Macroeconomics
Lecture 6 – Real Business Cycles

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Overview

1. Model specification
   - Households
   - Representative firm
   - Equilibrium
2. Solution techniques
3. Model results

Consumer optimization

Lagrangian of consumer problem

$\Omega_{t+1} - \Omega_t$

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(C_t, 1 - L_t)$$
$$+ \lambda_t [w_t L_t + (1 + r_t)\Omega_t + \Pi_t - C_t - \Omega_{t+1}]\}$$

Budget constraint holds as equality for all $t \Rightarrow \lambda_t > 0$ for all t.
First-order conditions in period t: (for agent living in t)

$$(1) \quad \frac{\partial \mathcal{L}}{\partial C_t} = 0 : \quad u'_c(C_t, 1 - L_t) - \lambda_t = 0$$

$$(2) \quad \frac{\partial \mathcal{L}}{\partial L_t} = 0 : \quad -u'_{1-L}(C_t, 1 - L_t) + \lambda_t w_t = 0$$

$$(3) \quad \frac{\partial \mathcal{L}}{\partial \Omega_{t+1}} = 0 : \quad -\lambda_t + \beta E_t[\lambda_{t+1}(1 + r_{t+1})] = 0$$

$\boxed{\lambda_t = u'_c(C_t, 1 - L_t)}$ follows from (1): λ_t is marginal utility of consumption at t, also known as **shadow price** of wealth.

Deriving $\mathcal{L}$ w.r.t. Ω_{t+1}:

$$\mathcal{L} = \sum_{s=0}^{t-2} \beta^s \Big[u(C_s, 1-L_s) + \lambda_s \big(w_s L_s + (1+r_s)\Omega_s + \Pi_s - C_s - \Omega_{s+1} \big) \Big]$$

$$+ \beta^{t-1} \Big[u(C_{t-1}, 1-L_{t-1}) + \lambda_{t-1} \big(w_{t-1} L_{t-1} + (1+r_{t-1})\Omega_{t-1} + \Pi_{t-1} - C_{t-1} - \Omega_t \big) \Big]$$

$$+ \beta^t \Big[u(C_t, 1-L_t) + \lambda_t \big(w_t L_t + (1+r_t)\Omega_t + \Pi_t - C_t - E_t \Omega_{t+1} \big) \Big]$$

$$+ E_t \beta^{t+1} \Big[u(C_{t+1}, 1-L_{t+1}) + \lambda_{t+1} \big(w_{t+1} L_{t+1} + (1+r_{t+1})\Omega_{t+1} + \Pi_{t+1} - C_{t+1} - \Omega_{t+2} \big) \Big]$$

$$+ \ldots$$

$$\boxed{-\lambda_t + \beta E_t \lambda_{t+1}(1+r_{t+1}) = 0}$$

Labour supply

- Eliminate λ_t from FOC (1), (2) to get the **consumption-labor optimality condition**:

$$\frac{u_l(C_t, 1 - L_t)}{u_c(C_t, 1 - L_t)} = w_t$$

- defines (implicity) the function of **labour supply for a given level of consumption**

$$L_t = L^s(w_t, C_t)$$

- features substitution and income effects: sign of w_t derivative is ambiguous
- number (or share) of hours worked is the *intensive margin* of labor supply. We do not have the *extensive margin* (work vs. unemployment) in this model. If interested, look at Ch. 10 of Romer textbook.

