

# Macroeconomics

## Lecture 8 – New Keynesian Model

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# Outline

1 Household

2 Firms

- Flexible prices

# The basic New Keynesian model

- ▷ Also uses the **microfoundations** as in RBC framework
  - ▷ rational expectations
  - ▷ representative, infinitely lived agents
  - ▷ optimizing behaviour
- ▷ But **important differences**
  - ▷ a large number (continuum) of consumption goods
    - ⇒ not perfectly substitutable for HH ⇒ no perfect competition → **monopolistic competition**
  - ▷ prices for goods not flexible → **nominal rigidities**
- ▷ We will also make simplifications w.r.t. RBC: no capital accumulation → production with labor only
- ▷ Versions of this model widespread in central banks, commercial banks, public authorities, international organizations...

# The basic New Keynesian model

- ▷ **Households**

- ▷ consume **a bundle of diversified goods**
- ▷ supply labour
- ▷ make saving in a **nominal bond** (zero in equilibrium)

- ▷ **Firms**

- ▷ a continuum of firms of measure one
- ▷ each **producing a single, imperfectly substitutable good**
- ▷ only using labour as factor input
- ▷ pricing the good
  - ▷ under monopolistic competition
  - ▷ given **nominal rigidities** (but we start with a **flexible price** version today)

# Outline

## 1 Household

## 2 Firms

- Flexible prices

## Household

Household utility has the form  $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t)$ , and we will work with *isoelastic* utility for both  $C$  and  $L$ :

$$U(C_t, L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta}$$

where  $C_t$  is a **consumption indicator** constructed with a large number of goods, each having index  $i$ .

$C_t$  calculated with **aggregator function** proposed by Dixit and

Stiglitz:

$$C_t \equiv \left( \sum_{i=1}^N C_{it}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

↳ N discrete goods case

where  $C_t(i)$  is the quantity of **good  $i$**  consumed by the household.

Integrals reminder :

$$\int_0^1 C \cdot x(i) di = C \cdot \int_0^1 x(i) di$$

$$\int_0^1 x(i) x(j) di \geq x(j) \int_0^1 x(i) di$$

Each good has its own price  $P_t(i)$  set by a firm producing the good.

## Differentiated goods

- ▷ imperfectly-substitutable goods combined yield an aggregate good
  - ▷ Sometimes assumed that intermediary firms combine the goods for the household ⇒ the aggregator is their production function

$$C_t \equiv \left( \int_0^1 C_t(i)^{\frac{e-1}{e}} di \right)^{\frac{e}{e-1}}$$

(absolute value)

- ▷  $e$  is the **constant elasticity of substitution (CES)** between any pair of differentiated goods
- ▷ **Properties of the aggregator**
  - ▷ (1) symmetric, (2) strictly increasing, (3) strictly concave in all arguments, (4) homogeneous of degree one

## Household

Households maximize the consumption index  $C_t$  for any given level of expenditures  $\zeta_t \equiv \int_0^1 P_t(i)C_t(i)di$ . The solution yields a set of demand equations

$$C_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t \quad \text{for all } i \in [0, 1], \quad (1)$$

where  $P_t \equiv [\int_0^1 P_t(i)^{1-\varepsilon} di]^{1/(1-\varepsilon)}$  is an aggregate price index. This allows to write total consumption expenditure as

$$\int_0^1 P_t(i)C_t(i)di = P_t C_t$$

Optimization in period  $t$ :

- Notation: no period index

$$\frac{\varepsilon-1}{\varepsilon} - 1 = \frac{\varepsilon-1-\varepsilon}{\varepsilon} \quad ; i \text{ is index, not argument}$$

$$L = \left( \int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} + \lambda (S_0^1 - S_0^1 P_i C_i)$$

total income

$$\frac{\partial L}{\partial C_i} = \frac{\varepsilon}{\varepsilon-1} \cdot \left( \int_0^1 C_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{1}{\varepsilon-1}} \cdot \left( \frac{\varepsilon-1}{\varepsilon} \right) \cdot C_i^{-\frac{1}{\varepsilon}} - \lambda P_i$$

$$\boxed{\frac{d f(g(x))}{dx} = f'(g(x)) \cdot g'(x)}$$

$$\frac{\partial L}{\partial C_i} = 0 \Rightarrow C_i^{\frac{1}{\varepsilon}} C_i^{-\frac{1}{\varepsilon}} = \lambda P_i$$

