Macroeconomics
Lecture 7 – Real Business Cycles in Closed Economy

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Overview

1. Representative household
2. Representative firm
3. Equilibrium

Outline

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2. Representative firm
3. Equilibrium

Representative household

A **large number** (population size normalized to 1) of **identical** households populate the economy.

A **representative household**: the aggregate outcomes as result of one (big) agent's behaviour.

It represents many small ones $\Rightarrow$ the representative household cannot manipulate aggregate quantities and prices (wage, interest) $\Rightarrow$ **takes prices as given**

Households in RBC model **live forever**:

- ▷ Demographics ignored (unlike overlapping generations models)
- ▷ Imagine a sequence of generations of constant size , with children' utility entering parents' utility

Representative household – budget constraint

Household chooses work effort and earns wage w_t, but also has several types of financial income:

1. it invests in bonds: B_{t+1} is the the stock of bonds at held at the beginning of $t+1$, which is chosen in t. It pays the real interest rate r_t in $t+1$

2. it invests in physical capital: K_{t+1} is the stock at beginning of $t+1$, that is result of investment in t, I_t. Capital depreciates at rate δ:

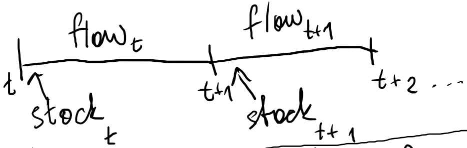
$$K_{t+1} = (1-\delta)K_t + I_t$$

   capital is **rented out to firms**, and in period t the stock of capital K_t pays the **rental rate of capital** R_t to the household

3. it owns the firms $\Rightarrow$ receives their profits Π_t

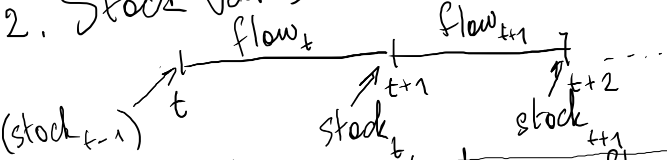
Stock-flow relationships reminder

1. Stock var's measured at beginning of per.:



$\Rightarrow$ $\boxed{stock_{t+1} = stock_t + flow_t}$ (used in this lecture)

2. Stock var's measured at end of period:



$\Rightarrow$ $\boxed{stock_t = stock_{t-1} + flow_t}$

(or $stock_{t+1} = stock_t + flow_{t+1}$)

(used in consumption lectures for S_t)

Representative household budget

The household budget constraint is then:

$$C_t + \underbrace{K_{t+1} - (1-\delta)K_t}_{I_t} + \underbrace{B_{t+1} - B_t}_{\text{bond saving}} = w_t N_t + R_t K_t + r_{t-1} B_t + \Pi_t$$

re-arranging terms, we get:

$$C_t + K_{t+1} + B_{t+1} = w_t N_t + (R_t + 1 - \delta)K_t + (1 + r_{t-1})B_t + \Pi_t$$

The convention about stock variables K, B here is **measuring stock variables at the beginning of period**.

Difference between bonds and capital as sources of income:

- the real interest rate is known one period in advance before it is paid on bonds (r_{t-1} on B_t, r_t on B_{t+1})
- rental rate of capital is determined at the same period when it is paid: the household does not know it in advance when investing in capital (R_{t+1} paid on K_{t+1} that is chosen in t)

Representative household maximization problem

We will make an additional assumption for the instantaneous utility function: it is **additively separable** between consumption and leisure utility:

$$u(C_t, 1-N_t) = u(C_t) + v(1-N_t)$$

We can now write the household problem with infinite horizon:

$$\max_{C_t, N_t, K_{t+1}, B_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t \left(u\left(C_t\right) + v\left(1-N_t\right) \right)$$

s.t. $C_t + K_{t+1} + B_{t+1} = w_t N_t + (R_t + 1 - \delta) K_t + (1 + r_{t-1}) B_t + \Pi_t$

$$\max_{C_t, N_t, K_t, B_t} E_0 \sum_{t=0}^{\infty} \beta^t \left(u(C_t) + v(1-N_t)\right)$$

$$\text{s.t.} \quad C_t + K_t + B_t = w_t N_t + (R_t + 1 - \delta) K_t + (1 + r_{t-1}) B_t + \Pi_t$$