Consumption-savings: the Euler equation

- ~~(1) and~~ **(3)** imply the **consumption-savings optimality condition**

$$1 = E_t \left[\left(\frac{\beta \lambda_{t+1}}{\lambda_t} \right) (1 + r_{t+1}) \right]$$

- **Euler equation**: key equation in modern macro models
- In finance, $E_t \frac{\lambda_{t+1}}{\lambda_t}$ known as **pricing kernel** or **stochastic discount factor**
  - important for finance theory + studied empirically
- Pricing kernel, Euler equation $\rightarrow$ intersection of macro and finance theory

Consumption-savings: the Euler equation

Substituting λ_t from FOC (1) and assuming no uncertainty (for this slide):

$$1 = \left(\frac{\beta u_c(C_{t+1}, 1 - L_{t+1})}{u_c(C_t, 1 - L_t)}\right)(1 + r_{t+1})$$

$$\Leftrightarrow \frac{u_c'(C_t, 1 - L_t)}{\beta u_c'(C_{t+1}, 1 - L_{t+1})} = 1 + r_{t+1}$$

- Left side: **marginal rate of substitution** (MRS) between period t and $t + 1$ consumption
- Right side: ratio of price of period t consumption $(= 1)$ to price of period $t + 1$ consumption $\left(= \frac{1}{1+r_{t+1}}\right)$
- Interpretation: consuming ε units less ($\varepsilon \to 0$, marginal amount) in t and $(1 + r)\varepsilon$ units more in $t + 1$ must keep utility unchanged

Euler equation with uncertainty

Bringing back uncertainty:

$$1 = E_t\left[\left(\frac{\beta u_c(C_{t+1}, 1 - L_{t+1})}{u_c(C_t, 1 - L_t)}\right)(1 + r_{t+1})\right]$$

- Utility of consuming ε (very small) in period t...
- ... equal to the **expected** utility of marginal savings with return on them, $(1 + r_{t+1})\varepsilon$, at $t + 1$
- Defines (implicitly) **demand for assets** or **savings supply**:

$$\Omega_{t+1} - \Omega_t = G(E_t r_{t+1})$$

Household choices - Summary

Representative household's period t optimal choices of C_t, L_t and Ω_{t+1} characterized by consumption-labor optimality condition, consumption-savings optimality condition and flow budget constraint:

$$
\begin{aligned}
1 =& E_t\left[\left(\frac{\beta u_c(C_{t+1}, 1-L_{t+1})}{u_c(C_t, 1-L_t)}\right)(1+r_{t+1})\right]\\
w_t =& \frac{u_n(C_t, 1-L_t)}{u_c(C_t, 1-L_t)}\\
C_t + \Omega_{t+1} =& w_t L_t + \underset{(1+r_t)}{\cancel{r_t}}\Omega_t + \Pi_t
\end{aligned}
$$

taking as given Ω_t (pre-determined), w_t, r_t, and Π_t

These define:

- demand side of period t goods market (depending on r_t) $\Rightarrow$ close to IS
- supply side of period t labour market
- supply side of period t asset/savings markets (will define **capital formation**)

Outline

1. Model specification
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Representative firm

A large number (a mass equal to 1) of identical firms in **perfect competition** $\Rightarrow$ study **representative firm**

Firms do not make intertemporal choices. They only:

1. rent factors of production (labour and capital) on markets
2. produce goods according to $Y_t = Z_t f(K_t, L_t)$, with Z_t **stochastic productivity** (same for all firms)

   total factor productivity
3. distribute profits (null in equilibrium) to households

No difference in model if firms live for 1 or ∞ periods

Perfect competition $\Rightarrow$ firm **takes prices** w_t, r_t **as given**

Production function f has $f'_L, f'_K > 0; f''_L, f''_K < 0$

Constant returns to scale in f $\Rightarrow$ **profits null in equilibrium**.

