

Macroeconomics

Lecture 6 – Real Business Cycles

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Overview

1 Model specification

- Households
- Representative firm
- Equilibrium

2 Solution techniques

3 Model results

Consumer optimization

Lagrangian of consumer problem

$$\begin{aligned}\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t & \{ u(C_t, 1 - L_t) \\ & + \lambda_t [w_t L_t + (1 + r_t) \Omega_t + \Pi_t - C_t - \Omega_{t+1}] \}\end{aligned}$$

Budget constraint holds as equality for all $t \Rightarrow \lambda_t > 0$ for all t .

First-order conditions in period t :

$$(1) \quad \frac{\partial \mathcal{L}}{\partial C_t} \quad u'_c(C_t, 1 - L_t) - \lambda_t = 0$$

$$(2) \quad \frac{\partial \mathcal{L}}{\partial L_t} : \quad u'_{1-L}(C_t, 1 - L_t) + \lambda_t w_t = 0$$

$$(3) \quad \frac{\partial \mathcal{L}}{\partial \Omega_{t+1}} : \quad -\lambda_t + \beta E_t[\lambda_{t+1}(1 + r_{t+1})] = 0$$

$\lambda_t = u'_c(C_t, 1 - L_t)$ follows from (1): λ_t is marginal utility of consumption at t , also known as **shadow price** of wealth.

Deriving $\frac{\partial \mathcal{L}}{\partial \Sigma_{t+1}} = 0$ (Euler equation) :

$$\mathcal{L} = \sum_{s=0}^{t-1} \beta^s [u(C_s, 1-L_s)]$$

$$+ \lambda_s (w_s L_s + (1+r_s) \Sigma_s + \Pi_s - C_s - \Sigma_{s+1})]$$

$$+ \beta^t [u(C_t, 1-L_t) + \lambda_t (w_t L_t + (1+r_t) \Sigma_t + \Pi_t - C_t - \Sigma_{t+1})]$$

$$+ E_t \beta^{t+1} [u(C_{t+1}, 1-L_{t+1}) + \lambda_{t+1} (w_{t+1} L_{t+1} + (1+r_{t+1}) \Sigma_{t+1} + \Pi_{t+1} - C_{t+1} - \Sigma_{t+2})]$$

$$+ E_t \sum_{j=t+1}^{\infty} \beta^j [u(C_j, 1-L_j) + \lambda_j (w_j L_j + (1+r_j) \Sigma_j + \Pi_j - C_j - \Sigma_{j+1})]$$

$$\frac{\partial \mathcal{L}}{\partial \Sigma_{t+1}} = 0 : -\beta^t \lambda_t + E[\beta^{t+1} \lambda_{t+1} (1+r_{t+1})] = 0 \rightarrow \text{divide both parts by } \beta^t$$

$$-\lambda_t + E[\beta \lambda_{t+1} (1+r_{t+1})] = 0$$

$$\Leftrightarrow -\lambda_t + \beta E_t [\lambda_{t+1} (1+r_{t+1})] = 0 \quad (\text{because } \beta \text{ determines } t(c))$$

Labour supply

- ▷ Eliminate λ_t from FOC (1), (2) to get the **consumption-labor optimality condition**:

$$\frac{u'_{1-L}(C_t, 1 - L_t)}{u'_c(C_t, 1 - L_t)} = w_t$$

- ▷ defines (implicitly) the function of **labour supply for a given level of consumption** (*Frisch labor supply*)

$$L_t = L^s(w_t, C_t)$$

- ▷ features substitution and income effects: sign of w_t derivative is ambiguous
- ▷ number (or share) of hours worked is the *intensive margin* of labor supply. We do not have the *extensive margin* (work vs. unemployment) in this model. If interested, look at Ch. 10 of Romer textbook.