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Big[u(C_t) + v(1-N_t) + \lambda_t \big(w_t N_t + (R_t + (1-\delta)) K_t + (1 + r_{t-1}) B_t + \Pi_t - C_t - K_{t+1} - B_{t+1} \big) \Big]$$

$$\mathcal{L}_t = \beta^t \Big[u(C_t) + v(1-N_t) + \lambda_t \big(w_t N_t + (R_t + (1-\delta)) K_t + (1 + r_{t-1}) B_t + \Pi_t - C_t - K_{t+1} - B_{t+1} \big) \Big]$$

$$+ E_t \beta^{t+1} \Big[u(C_{t+1}) + v(1-N_{t+1}) + \lambda_{t+1} \big(w_{t+1} N_{t+1} + (R_{t+1} + 1 - \delta) K_{t+1} + (1 + r_t) B_{t+1} + \Pi_{t+1} - C_{t+1} - K_{t+2} - B_{t+2} \big) \Big]$$

$$+ \ldots$$

from period t perspective

$$\mathcal{L}_t = \beta^t\Big[u(C_t) + v(1-N_t) + \lambda_t\big(w_t N_t + (R_t + (1-\delta))K_t + (1+r_{t-1})B_t + \Pi_t - C_t - K_{t+1} - B_{t+1}\big)\Big]$$

from period t perspective

$$+ E_t\,\beta^{t+1}\Big[u(C_{t+1}) + v(1-N_{t+1}) + \lambda_{t+1}\big(w_{t+1}N_{t+1} + (R_{t+1} + 1-\delta)K_{t+1} + (1+r_t)B_{t+1} + \Pi_{t+1} - C_{t+1} - K_{t+2} - B_{t+2}\big)\Big]$$

$$+ \ldots$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t\big(u'(C_t) + \lambda_t(-1)\big) = 0$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \beta^t\big(v'(1-N_t)\cdot(-1) + \lambda_t \cdot w_t\big) = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = \beta^t \cdot \lambda_t \cdot (-1) + E_t \beta^{t+1}\lambda_{t+1}(R_{t+1} + 1-\delta) = 0$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = \beta^t \lambda_t(-1) + E_t \beta^{t+1}\lambda_{t+1}(1+r_t)$$

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t\left(u'(C_t) + \lambda_t(-1)\right) = 0 \iff \begin{cases} u'(C_t) = \lambda_t \\ v'(1-N_t) = \lambda_t w_t \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = \beta^t\left(v'(1-N_t)\cdot(-1) + \lambda_t \cdot w_t\right) = 0 \iff$$

$$\frac{\partial \mathcal{L}}{\partial K_t} = \beta^t \cdot \lambda_t \cdot (-1) + E_t \beta^{t+1} \lambda_{t+1}\left(R_{t+1} + 1 - \delta\right) = 0 \iff \lambda_t = E_t\left[\beta \lambda_{t+1}(R_{t+1} + 1 - \delta)\right]$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = \beta^t \lambda_t (-1) + E_t \beta^{t+1} \lambda_{t+1}(1+r_t) \iff \lambda_t = E_t \beta \lambda_{t+1}(1+r_t)$$

$$\frac{u'(C_t)}{v'(1-N_t)} = \frac{1}{w_t}$$

$$\iff \boxed{\frac{v'(1-N_t)}{u'(C_t)} = w_t}$$

$$\boxed{u'(C_t) = \beta E_t\left[u'(C_{t+1})(R_{t+1} + 1 - \delta)\right]}$$

$$\boxed{u'(C_t) = \beta E_t u'(C_{t+1})(1+r_t)}$$

Household problem: solution

Lagrangian of household problem

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t [u(C_t) + v(1 - N_t)$$
$$+\lambda_t \left(w_t N_t + (R_t + 1 - \delta) K_t + (1 + r_{t-1}) B_t + \Pi_t - C_t - K_{t+1} - B_{t+1} \right)]$$

First-order conditions in period t (with information of $t \Rightarrow E_t[\cdot]$):

$$(1) \quad \frac{\partial \mathcal{L}}{\partial C_t}: \qquad u'(C_t) - \lambda_t = 0$$

$$(2) \quad \frac{\partial \mathcal{L}}{\partial N_t}: \qquad -v'(1 - N_t) + \lambda_t w_t = 0$$

$$(3) \quad \frac{\partial \mathcal{L}}{\partial B_{t+1}}: \qquad -\lambda_t + \beta E_t \lambda_{t+1}(1 + r_t) = 0$$

$$(4) \quad \frac{\partial \mathcal{L}}{\partial K_{t+1}}: \qquad -\lambda_t + \beta E_t[\lambda_{t+1}(R_{t+1} + 1 - \delta)] = 0$$