Firm profit maximization

Productivity Z_t observed at **beginning of period** t. Firms then optimize period t profit:

$$\Pi_t = Z_t f(K_t, L_t) - w_t L_t - r_t K_t$$

- Static maximization of profit function

$$\max_{\{L_t, K_t\}} Z_t f(K_t, L_t) - w_t L_t - r_t K_t$$

- First-order conditions

$$\begin{aligned} L_t : \quad & Z_t f_L(K_t, L_t) = w_t \\ K_t : \quad & Z_t f_K(K_t, L_t) = r_t \end{aligned}$$

FOCs define a downward sloping **labor demand function** $L^d(w_t, Z_t)$ and **capital demand function** $K^d(r_t, Z_t)$

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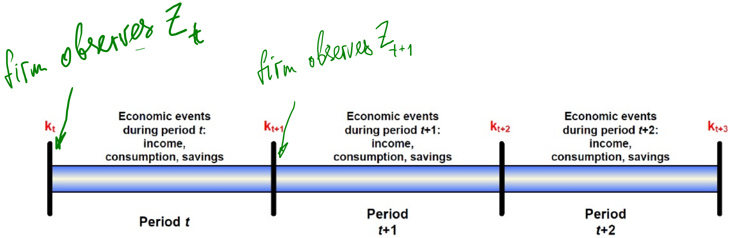
Dynamic equilibrium: diagram



Law of motion of productivity

Productivity has stochastic component, but can also have persistence: **autoregressive process of order 1** (in logs):

$$\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

where ϵ is the **white noise** component of the autoregressive process – the **source of randomness** in Z_t and **in the model** in general.

Productivity of period t is observed at beginning of period (or, equivalently, during $t-1$) – **before the firm hires and installs capital**.

Law of motion of capital

Capital lasts for more than one period. It partially depreciates and is increased by **investment**:

$$K_{t+1} = K_t \underbrace{-\delta K_t}_{\text{depreciation}} + I_t$$
$$= (1-\delta)K_t + I_t$$

Closed economy $\Rightarrow$ investment is financed only with households' **savings**:

$$I_t = \Omega_{t+1} - \Omega_t$$

Intertemporal equilibrium: period t

Capital market clearing

- Capital demand from firm's profit maximization: $K_{t+1} = K^d(r_{t+1}, Z_{t+1})$
- Capital supply is savings supply: $I_t = \underbrace{\Omega_{t+1} - \Omega_t}_{=G(E_t r_t)}$

$t+1$

- Combining with law of motion of capital:

$$K^d(r_{t+1}, Z_{t+1}) = (1-\delta)K_t + G(E_t r_{t+1})$$

Goods-market clearing

- Goods aggregate demand is $C_t + I_t \;\; (= C_t + K_{t+1} - (1-\delta)K_t)$
- Goods aggregate supply is Y_t, given by $Z_t f(K_t, L_t)$
- Goods market clearing

$$C_t + K_{t+1} - (1-\delta)K_t = Z_t f(K_t, L_t)$$

$\Rightarrow$ **aggregate resource constraint**

Labor market clearing

$$L^s(w_t, C_t) = L^d(w_t, Z_t)$$

States, controls, transversality

The economy can be characterized by dynamics of four variables (Z_t, K_t, C_t, L_t) that are divided in two types:

- Z_t, K_t are **state variables** – depend on the past
- C_t, L_t are **control** or **jump** variables – can change instantly depending on **expectations**
- State variables have **initial conditions**: $Z_0 (= 1), K_0$
- Control/jump variables need a **transversality condition**: information on where the economy ends up as $t \to \infty$

$$\lim_{T \to \infty} \frac{K_T}{\prod_{\tau=0}^{T} (1 + r_\tau - \delta)} = 0$$

- Obtained by substituting future incomes in the resource constraints

When transversality condition is satisfied, the resource constraint implies $K_0 = \sum_{s=0}^{\infty} \frac{C_s - w_s L_s - \Pi_s}{\prod_{\tau=0}^{s} (1 + r_\tau - \delta)}$ – initial capital in the economy is equal to **present discounted value** of all future **dissavings**.