Consumption-savings: the Euler equation

- ▷ (1) and (3) imply the **consumption-savings optimality condition**

$$1 = E_t \left[\left(\frac{\beta \lambda_{t+1}}{\lambda_t} \right) (1 + r_{t+1}) \right]$$

- ▷ **Euler equation:** key equation in modern macro models
- ▷ In finance, $E_t^{\frac{\beta \lambda_{t+1}}{\lambda_t}}$ known as **pricing kernel** or **stochastic discount factor**
 - ▷ important for finance theory + studied empirically
 - ▷ Pricing kernel, Euler equation → intersection of macro and finance theory

Consumption-savings: the Euler equation

Substituting λ_t from FOC (1) and assuming no uncertainty (for this slide):

$$1 = \left(\frac{\beta u'_c(C_{t+1}, 1 - L_{t+1})}{u'_c(C_t, 1 - L_t)} \right) (1 + r_{t+1})$$

$$\Leftrightarrow \frac{u'_c(C_t, 1 - L_t)}{\beta u'_c(C_{t+1}, 1 - L_{t+1})} = 1 + r_{t+1}$$

- ▷ Left side: **marginal rate of substitution** (MRS) between period t and $t + 1$ consumption
- ▷ Right side: ratio of price of period t consumption ($= 1$) to price of period $t + 1$ consumption ($= \frac{1}{1+r}$)
- ▷ Interpretation: consuming ε units less ($\varepsilon \rightarrow 0$, marginal amount) in t and $(1+r)\varepsilon$ units more in $t + 1$ must keep utility unchanged

Euler equation with uncertainty

Bringing back uncertainty:

$$1 = E_t \left[\left(\frac{\beta u'_c(C_{t+1}, 1 - L_{t+1})}{u'_c(C_t, 1 - L_t)} \right) (1 + r_{t+1}) \right]$$

- ▷ Utility of consuming ε (very small) in period t ...
- ▷ ... equal to the **expected** utility of marginal savings with return on them, $(1 + r_{t+1})\varepsilon$, at $t + 1$
- ▷ Defines (implicitly) **demand for assets** or **savings supply**:

$$\Omega_{t+1} - \Omega_t = G(E_t r_{t+1})$$

Household choices - Summary

Representative household's period t optimal choices of C_t , L_t and Ω_{t+1} characterized by consumption-labor optimality condition, consumption-savings optimality condition and flow budget constraint:

$$1 = E_t \left[\left(\frac{\beta u'_c(C_{t+1}, 1 - L_{t+1})}{u'_c(C_t, 1 - L_t)} \right) (1 + r_{t+1}) \right]$$
$$w_t = \frac{u'_{1-L}(C_t, 1 - L_t)}{u'_c(C_t, 1 - L_t)}$$

$$C_t + \Omega_{t+1} = w_t L_t + r_t \Omega_t + \Pi_t$$

taking as given Ω_t (pre-determined), w_t , r_t , and Π_t

These define:

- ▷ demand side of period t goods market (depending on r_t) \Rightarrow close to IS
- ▷ supply side of period t labour market
- ▷ supply side of period t asset/savings markets (will define **capital formation**)

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Representative firm

A large number (a mass equal to 1) of identical firms in **perfect competition** \Rightarrow study **representative firm**

Firms do not make intertemporal choices. They only:

1. rent factors of production (labour and capital) on markets
2. produce goods according to $Y_t = Z_t f(K_t, L_t)$, with Z_t **stochastic productivity** (same for all firms)
3. distribute profits (null in equilibrium) to households

No difference in model if firms live for 1 or ∞ periods

Perfect competition \Rightarrow firm **takes prices w_t, r_t as given**

Production function f has $f'_L, f'_K > 0$; $f''_L, f''_K < 0$

Constant returns to scale in $f \Rightarrow$ **profits null in equilibrium.**

Firm profit maximization

Productivity Z_t observed at **beginning of period** t . Firms then optimize period t profit:

$$\begin{aligned}\Pi_t &= Z_t f(K_t, L_t) - w_t L_t - (1 + r_t) K_t + (1 - \delta) K_t \\ &= Z_t f(K_t, L_t) - w_t L_t - (r_t + \delta) K_t\end{aligned}$$

- ▷ Static maximization of profit function

$$\max_{\{L_t, K_t\}} Z_t f(K_t, L_t) - w_t L_t - (r_t + \delta) K_t$$

- ▷ First-order conditions

$$L_t : \quad Z_t f_L(K_t, L_t) = w_t$$

$$K_t : \quad Z_t f_K(K_t, L_t) = r_t + \delta$$

FOCs define a downward sloping **labor demand function** $L^d(w_t, Z_t)$ and **capital demand function** $K^d(r_t, Z_t)$

Outline

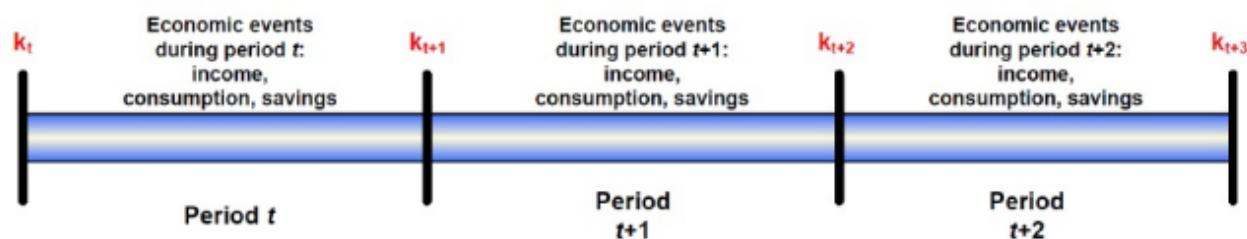
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Dynamic equilibrium: diagram



Law of motion of productivity

Productivity has stochastic component, but can also have persistence: **autoregressive process of order 1** (in logs):

$$\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

where ϵ is the **white noise** component of the autoregressive process – the **source of randomness** in Z_t and **in the model** in general.

Productivity of period t is observed at beginning of period (or, equivalently, during $t - 1$) – **before the firm hires and installs capital.**

Law of motion of capital

Capital lasts for more than one period. It partially depreciates and is increased by **investment**:

$$\begin{aligned} K_{t+1} &= K_t - \underbrace{\delta K_t}_{\text{depreciation}} + I_t \\ &= (1 - \delta)K_t + I_t \end{aligned}$$

Closed economy \Rightarrow investment is financed only with households' **savings**:

$$I_t = \Omega_{t+1} - \Omega_t$$

Intertemporal equilibrium: period t

Capital market clearing

- ▷ Capital demand from firm's profit maximization:

$$K_{t+1} = K^d(r_{t+1}, Z_{t+1})$$

- ▷ Capital supply is savings supply: $I_t = \underbrace{\Omega_{t+1} - \Omega_t}_{=G(E_t r_t)}$

- ▷ Combining with law of motion of capital:

$$K^d(r_{t+1}, Z_{t+1}) = (1 - \delta)K_t + G(E_t r_{t+1})$$

Goods-market clearing

- ▷ Goods aggregate demand is $C_t + I_t$ ($= C_t + K_{t+1} - (1 - \delta)K_t$)
- ▷ Goods aggregate supply is Y_t , given by $Z_t f(K_t, L_t)$
- ▷ Goods market clearing

$$C_t + K_{t+1} - (1 - \delta)K_t = Z_t f(K_t, L_t)$$

⇒ **aggregate resource constraint**

Labor market clearing

$$L^s(w_t, C_t) = L^d(w_t, Z_t)$$

States, controls, transversality

The economy can be characterized by dynamics of four variables (Z_t, K_t, C_t, L_t) that are divided in two types:

- ▷ Z_t, K_t are **state variables** – depend on the past
- ▷ C_t, L_t are **control** or **jump** variables – can change instantly depending on **expectations**
- ▷ State variables have **initial conditions**: $Z_0 (= 1), K_0$
- ▷ Control/jump variables need a **transversality condition**: information on where the economy ends up as $t \rightarrow \infty$

$$\lim_{T \rightarrow \infty} \frac{K_T}{\prod_{\tau=0}^T (1 + r_\tau - \delta)} = 0$$

- ▷ Obtained by substituting future incomes in the resource constraints

When transversality condition is satisfied, the resource constraint implies $K_0 = \sum_{s=0}^{\infty} \frac{C_s - w_s L_s - \Pi_s}{\prod_{\tau=0}^s (1 + r_\tau - \delta)}$ – initial capital in the economy is equal to **present discounted value** of all future **dissavings**.

Dynamic equilibrium: definition

A **dynamic equilibrium** is a sequence $\{C_t, L_t, K_{t+1}, r_t, w_t\}_{t=0}^{\infty}$ that, given K_0, Z_0 and the exogenous stochastic process $\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t$, satisfies the following:

- Given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{C_t, L_t, K_{t+1}\}_{t=0}^{\infty}$ satisfies the sequence of **consumption-savings** optimality conditions, **consumption-labor** optimality conditions, **consumer budget constraints** and the **transversality condition**
- Given $\{r_t, w_t\}_{t=0}^{\infty}$, $\{L_t, K_t\}_{t=0}^{\infty}$ satisfy **labor market clearing**: $L^s(w_t, C_t) = L^d(w_t, Z_t) = L_t$ and **capital (asset) market clearing** $G(r_t) = K^d(r_t, Z_t) = K_t$
- Goods market clears:
 $C_t + K_{t+1} - (1 - \delta)K_t = Z_t f(K_t, L_t) = Y_t;$

Dynamic equilibrium: equations

A dynamic equilibrium is a sequence $\{C_t, L_t, K_{t+1}, Z_{t+1}, r_t, w_t\}_{t=0}^{\infty}$ that, given K_0, Z_0 , satisfies

$$\frac{u'_{1-L}(C_t, 1 - L_t)}{u'_c(C_t, 1 - L_t)} = w_t \quad \text{and} \quad \frac{\beta u'_c(C_{t+1}, 1 - L_{t+1})}{u'_c(C_t, 1 - L_t)} (1 + r_{t+1}) = \mathbb{E}_t \left[\frac{\beta u'_c(C_{t+1}, 1 - L_{t+1})}{u'_c(C_t, 1 - L_t)} (1 + Z_{t+1} f'(K_{t+1}, L_{t+1}) - \delta) \right]$$
$$Z_t f'_L(K_t, L_t) = w_t \quad \text{and} \quad Z_t f'_K(K_t, L_t) = r_t + \delta$$
$$C_t + K_{t+1} = Z_t f(K_t, L_t) + (1 - \delta) K_t$$
$$\ln Z_t = \rho \ln Z_{t-1} + \epsilon_t$$

will be important later

for $t = 0, 1, 2, \dots$, as well as transversality condition as $t \rightarrow \infty$

How do we solve it? First, specify the functions $u(\cdot, \cdot)$, $f(\cdot, \cdot)$.

Can we then compute the solution on the board? **No!** 🤦

We can only solve RBC **numerically** – on the computer.

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Numerical solution: Blanchard-Kahn algorithm

Once functions $u(\cdot, \cdot)$, $f(\cdot, \cdot)$ chosen, we *can* do a couple of steps without the computer:

1. Find formulas for **steady state** values of variables
2. Find some **linearized** from of model equations (e.g. **log-linearized**)
3. Obtain the following **matrix form** of the linear model:

$$\begin{bmatrix} x_{t+1}^s \\ E_t x_{t+1}^c \end{bmatrix} = A \begin{bmatrix} x_t^s \\ x_t^c \end{bmatrix} + R\nu_{t+1}$$

with x_t^s the **vector of state variables**, x_t^c the **vector of control/jump variables**, ν_t vector of shocks

In practice, these 3 steps can be done with computer, too. Next steps are *supposed to be done with the computer*:

1. Find eigenvalues and eigenvectors of the A matrix
2. Check that number of **unstable** eigenvalues (absolute value ≥ 1) is same as number of control/jump variables
(Blanchard-Kahn condition)

Choosing functional forms

- ▷ **Preferences** – the $u(\cdot, \cdot)$:

$$u(C_t, L_t) = \ln C_t - \chi \frac{L_t^{1+\eta}}{1+\eta},$$

where η is the inverse of Frisch elasticity of labor supply, χ the weight of labor disutility in the utility function. Note the it is labor and not leisure that enters the function.

- ▷ **Production function** – the $f(\cdot, \cdot)$ – Cobb-Douglas:

$$Y_t = Z_t f(K_t, L_t) = Z_t K_t^\alpha L_t^{1-\alpha},$$

with α the capital share in total income.

