beta}$$

$$\sum_{t=0}^{T} \beta^t = \sum_{t=0}^{\infty} \beta^t - \sum_{t=T+1}^{\infty} \beta^t = \boxed{\frac{1-\beta^{T+1}}{1-\beta}}$$

Permanent consumption and income shocks: temporary

$$\bar{C} = \frac{1-\beta}{1-\beta^{T+1}}\left[(1+r_{-1})S_{-1} + \sum_{t=0}^{T}\beta^t Y_t\right]$$

where $\frac{1-\beta}{1-\beta^{T+1}} < 1$ **(check this using $\beta^{T+1} < \beta$)**

Derivative with respect to income in $t=0$: $\frac{\partial \bar{C}}{\partial Y_0} = \frac{1-\beta}{1-\beta^{T+1}}$
This is the **mpc** in the Keynesian sense: share of increase in **current** income that is consumed.

The mpc is smaller when the lifespan T is larger.
If $T \to \infty$, mpc $= 1-\beta$

Derivative with respect to income in any period t: $\frac{\partial \bar{C}}{\partial Y_t} = \beta^t \frac{1-\beta}{1-\beta^{T+1}}$
The **further** in the future the income shock, the **smaller the reaction** of permanent consumption

Permanent consumption and income shocks: permanent

$$\bar{C} = \frac{1-\beta}{1-\beta^{T+1}} \left[(1+r)S_{-1} + \sum_{t=0}^{T} \beta^t Y_t \right]$$

Take a total differential (with $dS_{-1} = 0$):

$$d\bar{C} = \frac{1-\beta}{1-\beta^{T+1}} \sum_{t=0}^{T} \beta^t dY_t$$

Now consider a **permanent shock**: $dY_0 = ... = dY_t = ... = dY_T$:

$$d\bar{C} = \frac{1-\beta}{1-\beta^{T+1}} \cdot dY_t \cdot \sum_{t=0}^{T} \beta^t = dY_t \cdot \underbrace{\frac{1-\beta}{1-\beta^{T+1}} \cdot \frac{1-\beta^{T+1}}{1-\beta}}_{=1}$$

We obtain $d\bar{C} = dY_t$, **a one-to-one reaction of consumption to a permanent change in income**

$$Y_{t+1} = \bar{Y} + \rho Y_t + \varepsilon_{t+1}$$

Savings response to temporary and permanent shocks

What happens to inital period savings S_0 under different income shocks?

$= \bar{C}$

Use formula $S_0 = (1+r)S_{-1} + Y_0 - C_0$:

1. Temporary shock of current income $Y_0 \uparrow$:
   $C_0 \uparrow$, but $dY_0 > dC_0$ because mpc $< 1 \Rightarrow$ $S_0 \uparrow$
2. Temporary shock of future income Y_t $(t > 0)$:
   only $C_0 \uparrow$ in period 0 $\Rightarrow$ $S_0 \downarrow$
3. Permanent income shock:
   one-to-one change in C_0 and $Y_0 \Rightarrow$ S_0 **unchanged**

Table of Contents

Uncertain income

Suppose now that income $\{Y_t\}_{t=0}^{T}$ is a random variable

Consumer then maximizes **expected utility**

$E_t X_k$ is expectation of period k variable under information available in t

The expected utility in period 0 is $E_0 \sum_{t=0}^{T} \beta^t u(C_t)$

The Lagrangian then also has an expectation:

$$\mathcal{L} = E_0 \sum_{t=0}^{T} \beta^t [u(C_t) + \lambda_t (Y_t + (1 + r_{t-1}) S_{t-1} - C_t - S_t)]$$

We will write the FOC for S_t under **period t information**:

$$\lambda_t = \beta E_t[(1 + r_t)\lambda_{t+1}] \;\Leftrightarrow\; u'(C_t) = \beta E_t[(1 + r_t) u'(C_{t+1})]$$

Permanent income under uncertainty

Assume again $r_t = r$ and $\beta = \frac{1}{1+r}$

Then, the Euler equation with information of period t becomes:

$$u'(C_t) = E_t u'(C_{t+1})$$

If we also assume a quadratic u, then we get $C_t = E_t C_{t+1} \Rightarrow$ consumer **expects** to have constant consumption; consumption can only change due to **unexpected shocks**

Table of Contents

Multi-period IIP dynamics

A small open economy exists for $T+1$ periods $t = 0, 1 \ldots T$. IIP_t is **IIP of beginning of period** t. World interest rate is constant at r. IIP dynamics depends on TB as before:

$$\begin{aligned}
IIP_1 &= (1+r)IIP_0 + TB_0 \\
&\ldots \\
IIP_{t+1} &= (1+r)IIP_t + TB_t \\
IIP_{t+2} &= (1+r)IIP_{t+1} + TB_{t+1} \\
&\ldots \\
IIP_T &= (1+r)IIP_{T-1} + TB_{T-1}
\end{aligned}$$

Country cannot have debts in last period ($IIP_T \geq 0$) nor hold assets in other countries ($IIP_T \leq 0$), so $IIP_T = 0$

IPP and future trade balances step by step

In order to obtain the relationship of initial IIP and all the trade balances, same approach as with inter-temporal budget constraint:

1. Solve for IIP_1 in equation with TB_1:

$$IIP_2 = (1+r)IIP_1 + TB_1 \Leftrightarrow IIP_1 = \frac{-TB_1 + IIP_2}{1+r}$$

2. Plug the obtained expression of IIP_1 in the equation with TB_0. IIP_1 is eliminated, but now IIP_2 in the equation

3. Do step 1 for IIP_2 in equation with TB_2

4. Plug obtained expression of IIP_2 back into initial equation. IIP_2 eliminated, but now IIP_3 in the equation

5. Repeat for $IIP_3, IPP_4, ...IIP_T$

Multi-period economy: trade balance and solvency

$$IIP_0 = \sum_{t=0}^{T} \frac{-TB_t}{(1+r)^{t+1}} + \underbrace{\frac{IIP_T}{(1+r)^{T+1}}}_{=0}$$

According to the initial value of International Investment Position, several scenarios for values of trade balance:

- If $IIP_t = 0$, then trade deficit in one period must be compensated by a surplus in **at least one** other period
- If $IIP_t > 0$, it is possible to have a trade deficit in each period and remain solvent
- If $IIP_t < 0$, the economy may have to run trade surpluses every period to remain solvent. The earlier the surplus, the bigger role it has for debt repayment (interest accumulation)

Transversality condition for $T \to \infty$: $\lim_{T\to\infty} \frac{IIP_T}{(1+r)^{T+1}} = 0$

Summary

- We studied microeconomics of consumption choice and dynamics of small open economies
- A consumer has forward-looking behaviour that leads to mpc< 1 and reaction to future income shocks
- Under specific assumptions on interest rate and discount factor, consumption is constant and reacts one-to-one to permanent income shocks
- Under uncertainty, expected utility is maximized subject to current information. Under simplifying assumptions, consumption changes only after unexpected shocks
- A small open economy has international investment position that changes with current account flows. In a representative consumer economy, IIP has similar dynamics to consumer savings