Dynamic equilibrium: definition

A **dynamic equilibrium** is a sequence $\{C_t, L_t, K_{t+1}, r_t, w_t\}_{t=0}^{\infty}$ that, given K_0, Z_0 and the exogenous stochastic process $\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t$, satisfies the following:

1. Given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ satisfies the sequence of **consumption-savings** optimality conditions, **consumption-labor** optimality conditions, **consumer budget constraints** and the **transversality condition**
2. Given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{L_t, K_t\}_{t=0}^{\infty}$ satisfy **labor market clearing**: $L^s(w_t, C_t) = L^d(w_t, Z_t) = L_t$ and **capital (asset) market clearing** $G(r_t) = K^d(r_t, Z_t) = K_t$
3. Goods market clears: $C_t + K_{t+1} - (1-\delta)K_t = Z_t f(K_t, L_t) = Y_t$;

Dynamic equilibrium: equations

A dynamic equilibrium is a sequence $\{C_t, L_t, K_{t+1}, Z_{t+1}, r_t, w_t\}_{t=0}^{\infty}$ that, given K_0, Z_0, satisfies

$$
\begin{aligned}
1 =& E_t\left[\left(\frac{\beta u_c(C_{t+1}, 1-L_{t+1})}{u_c(C_t, 1-L_t)}\right)(1+r_{t+1})\right] \\
\frac{u_\ell(C_t, 1-L_t)}{u_c(C_t, 1-L_t)} =& w_t \\
Z_t f_\ell(K_t, L_t) =& w_t \\
Z_t f_k(K_t, L_t) =& r_t \\
C_t + K_{t+1} =& Z_t f(K_t, L_t) + (1-\delta)K_t \\
\ln Z_t =& \rho \ln Z_{t-1} + \epsilon_t
\end{aligned}
$$

$1-L$

L

$t+1 \quad t \quad t+1$

for $t = 0, 1, 2, \ldots$, as well as transversality condition as $t \to \infty$

We can eliminate the prices w_t, r_t from the system (see next slide)

Dynamic equilibrium – prices eliminated

A dynamic equilibrium is a sequence $\{C_t, L_t, K_{t+1}, Z_{t+1}\}_{t=0}^{\infty}$ that, given K_0, Z_0, satisfies

1-L

$$
\begin{aligned}
1 =& E_t\left[\left(\frac{\beta u_c(C_{t+1}, 1-L_{t+1})}{u_c(C_t, 1-L_t)}\right)(1 + Z_{t+1} f_k(K_{t+1}, L_{t+1}))\right] \\
\frac{u_n(C_t, 1-L_t)}{u_c(C_t, 1-L_t)} =& Z_t f_n(K_t, L_t) \\
C_t + K_{t+1} =& Z_t f(K_t, L_t) + (1-\delta)K_t \\
\ln Z_t =& \rho \ln Z_{t-1} + \epsilon_t
\end{aligned}
$$

for $t = 0, 1, 2, \ldots$, as well as transversality condition as $t \to \infty$

How do we solve it? First, specify the functions $u(\cdot,\cdot), f(\cdot,\cdot)$.
Can we then compute the solution on the board? **No!** 😱

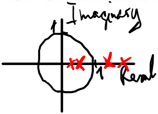
We can only solve RBC **numerically** – on the computer.

Outline

Numerical solution: Blanchard-Kahn algorithm

Once functions $u(\cdot,\cdot), f(\cdot,\cdot)$ chosen, we *can* do a couple of steps without the computer:

1. Find formulas for **steady state** values of variables
2. Find the **log-linearized** from of model equations
3. Obtain the following **matrix form** of the linear model:



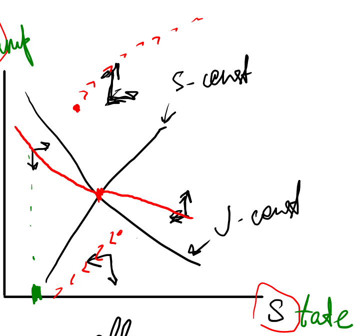
$$\begin{bmatrix} x^s_{t+1} \\ E_t x^c_{t+1} \end{bmatrix} = A \begin{bmatrix} x^s_t \\ x^c_t \end{bmatrix} + R\nu_{t+1}$$

with x_t^s the **vector of state variables**, x_t^s the **vector of control/jump variables**, ν_t vector of shocks