Household problem: solution

- From $\frac{\partial \mathcal{L}}{\partial C_t} = 0$ and $\frac{\partial \mathcal{L}}{\partial N_t} = 0$, **consumption-leisure optimality** or labor supply conditional on consumption level:

$$\frac{v'(1-N_t)}{u'(C_t)} = w_t, \text{ or } N_t = N^s(w_t, C_t)$$

$\frac{v'(1-N_t)}{u'(C_t)}$ is $\frac{-u'_L(C_t,1-N_t)}{u'_C(C_t,1-N_t)}$ in previous notation

- From $\frac{\partial \mathcal{L}}{\partial C_t} = 0$ and $\frac{\partial \mathcal{L}}{\partial B_t} = 0$, the **Euler equation**:

$$u'(C_t) = \beta E_t u'(C_{t+1})(1+r_t)$$

- From $\frac{\partial \mathcal{L}}{\partial C_t} = 0$ and $\frac{\partial \mathcal{L}}{\partial K_t} = 0$, **capital holding optimality**:

$$u'(C_t) = \beta E_t[u'(C_{t+1})(R_{t+1} + 1 - \delta)]$$

where E_t applies to product of $u'(C_{t+1})$ and R_{t+1}, both uncertain, and $E(XY) \neq E(X)E(Y)$ in general, so we **cannot conclude that $r_t = E_t R_{t+1} - \delta$!**

Euler equation and Stochastic Discount Factor

Define the **stochastic discount factor** M_t as

$$M_t = \frac{\beta^t E_0 u'(C_t)}{u'(C_0)}$$

Multiplying uncertain future income in period t by M_t gives its **present discounted value**

Why? Use the **Euler equation**:

$$u'(C_t) = \beta E_t u'(C_{t+1})(1+r_t) \Leftrightarrow \frac{u'(C_t)}{\beta E_t u'(C_{t_1})} = 1 + r_t$$

$$\Leftrightarrow \frac{\beta E_t u'(C_{t+1})}{u'(C_t)} = \frac{1}{1+r_t}$$

and:

$$M_t = \frac{\beta^t E_0 u'(C_t)}{u'(C_0)} = E_0[\underbrace{\frac{\beta u'(C_1)}{u'(C_0)}}_{1/(1+r_0)} \cdot \underbrace{\frac{\beta u'(C_2)}{u'(C_1)}}_{1/(1+r_1)} \cdots \underbrace{\frac{\beta u'(C_t)}{u'(C_{t-1})}}_{1/(1+r_{t-1})}]$$

$$= \frac{1}{1+r_0} \cdot E_0\left[\frac{1}{1+r_1}\frac{1}{1+r_2}\cdots\frac{1}{1+r_{t-1}}\right] = E_0\frac{1}{R_{0,t}} \text{ (see Lec. 5)}$$

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Representative firm

A continuum of small, identical firms $\Rightarrow$ study **representative firm**

Production function $A_t F(K_t, N_t)$ with **constant returns to scale** and TFP A_t that is **known at beginning of** t

$F(aK_t, aN_t) = aF(K_t, N_t)$

Maximize present discounted value of profits:

Revenue from production (= production because $P = 1$)

– Cost of production: wage for labor, rental rate for capital

Perfect competition $\Rightarrow$ firms takes factor prices w_t, R_t as given

Firm choices turn out to be static, i.e. solving a one-period problem is enough

In the textbook, read section 2.2 Households own capital stock and ignore debt variable D_t that is shown to be neutral

Firm profit maximization

Maximize discounted future profits (= firm value):

$$V_0 = \sum_{t=0}^{\infty} M_t \Pi_t = \sum_{t=0}^{\infty} M_t \left(A_t F(K_t, N_t) - w_t N_t - R_t K_t \right)$$

Where households' stochastic discount factor M_t is used because households are firm owners

Every period t has variables t only $\Rightarrow$ firm problem is static:
$\max_{N_t, K_t} A_t F(K_t, N_t) - w_t N_t - R_t K_t$

$F'_k > 0, F''_k > 0$

First-order conditions:

$$K_t : \quad A_t F'_K(K_t, N_t) - R_t = 0 \Rightarrow \mathrm{K}_t = K^d(\underset{+}{A_t}, \underset{-}{R_t})$$

$$N_t : \quad A_t F'_N(K_t, N_t) - w_t = 0 \Rightarrow \mathrm{N}_t = N^d(\underset{+}{A_t}, \underset{-}{w_t})$$

$F'_N > 0, F''_N < 0$

FOCs define a **capital demand function** $K^d(A_t, R_t)$ and a **labor demand function** $N^d(A_t, w_t)$

Perfect competition, constant returns to scale $\Rightarrow \Pi_t = 0$

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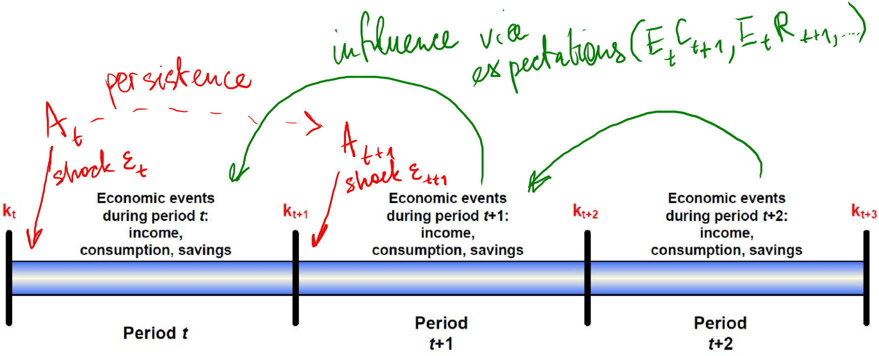
Dynamic equilibrium: diagram



Market clearing conditions

Capital market clearing

- Capital demand from firm's profit maximization:
  $K_t = K^d(A_t, R_t)$
- Capital supply is given by HH's consumption-saving decisions:
  $K_t = K^s(C_{t-1}, E_{t-1}C_t, E_{t-1}R_t)$
- Combining: $K_t = K^d(A_t, R_t) = K^s(C_{t-1}, E_{t-1}C_t, E_{t-1}R_t)$

Goods market clearing

- Goods supply, or GDP, is $Y_t = A_t F(K_t, N_t)$
- Goods demand is $C_t + I_t$, with $I_t = K_{t+1} - (1-\delta)K_t$
- Combining: $C_t + K_{t+1} - (1-\delta)K_t = A_t F(K_t, N_t)$
  $\Rightarrow$ **aggregate resource constraint**

Labor market clearing: $N^s(w_t, C_t) = N^d(A_t, w_t)$

Bond market clearing: $B_t = 0$ since all households are the same (no borrowers, no savers)

Dynamic equilibrium: definition

A dynamic equilibrium is a sequence $\{C_t, N_t, K_t, B_t, w_t, r_t, R_t,\}_{t=0}^{\infty}$ that, given exogenous initial capital $K_0 > 0$ and bonds $B_0 = 0$ and the exogenous stochastic processes for $\{A_t\}_{t=0}^{\infty}$, satisfies:

1. Given $\{w_t, r_t, R_t\}_{t=0}^{\infty}$, $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ satisfies the sequence of consumption-leisure optimality conditions, consumption-saving optimality conditions, and consumer budget constraints

   1. Euler eq.
   2. Capital holding optimality

2. Given $\{w_t, R_t\}_{t=0}^{\infty}$, $\{N_t, K_t\}_{t=0}^{\infty}$ satisfies the sequence of labor demand functions and capital demand functions

3. All markets clear: $Y_t = C_t + K_{t+1} - (1-\delta)K_t = A_t F(K_t, N_t)$; $N_t^s = N_t^d = N_t$; $K_t^s = K_t^d = K_t$, $B_t = 0$

Dynamic equilibrium: system of equations

At each $t \geq 0$, $(C_t, N_t, K_t, w_t, r_t, R_t, A_t)$ satisfies

$$\frac{v'(1-N_t)}{u'(C_t)} = w_t$$
$$u'(C_t) = \beta E_t u'(C_{t+1})(1+r_t)$$
$$u'(C_t) = \beta E_t[u'(C_{t+1})(R_{t+1}+1-\delta)]$$
$$A_t F'_N(K_t, N_t) = w_t$$
$$A_t F'_K(K_t, N_t) = R_t$$
$$C_t + K_{t+1} - (1-\delta)K_t = A_t F(K_t, N_t)$$
$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t$$

Prices w_t, r_t, R_t can be eliminated from the system

$\hookrightarrow$ appears in system once $\Rightarrow$ solution not affected

Dynamic equilibrium – prices eliminated

At each $t \geq 0$, (C_t, N_t, K_t, A_t) satisfies

$$\begin{aligned}
\frac{v'(1-N_t)}{u'(C_t)} &= A_t F'_N(K_t, N_t) \\
u'(C_t) &= \beta E_t[u'(C_{t+1})(A_{t+1} F'_K(K_{t+1}, N_{t+1}) + 1 - \delta)] \\
C_t + K_{t+1} - (1-\delta)K_t &= A_t F(K_t, N_t) \\
\ln A_t &= \rho \ln A_{t-1} + \varepsilon_t
\end{aligned}$$

This summarizes the behaviour of all the macroeconomic aggregates of the model