$$C_i^{\frac{1}{\varepsilon}} C_j^{-\frac{1}{\varepsilon}} = \lambda P_i$$

$$\frac{C_i}{C_j} = \left( \frac{P_j}{P_i} \right)^{-\varepsilon}$$

$$\boxed{\left( \frac{C_i}{C_j} \right)^{-\frac{1}{\varepsilon}} = \frac{P_i}{P_j}}$$

$$S = S_0^1 P_i C_i = S_0^1 P_i \underbrace{\left(\frac{P_i}{P_j}\right)^{\varepsilon}}_{C_j} C_j = \left(\frac{1}{P_j}\right)^{\varepsilon} C_j \int_0^1 P_i^{1-\varepsilon} d_i$$

$$= P_i^\varepsilon C_j \int_0^1 P_i^{1-\varepsilon} d_i \Rightarrow C_j = \frac{S}{P_i^\varepsilon S_0^1 P_i^{1-\varepsilon} d_i}$$

~~$$C = \left( \int_0^1 C_j^{\frac{\varepsilon}{\varepsilon-1}} d_j \right)^{\frac{\varepsilon-1}{\varepsilon}}$$~~

$$= \left( \int_0^1 \frac{S^{\frac{\varepsilon-1}{\varepsilon}}}{P_i^{\varepsilon-1} (S_0^1 P_i^{1-\varepsilon} d_i)^{\frac{\varepsilon-1}{\varepsilon}}} d_j \right)^{\frac{\varepsilon-1}{\varepsilon}}$$

$$= S \frac{1}{S_0^1 P_i^{1-\varepsilon} d_i} \left( \int_0^1 \frac{1}{P_j^{\varepsilon-1}} d_j \right)^{\frac{\varepsilon-1}{\varepsilon}}$$

$$= S \left( \int_0^1 P_i^{1-\varepsilon} d_i \right)^{-1} \left( \int_0^1 P_i^{1-\varepsilon} d_i \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$= S \left( \int_0^1 P_i^{1-\varepsilon} d_i \right)^{\frac{1}{\varepsilon-1}} \xrightarrow[\text{some } i \text{ with } i]{\text{replace } j \text{ with } i}$$

$$C = S \left( \int_0^1 P_i^{1-\varepsilon} d_i \right)^{\frac{1}{\varepsilon-1}} \Rightarrow S = C \underbrace{\left( \int_0^1 P_i^{1-\varepsilon} d_i \right)^{\frac{1}{1-\varepsilon}}}_{P}$$

$$P = \left( \int_0^1 P_i^{1-\varepsilon} d_i \right)^{1/\varepsilon} \text{(see bottom of prev. slide)}$$

$$\xi = \underline{PC}$$

$$\begin{aligned}\xi &= \int_0^1 P_i C_i d_i = \int_0^1 P_i \left( \frac{P_j}{P_i} \right)^\varepsilon C_j d_i \\ &= P_j^\varepsilon C_j \int_0^1 P_i^{1-\varepsilon} d_i = P_j^\varepsilon C_j P^{1-\varepsilon}\end{aligned}$$

$$= \left( \frac{P_j}{P} \right)^\varepsilon C_j P$$

$$PC = \left( \frac{P_j}{P} \right)^\varepsilon C_j P \Rightarrow \boxed{C_j = C \left( \frac{P_j}{P} \right)^{-\varepsilon}}$$

## Household budget constraint

The flow budget constraint is

(\*)

$$\int_0^1 P_t(i) C_t(i) di + B_{t+1}^N \leq (1 + i_t) B_t^N + W_t^N L_t + \Pi_t^N$$

with  $C_t(i)$  period  $t$  consumption of good  $i$ ,  $P_t(i)$  price of good  $i$ ,  
 $L_t$  hours of work,  $W_t^N$  nominal (i.e. in units of currency) wage,  
 $B_t^N$  nominal value of bonds held at beginning of  $t$ ,  $i_t$  the nominal  
interest rate,  $\Pi_t^N$  nominal profits.

(\*): A new timing assumption (for this model only):  $i_{t-1}$  is return on the bonds that  
are held in beginning of period  $t$  (so they have been chosen in  $t-1$ ).  
Using consumption aggregator and price indicator, the constraint  
can be rewritten:

$$P_t C_t + B_{t+1}^N \leq (1 + i_t) B_t^N + W_t^N L_t + \Pi_t^N$$
$$\max_{\{C_t, L_t, B_{t+1}^N\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left( \frac{C_t^{1-\delta}}{1-\delta} - \frac{L_t^{1+\eta}}{1+\eta} \right) \quad s.t.$$