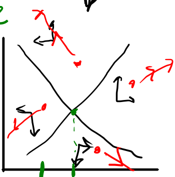
In practice, these 3 steps can be done with computer, too. Next steps are *supposed to be done with the computer*:

1. Find eigenvalues and eigenvectors of the A matrix
2. Check that number of **unstable** eigenvalues (absolute value ≥ 1) is same as number of control/jump variables (Blanchard-Kahn condition)

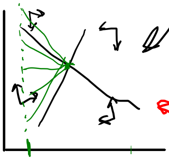
Blanchard Kahn conditions satisfied (with 1 state and 1 jump var.)



all eigenvalues unstable



all eigenvalues stable



Blanchard-Kahn condition violated

Choosing functional forms

- **Preferences** – the $u(\cdot,\cdot)$:

$$u(\underset{+}{C_t}, \underset{-}{L_t}) = \ln C_t - \chi \frac{L_t^{1+\eta}}{1+\eta},$$

where η is the inverse of Frisch elasticity of labor supply, χ the weight of labor disutility in the utility function. Note the it is labor and not leisure that enters the function.

- **Production function** – the $f(\cdot,\cdot)$ – Cobb-Douglas:

$$Y_t = Z_t f(K_t, L_t) = Z_t K_t^{\alpha} L_t^{1-\alpha},$$

with α the capital share in total income.

Equilibrium equations with chosen functions

Consumption-savings condition (Euler equation):

$$1 = E_t \left[\frac{\beta C_{t+1}}{C_t} (1 + r_{t+1}) \right]$$

Consumption-labor condition:

$$\chi L_t^{\eta} C_t = w_t$$

Firms' **capital and labor demands**:

$$\alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} = r_t$$
$$(1-\alpha) Z_t K_t^{\alpha} L_t^{-\alpha} = w_t$$

Resource constraint:

$$C_t + K_{t+1} - (1-\delta) K_t = Z_t K_t^{\alpha} L_t^{1-\alpha}$$

Steady state

Substitute constant value for all the variables and the null expected value for the random variable $\epsilon \Rightarrow$ a system of equations is obtained:

$$
\begin{aligned}
1 = \left[\frac{\beta C_{ss}}{C_{ss}}(1+r_{ss})\right] \quad \Rightarrow \quad \frac{1}{\beta} - 1 &= r_{ss} \Rightarrow 1 + r_{ss} = \frac{1}{\beta} \\
\chi L_{ss}^{\eta} C_{ss} &= w_{ss} \\
Z_{ss} &= 1 \\
\alpha (K_{ss}/L_{ss})^{\alpha-1} &= r_{ss} \\
(1-\alpha)(K_{ss}/L_{ss})^{\alpha} &= w_{ss} \\
C_{ss} + \delta K_{ss} &= K_{ss}^{\alpha} L_{ss}^{1-\alpha}
\end{aligned}
$$

Calibration

β	α	η	χ	δ	ρ	σ_ϵ
0.99	0.33	1	8	0.025	0.8	0.01

Where do the values come from?

- discount factor β to yield real *annual* interest rate of 4% $\Rightarrow$ $\beta = 1/(1 + r_{ss}) = 1/(1 + 0.04/4) \approx 0.99$ (we divide 0.04 by 4 in formula because model periods are quarters)
- Cobb-Douglas production function: $\alpha = 1/3$ a long-run estimate of capital share in national income
- Capital depreciation rate is 10% annually ($\delta = 0.1/4 = 0.025$)
- χ is an unobserved parameter that is chosen to match a **target**: $L_{ss} = 0.33$
- ρ can be manipulated to obtain more or less persistent productivity; σ_ϵ for more or less strong shocks

Log-linearization of equations

Use the following substitution for rules for each variable x_t:

$$x_t = x_{ss} e^{\tilde{x}_t}, \quad \text{with} \quad \tilde{x}_t \equiv \ln x_t - \ln x_{ss}$$

$\tilde{x}_t$ is a relative (measured in %) **deviation of x_t from its steady state**. Reminder: $\ln x_t - \ln x_{ss} \approx \frac{x_t - x_{ss}}{x_{ss}}$.