Equilibrium equations with chosen functions

Consumption-savings condition (Euler equation):

$$1 = E_t \left[\frac{\beta C_{t+1}}{C_t} (1 + r_{t+1}) \right]$$

Consumption-labor condition:

$$\chi L_t^\eta C_t = w_t$$

Firms' capital and labor demands:

$$\alpha Z_t K_t^{\alpha-1} L_t^{1-\alpha} = r_t + \delta$$

$$(1 - \alpha) Z_t K_t^\alpha L_t^{-\alpha} = w_t$$

Resource constraint:

$$C_t + K_{t+1} - (1 - \delta) K_t = Z_t K_t^\alpha L_t^{1-\alpha}$$

Steady state

Substitute constant value for all the variables and the null expected value for the random variable $\epsilon \Rightarrow$ a system of equations is obtained:

$$1 = \left[\frac{\beta C_{ss}}{C_{ss}} (1 + r_{ss}) \right] \Rightarrow \frac{1}{\beta} - 1 = r_{ss}$$

$$\chi L_{ss}^{\eta} C_{ss} = w_{ss}$$

$$Z_{ss} = 1$$

$$\alpha(K_{ss}/L_{ss})^{\alpha-1} = r_{ss} + \delta$$

$$(1 - \alpha)(K_{ss}/L_{ss})^{\alpha} = w_{ss}$$

$$C_{ss} + \delta K_{ss} = K_{ss}^{\alpha} L_{ss}^{1-\alpha}$$

Calibration

β	α	η	χ	δ	ρ	σ_ϵ
0.99	0.33	1	8	0.025	0.8	0.01

Where do the values come from?

- ▷ discount factor β to yield real *annual* interest rate of 4% $\Rightarrow \beta = 1/(1 + r_{ss}) = 1/(1 + 0.04/4) \approx 0.99$ (we divide 0.04 by 4 in formula because **model periods are quarters**)
- ▷ Cobb-Douglas production function: $\alpha = 1/3$ a long-run estimate of capital share in national income
- ▷ Capital depreciation rate is 10% annually ($\delta = 0.1/4 = 0.025$)
- ▷ χ is an unobserved parameter that is chosen to match a **target**: $L_{ss} = 0.33$
- ▷ ρ can be manipulated to obtain more or less persistent productivity; σ_ϵ for more or less strong shocks

Log-linearization of equations

Log-linearization – most widespread **linearization** technique when done by hand. Practical – variables become relative deviations from s.s. value.

Use the following substitution for rules for each variable x_t :

$$x_t = x_{ss} e^{\tilde{x}_t}, \quad \text{with} \quad \tilde{x}_t \equiv \ln x_t - \ln x_{ss}$$

\tilde{x}_t is a relative **deviation of x_t from its steady state** (as a share of s.s. value), because $\ln x_t - \ln x_{ss} \approx \frac{x_t - x_{ss}}{x_{ss}}$.

Then, use the following approximations:

1. $e^{\tilde{x}_t} \approx 1 + \tilde{x}_t \ (\Leftrightarrow \ln(1 + \tilde{x}_t) \approx \tilde{x}_t)$
2. $\tilde{x}_t \tilde{y}_t \approx 0$ (we are doing a **first-order** approximation)

There are many tricks that follow from these rules and simplify the process. See, for example, J. Zietz (2006) "Log-Linearizing Around the Steady State: A Guide with Examples"

Log-linearized model

$$E_t \left[\tilde{C}_{t+1} - \tilde{C}_t - \frac{r_{ss}}{1+r_{ss}} \tilde{r}_{t+1} \right] = 0$$
$$\tilde{C}_t + \chi \tilde{L}_t = \tilde{w}_t$$

$$\tilde{Z}_t + (\alpha - 1) \tilde{K}_t + (1 - \alpha) \tilde{L}_t = \frac{r_{ss}}{r_{ss} + \delta} \tilde{r}_t$$

$$\tilde{Z}_t + \alpha \tilde{K}_t - \alpha \tilde{L}_t = \tilde{w}_t$$

$$C_{ss} \tilde{C}_t + K_{ss} (\tilde{K}_{t+1} - (1 - \delta) \tilde{K}_t) = K_{ss}^\alpha L_{ss}^{1-\alpha} (\tilde{Z}_t + \alpha \tilde{K}_t + (1 - \alpha) \tilde{L}_t)$$
$$\tilde{Z}_{t+1} = \rho \tilde{Z}_t + \epsilon_t$$

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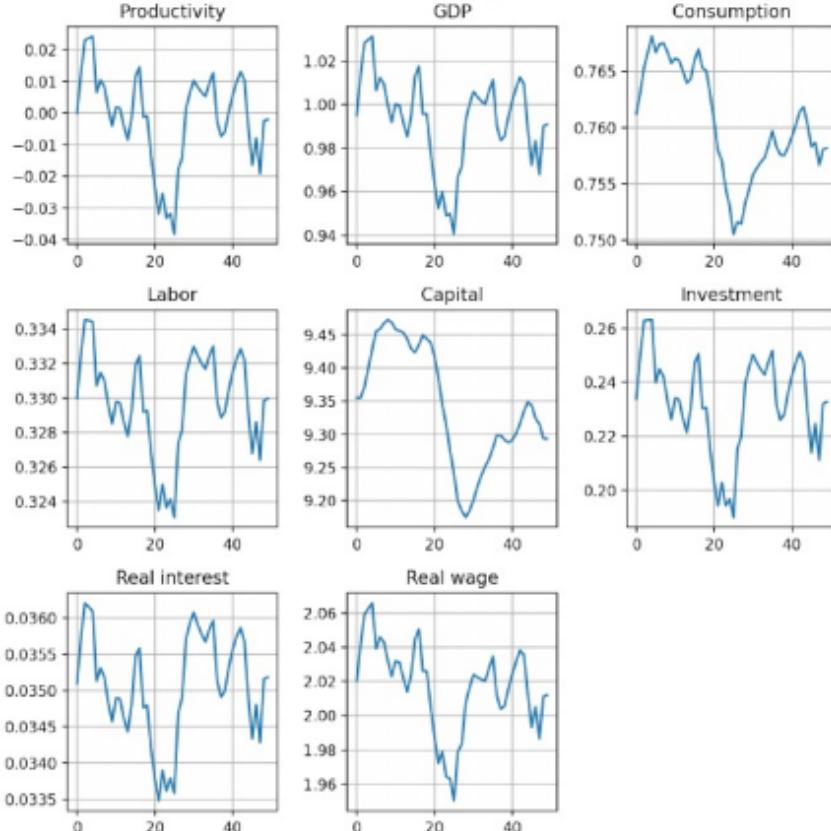
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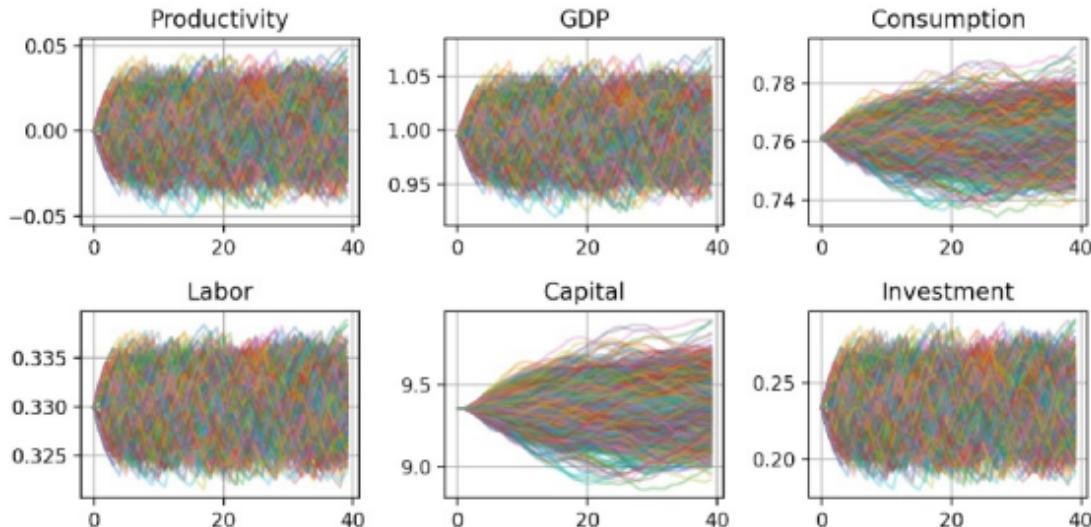
Stochastic simulation – an example



Attention: here and henceforth model linearized, not **log-linearized** by computer \Rightarrow variables in **levels**, not **relative deviations** from s.s.

Stochastic simulation – 1000 examples

We simply run the above simulation 1000 times. **Sequence of shocks is different every time** \Rightarrow dynamics different. Interest and wages not plotted.



Model variances vs. empirical variances

Compare average standard deviations of 3 variables: GDP, consumption, investment: $\sigma_Y, \sigma_C, \sigma_I$.

Model results using 1000 simulations:

1. $\sigma_C = 0.15\sigma_Y$ (*how does it follow from consumer behavior?*)
2. $\sigma_I = 4.5\sigma_Y$

Same statistics in US data (King&Rebelo(1999) Table 1):

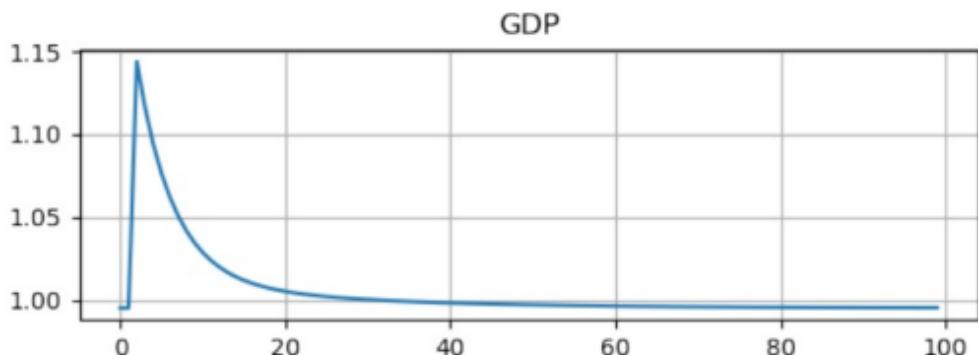
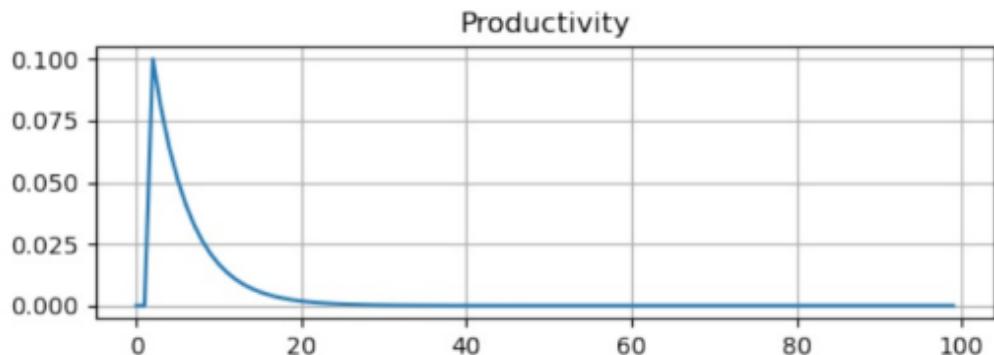
1. $\sigma_C = 0.74\sigma_Y$
2. $\sigma_I = 2.9\sigma_Y$

The model exaggerates the difference in variances, but **reproduces the pattern**: $\sigma_C < \sigma_Y < \sigma_I$

Easy to improve quantitative performance with more advanced functional forms.

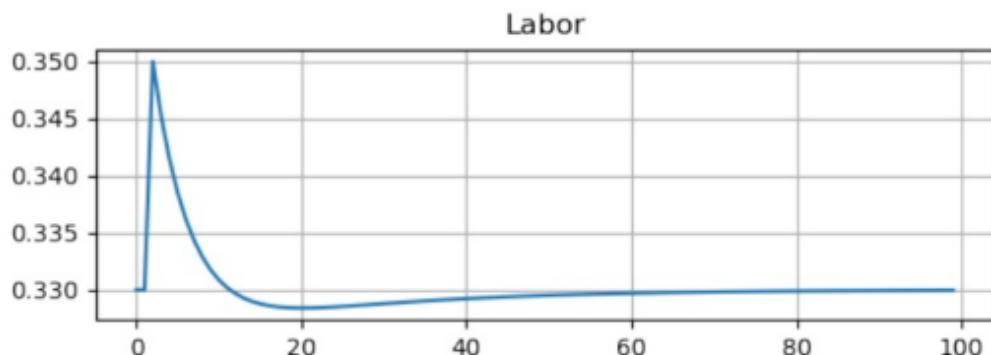
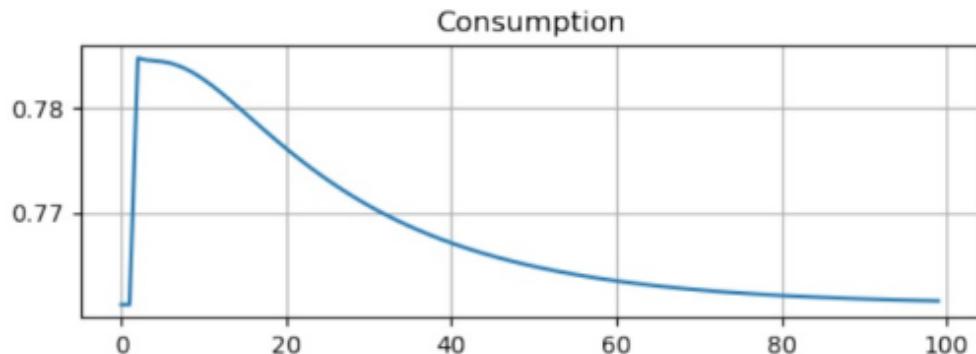
Impulse response functions — Z, Y

Transitory productivity shock: ϵ rises from 0 to σ_ϵ at $t = 2$ and is at 0 afterwards:



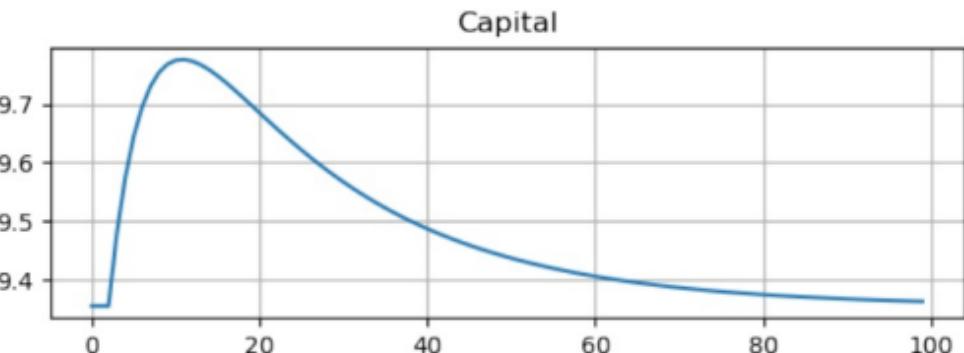
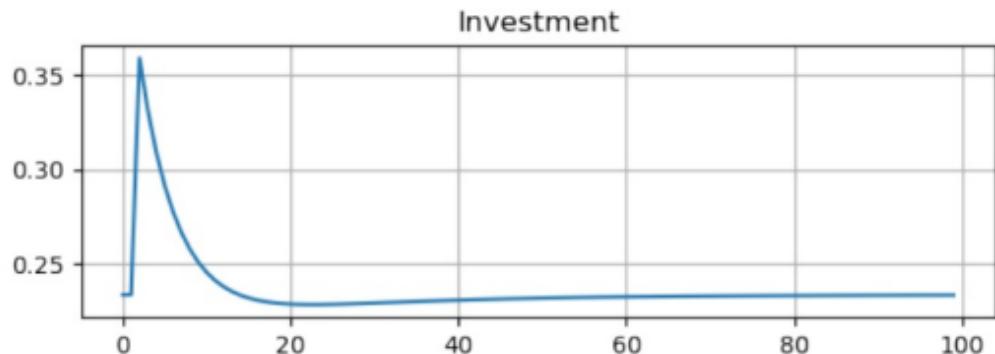
Impulse response functions — C, L

Transitory productivity shock: ϵ rises from 0 to σ_ϵ at $t = 2$ and is at 0 afterwards:



Impulse response functions — I, K

Transitory productivity shock: ϵ rises from 0 to σ_ϵ at $t = 2$ and is at 0 afterwards:



Transitory vs. permanent productivity shock

Transitory shock (blue) – as before;

permanent, anticipated shock (orange) – $\epsilon_t > 0$ constant for $t \geq 2$, such that steady-state Z is σ_ϵ :