$$\mathcal{Z} = E_0 \sum_{t=0}^{\infty} \left( \frac{C_t^{1-\beta}}{1-\beta} - \frac{L_t^y}{\gamma + \delta} + \right. \\ \left. + \lambda_t ((1+i_{t+1})B_t^N + W_t^N L_t^y + \Gamma_t^N - PC_t - B_{t+1}^N) \right)$$

$$\frac{\partial \mathcal{Z}}{\partial C_t} = C_t^{-\beta} - \lambda_t P_t = 0$$

$$\frac{\partial \mathcal{Z}}{\partial L_t^y} = -L_t^y + \lambda_t W_t^N = 0$$

$$-\frac{C_t^{-\beta}}{L_t^y} = -\frac{P_t}{W_t^N} \iff C_t^{-\beta} L_t^y = \frac{P_t}{W_t^N} \iff C_t L_t^y = \frac{W_t^N}{P_t}$$

$$\ln C_t = c_t \quad \text{in logs: } \beta \ln C_t + \gamma \ln L_t = \ln W_t^N - \ln P_t$$

$$\boxed{\beta c_t + \gamma l_t = w_t^N - p_t}$$

real wage

Euler equation derivation (attention to timing of  $i$  !)

$$\begin{aligned} \mathcal{L} = & \sum_{s=0}^{t-1} \beta \left[ \frac{C_s^{1-\delta}}{1-\delta} - \frac{L_s}{1+i_s} + \lambda_s ((1+i_{s-1}) B_s^N + W_s^N L_s + \Pi_s^N - P_s C_s - B_{s+1}^N) \right] \\ & + \beta^t \left[ \frac{C_t^{1-\delta}}{1-\delta} - \frac{L_t}{1+i_t} + \lambda_t ((1+i_{t-1}) B_t^N + W_t^N L_t + \Pi_t^N - P_t C_t - B_{t+1}^N) \right] \\ & + \beta^{t+1} E_t \left[ \frac{C_{t+1}^{1-\delta}}{1-\delta} - \frac{L_{t+1}}{1+i_{t+1}} + \lambda_{t+1} ((1+i_t) B_{t+1}^N + W_{t+1}^N L_{t+1} + \Pi_{t+1}^N - P_{t+1} C_{t+1} - B_{t+2}^N) \right] \\ & + \sum_{k=t+2}^{\infty} \beta^k E_t \left[ \frac{C_k^{1-\delta}}{1-\delta} - \frac{L_k}{1+i_k} + \lambda_k ((1+i_{k-1}) B_k^N + W_k^N L_k + \Pi_k^N - P_k C_k - B_{k+1}^N) \right] \end{aligned}$$

"divide both sides by"

$$\frac{\partial \mathcal{L}}{\partial B_{t+1}^N} = 0 : -\beta^t \lambda_t + \beta^{t+1} E_t \lambda_{t+1} (1+i_t) = 0 \quad | : (\beta^t \lambda_t (1+i_t))$$

$$-\frac{1}{1+i_t} + \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \right] = 0$$

$$\text{Use } \lambda_t = C_t^{-\delta} / P_t \text{ from } \frac{\partial \mathcal{L}}{\partial C_t} = 0 : \beta E_t \left[ \frac{P_t C_{t+1}^{-\delta}}{P_{t+1} C_t^{-\delta}} \right] = \frac{1}{1+i_t}$$

## Households' optimization

Using same approach as in the RBC, we obtain the FOCs:

$$\beta E_t \left[ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right] = \frac{1}{1 + i_t}$$
$$c_t^{\sigma} l_t^{\eta} = \frac{w_t^N}{P_t}$$

We will use lowercase letters for logs of variables:

$c_t = \ln C_t$ ,  $l_t = \ln L_t$ , etc.:

$$c_t = E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho)$$

$$\sigma c_t + \eta l_t = w_t^N - p_t$$

with  $\rho = -\ln \beta$  the discount **rate** (used in continuous time models)

$$\beta^t \rightarrow e^{-\rho t}$$

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## Firms

- ▷ Continuum of firms indexed by  $i \in [0, 1]$  (1 firm – 1 good)
- ▷ Production with common exogenous productivity for all firms  $A_t$  and labor:  $Y_t(i) = A_t L_t(i)^{1-\alpha} \Rightarrow$  labor demand trivial:  
$$L_t(i) = \left( \frac{Y_t(i)}{A_t} \right)^{\frac{1}{1-\alpha}}$$
- ▷ Differentiated goods  $\Rightarrow$  monopoly power, setting price  $P_t(i)$ :  
  - ▷ demand function given by  
$$Y_t = (\sum_0^1 Y_k(i)^{\frac{\varepsilon-1}{\varepsilon}} d_i)^{\frac{\varepsilon}{\varepsilon-1}}$$
  - ▷ continuum of goods  $\Rightarrow$  firm  $i$  doesn't influence  $Y_t$ ,  $C_t$ ,  $P_t$

We will look at firm optimization and model equilibrium under **flexible prices** and **sticky prices (Calvo pricing)** in turn.