Then, use the following approximations:

1. $e^{\tilde{x}_t} \approx 1 + \tilde{x}_t$ ($\Leftrightarrow \ln(1 + \tilde{x}_t) \approx \tilde{x}_t$)
2. $\tilde{x}_t \tilde{y}_t \approx 0$ (we are doing a **first-order** approximation)

There are many tricks that follow from these rules and simplify the process. See, for example, J. Zietz (2006) "Log-Linearizing Around the Steady State: A Guide with Examples"

Log-linearized model

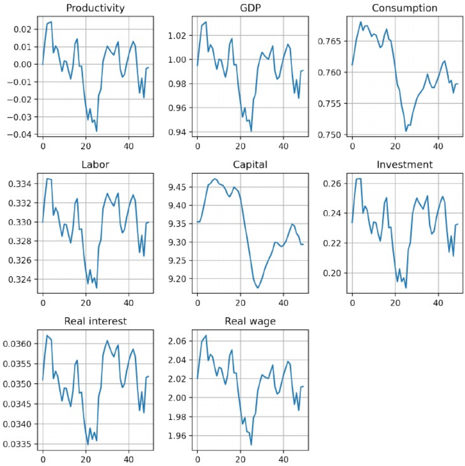
$$E_t\left[\tilde{C}_{t+1}-\tilde{C}_t-\frac{r_{ss}}{1+r_{ss}}\tilde{r}_{t+1}\right]=0$$
$$\tilde{C}_t+\chi\tilde{L}_t=\tilde{w}_t$$
$$\tilde{Z}_t+(\alpha-1)\tilde{K}_t+(1-\alpha)\tilde{L}_t=\tilde{r}_t$$
$$\tilde{Z}_t+\alpha\tilde{K}_t-\alpha\tilde{L}_t=\tilde{w}_t$$
$$C_{ss}\tilde{C}_t+K_{ss}(\tilde{K}_{t+1}-(1-\delta)\tilde{K}_t)=K_{ss}^{\alpha}L_{ss}^{1-\alpha}(\tilde{Z}_t+\alpha\tilde{K}_t+(1-\alpha)\tilde{L}_t)$$
$$\tilde{Z}_{t+1}=\rho\tilde{Z}_t+\epsilon_t$$

$$\tilde{x}^s_t=\begin{bmatrix}\tilde{Z}_t\\ \tilde{K}_t\end{bmatrix}\qquad \tilde{x}^c_t=\begin{bmatrix}\tilde{C}_t\\ \tilde{L}_t\end{bmatrix}$$

$$\Rightarrow \underbrace{}_{A_1}\begin{bmatrix}\tilde{x}^s_{t+1}\\ E_t\tilde{x}^c_{t+1}\end{bmatrix}=\underbrace{A}_{A_2}\begin{bmatrix}\tilde{x}^s_t\\ \hat{x}^c_t\end{bmatrix}+\underbrace{R}_{A_3}\begin{bmatrix}\epsilon_t\\ 0\end{bmatrix}$$

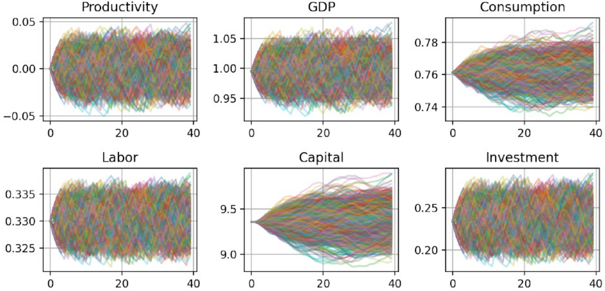
Outline

Stochastic simulation – an example



Stochastic simulation – 1000 examples

We simply run the above simulation 1000 times. **Sequence of shocks is different every time** ⇒ dynamics different. Interest and wages not plotted.