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price setting problem (micro)

$$\max_P P \cdot Y(P) - TC^N(Y(P)) \frac{d(f(x)g(x))}{dx} =$$

FOC:  $Y(P) + Y'(P) \cdot P - MC^N(Y(P)) \cdot Y'(P) = Q \quad (\because Y(P))$

$$\frac{\partial TC^N}{\partial Y} = MC^N \quad 0 = 1 + \frac{Y'(P)P}{Y(P)} - MC^N(Y(P)) \frac{Y(P)'}{Y(P)}$$

$$\frac{Y'(P) \cdot P}{Y(P)} = -\varepsilon \quad 1 - \varepsilon + MC^N(Y(P)) \cdot \frac{\varepsilon}{P} = 0$$
$$P - \varepsilon P = -\varepsilon \cdot MC^N(Y(P))$$
$$P = \frac{-\varepsilon}{1+\varepsilon} \cdot MC^N(Y(P))$$
$$\frac{\varepsilon}{\varepsilon-1}$$

## Firm optimization – flexible prices

Maximize profits:

$$\max_{P_t(i), Y_t(i)} P_t(i) Y_t(i) - TC^N(Y_t(i))$$

Where:

- ▷  $TC^N$  is nominal cost function:

$$TC^N(Y_t(i)) = W_t^N L_t^d = W_t^N \left( \frac{Y_t(i)}{A_t} \right)^{\frac{1}{1-\alpha}}$$

- ▷  $Y_t(i)$  related to  $P_t(i)$  via demand:  $Y_t(i) = \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} Y_t$ .

Unusual notation, but a familiar problem of monopolistic pricing.

Solution:

$$P_t(i) = \frac{\varepsilon}{\varepsilon - 1} MC^N(Y_t(i))$$

## Symmetric solution

All firms symmetric in flexible price equilibrium  $\Rightarrow$  drop the  $i$  index:

$$P_t = \frac{\varepsilon}{\varepsilon - 1} MC^N(Y_t)$$

and we can get the marginal cost as derivative of total cost:

$$MC^N(Y_t) = \frac{dTC^N(Y_t)}{dY_t} = \frac{d(W_t^N L^d(Y_t))}{dY_t} = \frac{1}{1-\alpha} W_t^N A_t^{\frac{1}{\alpha-1}} Y_t^{\frac{\alpha}{1-\alpha}}$$

so we can use it in the optimal price equation:

$$P_t = \frac{\varepsilon}{\varepsilon - 1} \frac{1}{1-\alpha} W_t^N A_t^{\frac{1}{\alpha-1}} Y_t^{\frac{\alpha}{1-\alpha}}$$

or  $p_t = \mu - \ln(1-\alpha) + w_t^N + \left(\frac{1}{\alpha-1}\right) a_t + \left(\frac{\alpha}{1-\alpha}\right) y_t$  in logs

where  $\mu$  is log of the price markup:  $\mu \equiv \ln\left(\frac{\varepsilon}{\varepsilon-1}\right)$

## Flexible price equilibrium

A flexible price equilibrium is a sequence of variables  $\{Y_t(i), C_t(i), P_t(i), L_t(i), W_t^N, A_t\}_{t=0}^\infty$  and aggregates  $c_t$

$\{Y_t(i), C_t(i), P_t(i), L_t(i), W_t^N, A_t\}_{t=0}^\infty$  and aggregates

$$C_t = \left( \int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}, \quad Y_t = \left( \int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

$P_t = \left( \int_0^1 P_t(i)^{1-\varepsilon} di \right)^{1/(1-\varepsilon)}, \quad L_t = \int_0^1 L_t(i) di$  such that, given an exogenous process for  $A_t$ :

1. The **Euler equation** holds:  $\beta E_0 \left[ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right] = \frac{1}{1+i_t}$

2. **Consumption-labor optimality** holds:  $\frac{C_t}{L_t} = \frac{W_t^N}{P_t}$

3. **Optimal price** is set by each firm:

$$P_t(i) = \frac{\varepsilon}{\varepsilon-1} \frac{1}{1-\alpha} W_t^N A_t^{\frac{1}{\alpha-1}} Y_t(i)^{\frac{\alpha}{1-\alpha}}$$

4. Goods market clears:  $Y_t(i) = C_t(i) \Rightarrow Y_t = C_t$ , with  
 $Y_t(i) = A_t L_t(i)^{1-\alpha}$

5. Bonds market clears:  $B_t^N = 0$

Technically, we also need to impose a transversality condition in households' optimization:  $\lim_{T \rightarrow \infty} E_t[B_t^N] \geq 0$

## Flexible price equilibrium: monetary neutrality

As in RBC, nothing depends on nominal variables  $P_t, W_t^N, i_t$  in equilibrium. Consider equilibrium conditions (2)-(4) in logs (written without goods index  $i$ ):

$$\sigma c_t + \eta l_t = w_t^N - p_t \quad \text{Real wage : } \frac{w_t^N}{P_t}$$

*log real wage*

$$p_t = \mu - \ln(1 - \alpha) + w_t^N + \left(\frac{1}{\alpha - 1}\right) a_t + \left(\frac{\alpha}{1 - \alpha}\right) y_t$$

$$y_t = c_t$$

$$y_t = a_t + (1 - \alpha)l_t,$$

where the last equation is the production function in logs.

$w_t \equiv w_t^N - p_t$  can be introduced in the first two equations. We then have 4 equations, 4 unknowns  $y_t, c_t, l_t, w_t$ , that have a static solution each period that depends on  $a_t$ . Solution for log GDP is:

$$y_t = \frac{1 - \alpha}{(1 - \alpha)\sigma + \eta + \alpha} \left( -\mu + \ln(1 - \alpha) - \frac{1 + \eta}{1 - \alpha} a_t \right)$$

## The real interest rate

Real interest rate is a real quantity that can also be obtained in equilibrium using the log Euler equation:

$$c_t = E_t[c_{t+1}] - \frac{1}{\sigma} (i_t - E_t[\pi_{t+1}] - \rho)$$

$y_t$        $y_{t+1}$        $r_t$

Then, recall the definition of the **real interest rate**, a.k.a. the **Fischer equation**:

$$r_t = i_t - E_t\pi_{t+1}$$

combine the two and  $y_t = c_t$  to obtain

$$\begin{aligned} r_t &= \rho + \sigma E_t \Delta y_{t+1} \\ &= \rho + \sigma \frac{1 + \eta}{\sigma(1 - \alpha) + \eta + \alpha} E_t \Delta a_{t+1}, \quad (\text{using the solution for } y) \end{aligned}$$

So the real interest rate is, too, driven by productivity. In a steady state,  $\Delta a_t = 0$ , so  $r_t = \rho$ , the real interest rate is the discount factor. ~~rate~~

## Central Bank in a neutrality economy

Suppose you only know the flexible price model (the sticky price one is much harder!), but your employer **really** wants you to say something about prices, interest rates, central bank, etc.

A neutral central bank with an inflation targeting Taylor Rule can be introduced:

$$i_t = \rho + \phi_\pi \pi_t, \quad \text{with } \rho = -\ln \beta, \text{ the discount factor rate}$$

and combine ~~the two~~ with the Fischer equation:

$$\phi_\pi \pi_t = E_t \pi_{t+1} + \hat{r}_t \quad \text{with } \hat{r}_t \equiv r_t - \rho$$

$\hat{r}_t$  is the deviation of the real interest from its steady-state value  $\rho$ .

## Inflation determinacy – the Taylor Principle

$$\phi_\pi \hat{\pi}_{t+1} = E_{t+1} \hat{\pi}_{t+2} + \hat{r}_{t+1}$$
$$\phi_\pi \pi_t = E_t \pi_{t+1} + \hat{r}_t \quad \text{with } \hat{r}_t \equiv r_t - \rho$$

If  $\phi_\pi > 1$ , the level of inflation is **determined** as a discounted sum of expected  $\hat{r}_t$ :

$$\pi_t = \sum_{s=0}^{\infty} \phi_\pi^{-(s+1)} E_t \hat{r}_{t+s}$$

Otherwise, we can write inflation dynamics as an AR(1)-type process:

$$\pi_{t+1} = \phi_\pi \pi_t - \hat{r}_t + \xi_{t+1}$$

Where  $\xi$  is a random variable with  $E_t \xi_{t+1} = 0$  and no economic meaning. This is a **sunspot shock** – a random factor affecting economic outcomes such as inflation, but with no economic explanation.

Bottom line: an **active Taylor rule** ( $\phi_\pi > 1$ ) allows to determine level of inflation, otherwise – uncontrollable **sunspot shocks**. Not specific to neutral flexible price economy, – also with nominal rigidity economy, where monetary variables have real effects.