Model variances vs. empirical variances

Compare average standard deviations of 3 variables: GDP, consumption, investment: $\sigma_Y, \sigma_C, \sigma_I$.

Model results using 1000 simulations:

1. $\sigma_C = 0.15\sigma_Y$ (*how does it follow form consumer behavior?*)
2. $\sigma_I = 4.5\sigma_Y$

Same statistics in US data (King&Rebelo(1999) Table 1):

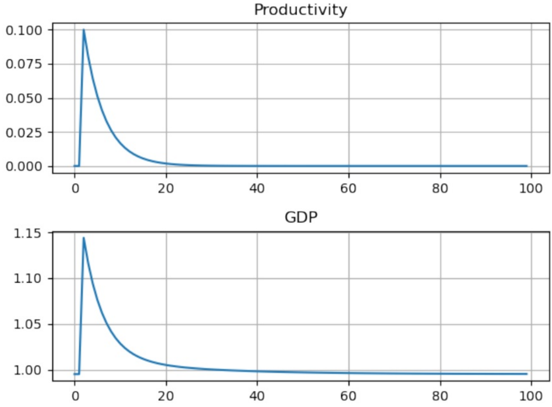
1. $\sigma_C = 0.74\sigma_Y$
2. $\sigma_I = 2.9\sigma_Y$

The model exaggerates the difference in variances, but **reproduces the pattern**: $\sigma_C < \sigma_Y < \sigma_I$

Easy to improve quantitative performance with more advanced functional forms.

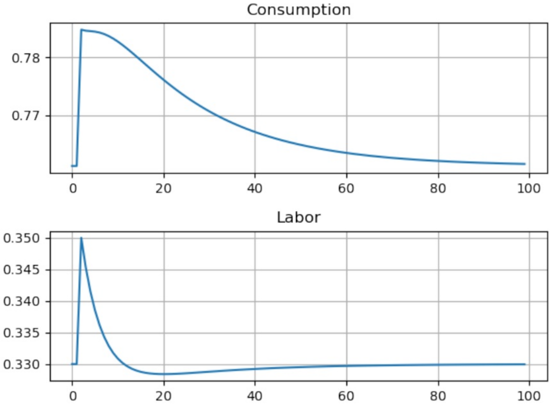
Impulse response functions — Z, Y

Transitory productivity shock: ϵ rises from 0 to σ_ϵ at $t = 2$ and is at 0 afterwards:



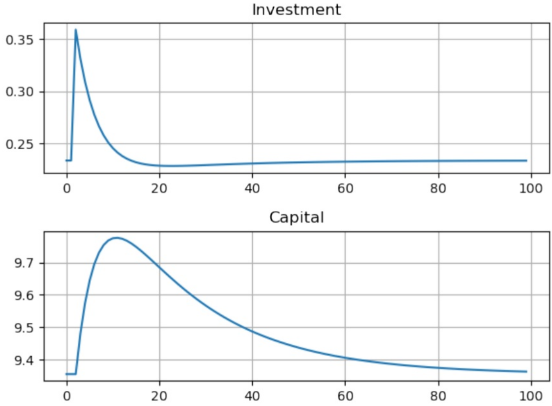
Impulse response functions — C, L

Transitory productivity shock: ϵ rises from 0 to σ_ϵ at $t = 2$ and is at 0 afterwards:



Impulse response functions — I, K

Transitory productivity shock: ϵ rises from 0 to σ_ϵ at $t = 2$ and is at 0 afterwards:



Transitory vs. permanent productivity shock

Transitory shock (blue) – as before; permanent shock (orange) – ϵ always positive, such that steady-state Z is σ_ϵ: