Intergenerational Redistribution with Endogenous Constraints to Private Debt

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Abstract

I study decentralization of optimal allocations in an endowment OLG economy where private agents face endogenous borrowing constraints. Lenders constrain credit because of limited commitment of borrowers: they can declare bankruptcy and only lose the ability to make savings for retirement in the future. The social planner can use lump-sum transfers and government debt in order to decentralize optimal allocations of consumption goods. I show that policies using government debt decentralize a larger set of optimal allocations, as opposed to balanced-budget policies with the same number of instruments. In addition, government debt rules out suboptimal equilibria where the private credit market does not operate. The results are explained by incentives to saving and debt repayment under limited commitment, so they do not hold under common specifications of exogenous debt constraints.

1 Introduction

The financial system and redistribution policies are both meant to enhance consumption smoothing, but the two systems can provide conflicting incentives to households. Consider an economy with a developed personal bankruptcy system, such as the United States. The opportunity for households to declare bankruptcy may create moral hazard, which is associated with credit constraints in unsecured loan markets (Jaffee and Russell, 1976; Kehoe and Levine, 1993). Redistribution policies can magnify the moral hazard if they alleviate the loss of consumption smoothing that follows bankruptcy. The focus of this article is the interaction of personal bankruptcy decisions and long-term policies that redistribute resources between generations. I find that an optimal policy — one that takes into account households' incentives for bankruptcy — is likely to use government debt, as opposed to running a balanced system of inter-generational transfers. The result is obtained by comparing policies with an equal number of fiscal instruments.

To understand the influence of redistribution policies on borrowing constraints, consider the bankruptcy-repayment trade-off of an indebted individual. In a personal bankruptcy described by Chapter 7 of the US Bankruptcy Code[1], borrowers' their unsecured loans

[1]The US personal bankruptcy law offers some of the highest levels of income and asset protection from

are written off and their subsequent labor earnings are protected from creditors. On the other hand, consumption smoothing becomes more difficult: if the bankrupt individual manages to accumulate savings after the bankruptcy, these must be used to repay past creditors. Borrowers therefore have limited commitment to debt repayment, meaning that a bankruptcy might be their optimal choice. Limited commitment to unsecured debt repayment has been introduced in a general equilibrium setting by Kehoe and Levine (1993). In their framework, lenders constrain the amount of credit just enough to make bankruptcy suboptimal for the borrower. Azariadis and Lambertini (2003) study this friction in an economy with overlapping generations, where individuals borrow early in life and save later on. If an agent decides not to repay her debts, she loses the opportunity to save for her old age.

I study optimal intergenerational redistribution in the framework of Azariadis and Lambertini (2003). The focus of the analysis is the different incentives to debt repayment created by lump-sum transfers on the one hand and government debt accumulation on the other. In spirit of Barro (1974), I question the equivalence of government debt and lump-sum taxes on prime-age agents as redistribution instruments. On the one hand, models with overlapping generations are known to produce excess of savings in equilibrium that the social planner can correct by a pay-as-you-go pension scheme (de la Croix and Michel, 2002) or by accumulating government debt (Diamond, 1965). On the other hand, in the framework with limited commitment, the pay-as-you-go redistribution from prime-age to retired agents affects adversely the endogenous borrowing constraint for the young. Taxing prime-age agents' income and paying out transfers to the old acts like doing savings on behalf of the prime-aged; crucially, these "savings" are not lost in case of personal bankruptcy[2]. This means that the inability to accumulate assets after personal bankruptcy is less costly under a large pay-as-you-go system, creating moral hazard and making borrowing constraints tighter in equilibrium. The effort of the state to better allocate goods between generations undermines the ability of credit markets to serve the same aim. This article therefore provides a credit-based argument for non-neutrality of government debt.

The view of government debt being equivalent to lump-sum taxes dates back to David Ricardo and is formalized in Barro (1974). Most relevant to this paper is the form of equivalence presented in (Buiter and Kletzer, 1992): government debt and lump-sum taxes or transfers are equivalent when a social planner decentralizes optimal consumption allocations without financial frictions. A substantial literature following Barro (1974) (Woodford, 1990), has explored how financial frictions make the Ricardian equivalence fail. In particular, when taxes decrease and government debt increases, welfare can be improved, since disposable income of households might increase in periods, when their savings are constrained from below – either in a credit or bequest context.

creditors. White (2007) compares it to the systems in EU countries: for instance, in France a large part of incomes and assets must be used for repayment of creditors. In Germany, income protection is stronger, but most assets are not protected.

[2]US federal law protects main classes of retirement accounts from creditors of a bankrupt borrower.

I show that limited commitment to debt repayment of young agents in an OLG economy makes a novel case for using government debt. It relaxes the borrowing constraint on the young by affecting the credit supply and not the credit demand (demand for negative bequest) as in Woodford (1990) (Barro, 1974). In my setup, the government does not levy taxes nor give transfers directly to the young: there are three periods of life and the policy only redistributes incomes of the second and the third period. Taxes and transfers affect the future, and not the current, income of the young. However, future incomes have a strong effect on the endogenous credit constraint due to limited commitment: the income profile shapes incentives for debt repayment in the future.

In this article, I provide conditions for government debt to be a necessary instrument for long-run redistribution in a life-cycle economy with limited commitment. I compare two policies that have the same number of policy instruments: one makes transfers between prime-age agents and old agents; another accumulates government debt and levies taxes or gives transfers to the old. I call the first policy balanced-budget and the second a debt-based one. I compare the sets of optimal allocations of consumption that can be decentralized with the two policies. The main result is that the optimal allocations decentralized by the balanced-budget policy is a subset of those decentralized by the debt-based policy.

A second, complementary, result of this article is that government debt can eliminate a suboptimal equilibrium where private credit markets fail to operate due to limited commitment. Multiple equilibria are a common property of models following Kehoe and Levine (1993), and at least one of the equilibria is typically suboptimal in the sense of Pareto. In both Kehoe and Levine (1993) and Azariadis and Lambertini (2003) this suboptimal equilibrium converges to a steady state with small interest rates and severe credit constraints. The intuition for such an equilibrium is the same in both articles. If low interest rates are expected in future periods when an agent has high income, this agent does not plan to make (large) savings at that time. Therefore, an exclusion from financial markets at that time is not costly and the agent can declare bankruptcy on her current debts with a low opportunity cost. The lenders then constrain credit to the agent, so she enters the next period with small debts and will make large savings by the end of the period, leading to low interest rates. The expected decrease of interest rates is therefore self-fulfilling. In my setting, government debt prevents such an equilibrium because it pushes interest rates up by absorbing savings and crowding out the supply of credit.

The case for using government debt found in this article is based on the limited commitment along the life cycle and not on mechanisms identified previously in the literature. Firstly, I do not consider government debt as an additional instrument to a set of transfers; instead, the main results are obtained by comparing policies with an equal number of instruments. Second, I do not consider the influence of the policy on credit demand (Woodford, 1990), but focus on the endogenous borrowing constraints, i.e. the credit supply. In addition, I show that policies with and without government debt are equivalent if standard exogenous borrowing constraints are assumed in my setting with overlapping generations. Finally, by focusing on lump-sum taxation I do not study distortions of taxes in the sense of Barro (1979), but rather distortions due to limited commitment that span multiple periods.

The rest of the article is organized as follows. Section 2 relates this article to the literature. In Section 3 I introduce the setup, which is the one of Azariadis and Lambertini (2003) augmented with a government sector, and then describe the equilibria of the model with two alternative institutional settings: with and without the possibility of personal bankruptcy. Section 5 then analyzes the problem of a social planner that decentralizes optimal allocations (optimality is defined in Section 5.1), with and without government debt. Finally, Section 5.6 highlights the role of endogenous borrowing constraints in the analysis by comparing it to common models with exogenous constraints. Section 6 concludes; the Appendix contains proofs of the article's propositions.

2 Related literature

The main focus of literature following Woodford (1990) has been whether public debt improves private consumption smoothing when markets are incomplete (Aiyagari and McGrattan, 1998; Floden, 2001; Eggertsson and Krugman, 2012; Guerrieri and Lorenzoni, 2017). The main intuition is that the increase of interest rates due an increase of public debt makes it easy to accumulate enough precautionary savings to avoid borrowing constraints in low-income periods. In another words, public debt is an additional savings device.

Closely related to the present article are Rohrs and Winter (2015) and Antunes and Ercolani (2020), but both articles have results opposite to mine. These authors have studied the problem in the limited commitment setting of Kehoe and Levine (1993) and arrive to a conclusion that high government debt makes endogenous borrowing constraint more tight. Their mechanism works through interest rates that are increasing in the level of government debt. In these models, unlike in the present article, high interest rates are associated with tighter endogenous borrowing constraints, for the following reasons. The authors assume infinite life horizon and a Markov income process where a transition to low levels of income can happen with positive probability in any state. This latter property is crucial, given that the authors use the definition of the endogenous borrowing limit in stochastic settings that follows Alvarez and Jermann (2000). According to this definition, the credit limit must be such that the borrower doesn't go bankrupt in the following period in any state of the world. Therefore, the most relevant state of the world for setting the borrowing limit is the one with lowest borrower income, because marginal utility of consumption is then highest and so is the incentive to go bankrupt. Importantly, in this state of the world the borrower is also most likely to be a net borrower and not a net saver at the end of the period, if she doesn't go bankrupt. This has direct consequences for the effect of government debt: it makes interest rates higher, so that the value of having access to financial market is smaller in the worst state of the world and endogenous borrowing constraints become more tight. This result is reversed in my setting not because of deterministic incomes, but rather because of the life-cycle income pattern. Retirement is an anticipated, large negative income shock for individuals. This makes it more likely that a person expects to be a net saver in any state of the world next period, so the endogenous borrowing limit increases in the expected interest rates.

The contribution of Antinolfi et al. (2007) is closely related to mine in its focus on decentralization of equilibria under endogenous borrowing constraints. They study the problem of the social planner that decentralizes optimal allocations with lump-sum transfers and monetary policy, in an infinite horizon economy with limited commitment. Furthermore, money is equivalent to government debt in their framework. The two assumptions that make the model different with respect to the present article are infinite lives of the agents and possibility to renege on all the future taxes and transfers that make the agent worse off in sum. In this framework, government debt is found to crowd out private lending one for one — a property found in a more general setting by Hellwig and Lorenzoni (2009). Government debt, therefore, does not allow for additional opportunities to decentralize optimal allocations if lump-sum transfers are used, in contrast with my paper. Moreover, the authors' result on multiplicity of equilibria is opposite to mine — by using lump-sum transfers, the social planner can eliminate the autarkic steady state, while it is not the case if government debt is the only instrument.

Some of my findings parallel those that have been obtained recently in other types of models. Carapella and Williamson (2015) study the credit markets under limited commitment and the role of government debt in a Lagos and Wright (2005) type of framework. Their model has infinitely lived agents, asymmetric information and limited enforceability of taxes. Their main result is analogous to mine: the presence of government debt improves the endogenous borrowing limits, since it raises the opportunity cost of default. Mian et al. (2020) study a model with infinitely-lived agents and collateralized credit; they find that equilibrium multiplicity vanishes when government debt is sufficiently high. As in my framework, government debt eliminates an equilibrium with low private debts.

3 Model

3.1 Economy structure, policy

This article follows the framework of Azariadis and Lambertini (2003), with some changes of notation and an additional government sector. Time is discrete and starts at period $t = 0$. The economy is populated by generations of identical agents that live for three periods. I call the agents young, adult and old in their first, second and third period of life, respectively. Generation t is the set of agents that are born in $t-1$ and are adults in t. There are N_t identical agents in a generation t. Population grows at a constant rate n, so $N_{t+1} = (1+n)N_t$. There is one good in the economy and the agents can consume it when young, when adult and when old. Agents of each cohort are endowed with y_0, y_1 and y_2

units of the good in the three periods of life, with $(y_0, y_1, y_2) > 0$. The consumption good is not storable, so generations need to exchange claims on endowments if they need to smooth their consumption profile. The consumption in the first, second and third period of life of an agent of generation t is denoted c_{t-1}, d_t and e_{t+1}, respectively.

This agent's utility from the three periods' consumption is represented by the following function:

$$U(c_{t-1}, d_t, e_{t+1}) = u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1})$$

with $u(\cdot)$ of constant intertemporal elasticity of substitution (CIES) type: either $u(x) = \frac{x^{1-1/\sigma}}{1-1/\sigma}$ with $\sigma > 1$ or $u(x) = \log(x)$ ($\sigma \to 1$). The assumption $\sigma \geq 1$ is used to simplify the equilibrium properties of the model: Kehoe and Levine (1990) have shown that $\sigma < 1$ generates equilibrium multiplicity in endowment OLG economies with three periods of life, even in absence of financial frictions.

The government can make positive and negative lump-sum transfers to different generations and finance them by issuing government debt. Without loss of generality, I abstract from government spending. Moreover, I abstract from transfers to the young generation — this is motivated in the Section 5. Two kinds of transfers are done in period t: a transfer to an adult and to an old agent, denoted τ_t^1 and τ_t^2, respectively. Any transfer can be negative, in which case I call it a tax. A positive sum $N_t\tau_t^1 + N_{t-1}\tau_t^2$ means there is a primary deficit of the government budget, which can be financed by government debt. The end-of-period government debt stock per adult of period t is denoted g_t and the gross interest paid on this stock of debt in the following period is R_{t+1}. The value of g_t can generally be negative, but the main results of the article will be obtained for non-negative government debt. The government budget constraint is the following:

$$N_t g_t = R_t N_{t-1} g_{t-1} + N_t \tau_t^1 + N_{t-1}\tau_t^2$$

or, using the constant population growth:

$$g_t = \frac{R_t}{1+n} g_{t-1} + \tau_t^1 + \frac{\tau_t^2}{1+n} \tag{1}$$

There is no uncertainty in the model and expectations are rational, so agents have perfect foresight. The policy, that is, the sequence of transfers and government debt $(\tau_t^1, \tau_t^2, g_t)_{t\geq 0}$, is announced by the government at the beginning of period 0 and is implemented as announced forever after. In the central part of the article on decentralization, the policy is endogenous : it depends on the allocation that the social planner aims to decentralize. However, some auxiliary results are formulated in terms of the model outcomes for different choices of policy, as if it was exogenous.

A private agent can borrow and lend by issuing and buying one-period bond-like assets.

An agent of generation t then faces the following budget constraints:

$$\begin{cases} c_{t-1} + b^0_{t-1} & \leq y_0 \\ d_t + b^1_t & \leq y_1 + \tau^1_t + R_t b^0_{t-1} \\ e_{t+1} & \leq y_2 + \tau^2_{t+1} + R_{t+1} b^1_t \end{cases} \tag{2}$$

where b^0_{t-1}, b^1_t are end-of-period asset positions of the agent when young and adult, respectively; τ^1_t, τ^2_{t+1} are transfers or taxes paid when adult and when old; R_t is the interest factor on assets purchased or sold at $t-1$. Private and government debts are assumed to be perfect substitutes[3].

The following two assumptions are made for tractability of saving and borrowing behaviors:

Assumption 1. *The life-cycle endowment profile is hump-shaped:* $y_1 > y_0$; $y_1 > y_2$.

Assumption 2. *Transfers are such that disposable income is positive at each period:* $\tau^1_t > -y_1,\ \tau^2_t > -y_2,\ \forall t \geq 0$.

Assumption 1, together with the CIES preferences with discounting, guarantees that the young borrow and the adults save in an equilibrium without policy.

Assumption 2 is introduced because negative disposable incomes raise two issues. Firstly, it makes consumer behavior more complex in terms of demands. Secondly, it makes consumption negative in a situation where agents do not hold assets, whereas this situation must be studied in the environment of limited commitment, introduced below. The consequences of the Assumption are also twofold: first, it limits the domain of policies in the comparative statics analysis. Second, it limits the set of allocations that can be decentralized, which is covered in the second part of the paper.

I will analyze two versions of institutional environment for private debts in the model. In the first environment, the agents are obliged to use their endowments and transfers to repay their debts. In this case, there are no borrowing constraints: agents can borrow up to the present value of their future endowments and transfers. I will call this environment full commitment to loan repayment, or full commitment in short. In the second environment, agents can declare bankruptcy when adult or when old and have a right to keep their endowments and transfers afterwards. Such agents do not participate in the asset markets for their remaining life. For sufficiently high debt levels, exclusion from asset markets does not discourage borrowers from bankruptcy – this is a limited commitment environment. Lenders are assumed to limit lending as in Kehoe and Levine (1993), imposing a lower limit on b^0_t and b^1_t. The following section analyzes equilibria in the two environments in turn, starting with the simpler case of full commitment.

[3] Although possibility of personal bankruptcy is introduced in Section 3.3, the bankruptcy is ruled out in equilibrium by the borrowing constraints, so the private assets are risk-free, as the government bonds.

3.2 Equilibrium with full commitment

In this benchmark environment the agents face no credit constraints — they can borrow up to the present value of their incomes after transfers. Then, the utility maximization problem of an agent of generation t can be written with the intertemporal budget constraint:

$$\begin{aligned} \max_{(c_{t-1},d_t,e_{t+1})>0} \quad & u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1}) \\ \text{s.t.} \quad & c_{t-1} + \frac{d_t}{R_t} + \frac{e_{t+1}}{R_t R_{t+1}} = y_0 + \frac{y_1 + \tau_t^1}{R_t} + \frac{y_2 + \tau_{t+1}^2}{R_t R_{t+1}} \end{aligned} \tag{3}$$

The fact that $\lim_{x\to 0+} u'(\cdot) = +\infty$, following from the CIES type of the function, guarantees that the solution is interior. The resulting asset positions can be obtained from $b_t^0 = y_0 - c_t$ and $b_t^1 = y_1 + R_t b_{t-1}^0 + \tau_t^1 - d_t$.

The asset market clearing condition requires a null total asset position of the private and the government sectors:

$$N_{t+1} b_t^0 + N_t b_t^1 - N_t g_t = 0$$

$$\Leftrightarrow (1+n) b_t^0 + b_t^1 = g_t \tag{4}$$

When time begins, at $t = 0$, generations $-1, 0, 1$ are alive. The generation -1 is old and only consumes its endowment, the transfers and the assets with the interest income. For this generation, the assets and the interest, $R_0 b_{-1}^1$, are exogenous. The generation -1 makes no decisions in the model and their consumption is determined by exogenous parameters and the transfer τ_0^2:

$$e_0 = y_2 + \tau_0^2 + R_0 b_{-1}^1 \tag{5}$$

The generation 0 optimizes their adult and old age consumption, given that it has some exogenous assets and interest on them:

$$\begin{aligned} \max_{(d_0,e_1)>0,b_0^1} \quad & u(d_0) + \beta u(e_1) \\ \text{s.t.} \quad & \begin{cases} d_0 + b_0^1 & = y_1 + \tau_0^1 + R_0 b_{-1}^0 \\ e_1 & = y_2 + \tau_1^2 + R_1 b_0^1 \end{cases} \end{aligned} \tag{6}$$

Equilibrium with full commitment can now be defined:

Definition 1 (Equilibrium with full commitment)**.** *An equilibrium of an economy with a policy $(\tau_t^1, \tau_t^2, g_t)_{t\geq 0}$ and full commitment is a sequence of positive variables $(c_t, d_t, e_t, R_t)_{t\geq 0}$ such that:*

- $\forall t \geq 1, (c_{t-1}, d_t, e_{t+1})$ *solves* (3)
- $\forall t \geq 0,$ *the asset market clearing condition* (4) *holds*

- $\forall t \geq 0$, *the government budget constraint* (1) *holds*
- $R_0, b^0_{-1}, b^1_{-1}, g_{-1}$ *are exogenous and* e_0 *is given by* (5)*;* d_1 *and* e_2 *solve* (6)*.*

The focus of the article is decentralization of allocations, so I do not address the general question of existence and uniqueness of equilibria. However, Sections 4.2, 5.5 discuss an special type of equilibrium steady state — an autarky — that can co-exist with more generic steady-state equilibria under limited commitment.

Most of the analysis below assumes stationary policies and focuses on steady-state equilibria. A stationary policy is defined as follows:

$$\tau^1_t = \tau^1,\ \tau^2_t = \tau^2,\ g_t = g = \frac{(1+n)\tau^1 + \tau^2}{1+n-R}\ \forall t > 0$$

where R is the equilibrium steady-state interest that results from the stationary transfers τ^1, τ^2. Equilibrium steady states are characterized below.

Note that economies with a balanced-budget stationary policy are closely related to the ones without government, studied by Azariadis and Lambertini (2003). Namely, one can re-define the endowments such that the transfers are included. All the results of Azariadis and Lambertini (2003) then hold for such economies, provided that the "post-transfer" endowments satisfy the Assumption 1.

In order to identify the role of government debt in the economy, I will contrast policies with perpetual primary deficits, $(1+n)\tau^1 + \tau^2 > 0,\ g > 0$, to balanced-budget policies $\tau^2 = -(1+n)\tau^1,\ g = 0$.

In what follows, I use the following notation for marginal rates of substitution:

$$MRS_{cd}(c_t, d_{t+1}) = \frac{u'(c_t)}{\beta u'(d_{t+1})};\quad MRS_{de}(d_t, e_{t+1}) = \frac{u'(d_t)}{\beta u'(e_{t+1})}$$

One can now characterize steady state properties of economies with balanced-budget stationary policies.

Proposition 1. *Under balanced-budget stationary policies, a steady-state equilibrium with full commitment exists and is unique. The steady-state interest rate is decreasing in* τ^1*.*

Proof. See section 7.1 of the Appendix.

Finally, I introduce notation for a steady state interest rate in an economy without policy and with full commitment:

Definition 2 (Laissez-faire interest)**.** *For a given economy, a laissez-faire interest rate* R^{LF} *is the steady-state interest rate in absence of policy and with full commitment.*

3.3 Equilibrium with limited commitment

Suppose now that indebted agents have a possibility to declare personal bankruptcy. For an adult of generation t with $b^0_{t-1} < 0$, this decision would be made at the beginning of

t. Their debts are written off and neither their endowments nor transfers can be seized. The only consequence is that such an agent cannot borrow nor save after bankruptcy, as in Kehoe and Levine (1993)[4]. There is full information in the model, so lenders are aware of the incentives to declare bankruptcy and are assumed to limit lending. Namely, there are two types of borrowing constraints in the model: one for adults and one for the young. The one for adults is motivated by limited commitment, but is technically an exogenous no-borrowing constraint. The borrowing constraint for the young, which is the focus of this article, is fully endogenous in the sense that the borrowing limit is a function of endogenous variables and structural parameters of the economy. I first present the constraint for adults.

If an adult borrows any amount, she has debts outstanding when old. However, she can declare bankruptcy at no cost, since she has no more need for financial markets and her endowment cannot be seized. The following individual rationality constraint ensures that an agent of generation t does not declare bankruptcy in the third period of life:

$$u(e_{t+1}) \geq u(y_2 + \tau^2_{t+1}) \tag{IR2}$$

The number 2 is in the label (IR2) because this is a second individual rationality constraint that an individual faces during her life; the first one is discussed below. This individual rationality and the budget constraint imply together that adults cannot borrow:

$$b^1_t \geq 0 \tag{7}$$

This no-borrowing constraint for the adults is a consequence of the institutional environment of the model. However, it is not endogenous: the right hand side does not depend on other variables of the model. This borrowing constraint for the young, on the contrary, is endogenous.

The individual rationality constraint for the young is defined in terms of their consumption when adult and when old:

$$u(d_t) + \beta u(e_{t+1}) \geq u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1}) \tag{IR1}$$

The right hand side is the utility after bankruptcy: no debts are repaid in the second period, but no savings are made for the third one. Whenever a young agent borrows, her future expected utility must not be below this guaranteed level. Assumption 2 on the policy makes sure that the right hand sides of (IR1), (IR2) are always defined. The constraint (IR1) is directly related to the saving decisions that the agent makes in her youth. By the budget constraint, (IR1) cannot hold if b^0_{t-1} is too low. In order to obtain the endogenous borrowing limit for the young, consider the utility maximization problem

[4] Savings are not not prohibited, but can be used to compensate for creditors' losses.

of an adult individual of period t, taking the amount of b^0_{t-1} as given:

$$\max_{(d_t, e_{t+1})>0, b^t_1} \quad u(d_t) + \beta u(e_{t+1})$$

$$\text{s.t.} \begin{cases} d_t + b^1_t & = y_1 + \tau^1_t + R_t b^0_{t-1} \\ e_{t+1} & = y_2 + \tau^2_{t+1} + R_{t+1} b^1_t \\ b^1_t & \geq 0 \end{cases} \tag{8}$$

Denote the demands for consumption that result from this program $d(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1})$ and $e(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1})$ and the corresponding indirect utility $V(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1})$:

$$V(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1}) \equiv u(d(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1})) + \beta u(e(R_{t+1}, R_t b^c_{t-1}, \tau^1_t, \tau^2_{t+1})) \tag{9}$$

If this utility is lower than the utility after bankruptcy, the agent chooses bankruptcy on b^0_{t-1}. This leads to the individual rationality constraint (IR1), written for equilibrium consumption levels that maximize utility :

$$V(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1}) \geq u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1}) \tag{10}$$

In equilibrium, the endogenous borrowing constraint for the young is such that the incentives to declare bankruptcy and to repay are equal, as in Kehoe and Levine (1993). If the young were allowed to borrow more than this amount, they would declare bankruptcy. If the constraints were more strict than this amount, lenders would forego opportunities of risk-free lending. Both situations are assumed to be out of equilibrium. Then the lower limit on assets in youth b^c_{t-1} is given in equilibrium by

$$V(R_{t+1}, R_t b^c_{t-1}, \tau^1_t, \tau^2_{t+1}) = u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1}) \tag{11}$$

Thanks to the monotonicity of V in wealth, $R_t b^c_{t-1}$ can be expressed as a function of $R_{t+1}, \tau^1_t, \tau^2_{t+1}$.

Lemma 1. *Equation* (11) *defines* $R_t b^c_{t-1}$ *as a continuously differentiable function* f *of* $(R_{t+1}, \tau^1_t, \tau^2_{t+1})$ *with a domain* $R_{t+1} > 0$, $\tau^1_t > -y_1$, $\tau^2_{t+1} > -y_2$ *and:*

$$f(R_{t+1}, \tau^1_t, \tau^2_{t+1}) \left[\begin{array}{ll} < 0 & \text{if } R_{t+1} > MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1}), \\ = 0 & \text{if } R_{t+1} \leq MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1}) \end{array} \right.$$

Proof. See section 7.2 of the Appendix.

According to the range of f, the value of $R_t b^c_{t-1}$, and hence of b^c_{t-1}, is always non-positive. It means that b^c_{t-1} is only a credit limit and never forces agents to make positive savings. The credit limit is null when interest rates are too low: this is because the agents have no incentives to make savings when adults. In this case, the exclusion from financial markets after bankruptcy is not costly and the agents choose to declare bankruptcy on any debt. In this case, borrowing when young is then ruled out altogether.

The borrowing constraint for the young can be stated as follows:

$$b_{t-1}^0 \geq f(R_{t+1}, \tau_t^1, \tau_{t+1}^2)/R_t \tag{12}$$

The maximization problem of a young agent of generation $t \geq 1$ with limited commitment can be written:

$$\max_{(c_{t-1}, d_t, e_{t+1}) > 0, b_{t-1}^0, b_t^1} \quad u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1})$$
$$\text{s.t.} \begin{cases} c_{t-1} + b_{t-1}^0 & = y_0 \\ d_t + b_t^1 & = y_1 + \tau_t^1 + R_t b_{t-1}^0 \\ e_{t+1} & = y_2 + \tau_{t+1}^2 + R_{t+1} b_t^1 \\ b_{t-1}^0 & \geq f(R_{t+1}, \tau_t^1, \tau_{t+1}^2)/R_t \\ b_t^1 & \geq 0 \end{cases} \tag{13}$$

Initial generations have exogenous asset positions, which raises two complications. Firstly, generation -1 declares bankruptcy whenever their exogenous asset position is negative, $b_{-1}^1 < 0$. In what follows, I always assume an initial condition $b_{-1}^1 \geq 0$. Secondly, the generation 0 might declare bankruptcy on their exogenous debts when $b_{-1}^0 < 0$, but this depends on the policy and interest rates. A definition of equilibrium must then include the individual rationality constraint (10) verified for the generation 0.

This leads to the following definition of an economy with limited commitment:

Definition 3 (Equilibrium with limited commitment.)**.** *An equilibrium of an economy with policy* $(\tau_t^1, \tau_t^2, g_t)_{t \geq 0}$ *and limited commitment is a sequence of positive variables* $(c_t, d_t, e_t, R_t)_{t \geq 0}$ *such that:*

- $\forall t \geq 0, (c_{t-1}, d_t, e_{t+1})$ *solves* (13)
- $\forall t \geq 0,$ *asset market clearing condition* (4) *holds*
- $\forall t \geq 0,$ *government budget constraint* (1) *holds*
- $b_{-1}^1 \geq 0, R_0 b_{-1}^0, g_{t-1}$ *are exogenous and* e_0 *is given by* (5)*;* d_1 *and* e_2 *are given by* (6)
- (10) *holds for period* 0*: generation* 0 *does declare bankruptcy.*

As shown by Azariadis and Lambertini (2003), an economy without transfers can have multiple equilibria with limited commitment. This article focuses on the question of decentralization of allocations, or whether a given allocation can be *one of* the equilibria. I address the question of multiple equilibria partially in the following section.

4 Equilibrium effects of policy: comparative statics

Before adressing the central topic of decentralization of optimal equilibria, I establish some results that help to build intuition for the following parts. In this section, I show two effects of transfers and government debt on the endogenous borrowing constraint. First, I show how balanced-budget policies with constant transfers make borrowing constraints more or less tight. Second, I show that government debt rules out an equilibrium where borrowing constraints rule out credit completely.

4.1 Transfers and the endogenous borrowing limit for the young.

Focusing on stationary balanced-budget policies, I show the influence of the policy on the borrowing limit as a comparative statics exercise.

Proposition 2. *In an economy with limited commitment and stationary balanced budget policy* $\tau_t^1 = \tau^1$; $\tau_t^2 = -(1+n)\tau^1$, *the function* $f(R_{t+1}, \tau^1, -(1+n)\tau^1)$ *(negative of credit limit for the young of generation t) is decreasing (has a null derivative) in* τ^1 *if* $b_t^1 > (=)\ 0$.

Proof. See section 7.3 in the Appendix. □

The intuition of the proposition is as follows. A decrease in τ^1 with a corresponding increase in τ^2 crowds out savings. This means the opportunity to make savings for old age is valued less and the agents have more incentives to declare personal bankruptcy. The corresponding borrowing constraint is tighter in equilibrium — $R_t b_{t-1}^c$ increases.

Taken together with Proposition 1 about the steady state without borrowing constraints, this result implies credit conditions for the young are more favorable under larger τ^1 — smaller taxes on adult agents. On the one hand, the borrowing limits become less tight, making it less likely that the constraints bind in equilibrium. On the other hand, the interest rates become smaller in the unconstrained steady state, making borrowing less costly.

4.2 Government debt and autarky

Endogenous credit constraints can result in an equilibrium autarky, where credit is absent. Agents consume their endowments and transfers in each period of life and have null asset positions:

$$(c_t, d_t, e_t) = (y_0, y_1 + \tau_t^1, y_2 + \tau_t^2), \quad t \geq 0 \tag{14}$$

Since generation -1 and 0 have exogenous initial asset positions, the following initial conditions are necessary for the possibility of autarky as equilibrium:

$$b_{-1}^0 = b_{-1}^1 = g_{-1} = 0 \tag{15}$$

Under full commitment, autarky is not a generic equilibrium, even if (15) is verified. Firstly, it has been shown in Proposition 1 that economies without government debt and stationary transfers have a unique steady state, which is generically not autarkic. Second, autarky is impossible under policies that include government debt: according to equation (4), government debt is a counterpart of a nonzero private asset position.

Under limited commitment, autarky is a prevalent equilibrium of economies with null initial asset positions. However, it is ruled out if government debt is present in the economy:

Proposition 3. *In an economy with limited commitment and null initial asset positions given by* (15)*, any sequence of interest rates satisfying*

$$R_{t+1} \leq \min\{MRS_{cd}(y_0, y_1 + \tau^1_{t+1}),\ MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})\}, \quad \forall t \geq 0 \tag{16}$$

results in equilibrium autarky if, and only if, there is no government debt in the economy.

Proof. See Section 7.4 in the Appendix. □

This kind of equilibrium exists in a number of models with limited commitment that follow Kehoe and Levine (1993). Intuitively, if low interest rates are expected in the future when the agent has relatively high income, i.e. in the second period of life, the agent does not plan to make (large) savings at that time. Therefore, an exclusion from financial markets is not costly and the agent can declare bankruptcy on her current debts with a low opportunity cost. The lenders then constrain credit to the agent, so she enters the next period with small debts and will make large savings by the end of the period, leading to low interest rates. The expected decrease of interest rates is self-fulfilling. In the economy of Azariadis and Lambertini (2003) that does not have any policy, such an equilibrium always exists, but it is not the case when policy is introduced. While abstracting from global analysis of multiple equilibria, I show in the above Proposition that the autarkic equilibrium is ruled out if government is used.

Intuitively, government debt can rule out autarky in two ways. First one, not specific to the limited commitment environment, is raising the interest rates sufficiently high for all agents to become net savers. The second one is based on limited commitment and operates as follows. By crowding out credit supply and raising interest rates in equilibrium, government debt increases the demand for savings of the adults. This means that an exclusion from the financial markets in the event of bankruptcy becomes more costly. By Lemma 1, this implies nonzero credit limits for the same agents when they are young, which rules out autarky. Therefore, government debt both incentivizes savers to save and enables borrowers to borrow by raising interest rates.

While optimality of allocations has not been addressed yet, one can see that autarky is a weakly sub-optimal steady state. Assume that autarky co-exists with other possible equilibria in an economy. Since the autarkic allocation is in the agents' budget set under any interest rates, the existence of non-autarkic equilibria implies weak revealed preference for corresponding consumption allocations. I address optimal allocations and their decentralization in the remainder of the article.

5 Decentralization with transfers and government debt

In order to further analyze policies with and without governement debt, I study them in a context of decentralization. In other words, I study whether the use of transfers or debt can make a given optimal allocation an equilibrium.

There are two frictions, or inefficiencies, that the social planner has to correct with the policy instruments. The first one is the overlapping generations structure: an equilibrium without policy intervention can have too much savings, as shown in the seminal papers on OLG (Samuelson, 1958; Diamond, 1965). This can be addressed by crowding out the private savings with either a pay-as-you-go transfer scheme from adults to the old or government debt used with at least one tax/transfer instrument in order to balance the government budget.

The second friction, central to this article, is the limited commitment of agents to repay debts and the resulting borrowing constraints. The choice of policy instruments used for decentralization has a number of indirect effects on the ability of agents to borrow. Firstly, the choice between transfers and government debt affects the set of equilibrium interest rates. This has been discussed in Section 4.2. The second effect is produced by the transfers entering directly the right hand side of individual rationality constraints (IR1), (IR2) or, equivalently, the borrowing constraints (7), (12). Proposition 2 has illustrated this in comparative statics, showing that smaller transfers (or larger taxes) to adults and corresponding larger transfers (lower taxes) to the old make the endogenous borrowing constraint more tight.

The goal of this part of the article is to study the indirect effects of policy on borrowing constraints, so I do not consider transfers to the young as a policy instrument, which alleviate the constraint directly. Instead, I address a question of whether the policy of transfers to the adults and to the old allows the young to finance their desired consumption with credit. Under the constraints on transfers from Assumption 2, whether a given set of instruments is sufficient for decentralization is not trivial, even under full commitment. Indeed, even a full set of transfers would potentially fail to decentralize some allocations under such constraints. I address the problem in the following way: first I establish conditions for decentralization under full commitment in Section 5.2. For an economy with limited commitment, I then compare policies under the assumption that they are equally good for decentralization under full commitment, i.e. they satisfy Assumption 2. I abstract from the commitment problems of the social planner : the government debt is always repaid and transfers are always made as announced. Since there is full information in the model, and the government announces all the policy variables in period 0, private agents have perfect foresight on all the future transfers and government debt.

In what follows, I first describe the goal of the social planner: an optimal allocations of consumption between generations. Then I study the benchmark decentralization problem:

decentralization with full commitment. Finally, I address the same problem under limited commitment and contrast the results to the benchmark case and to standard models of exogenous borrowing constraints.

5.1 Optimal allocations

I focus on allocations that are homogeneous both within and across generations. The social planner only optimizes three variables: the consumption in youth, the consumption in adult age and consumption in old age, same for all agents.

All the allocations chosen by the social planner must be feasible, meaning that aggregate consumption of each given period must not exceed the aggregate endowment. This is described by the following the resource constraint:

$$\begin{aligned} &N_{t+1}c_t + N_t d_t + N_{t-1}e_t \leq N_{t+1}y_0 + N_t y_1 + N_{t+1}y_2 \quad \forall t \geq 0 \\ \Leftrightarrow\ & c_t + \frac{d_t}{1+n} + \frac{e_t}{(1+n)^2} \leq y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \quad \forall t \geq 0 \end{aligned} \tag{17}$$

In what follows I write the resource constraint as equality since the objective function will always be monotonic.

The baseline optimal allocation is such that all the generations born in $t \geq 0$ obtain the same level of lifetime utility and this utility is maximized. In production economies, such an allocation results from the Golden Rule of capital accumulation of Phelps (1961), so I label it the Golden Rule allocation. To obtain the Golden Rule allocation, one has to maximize the utility of the representative generation, subject to the resource constraint:

$$\max_{(c,d,e)} \ u(c) + \beta u(d) + \beta^2 u(e) \quad \text{s.t.} \ \ c + \frac{d}{1+n} + \frac{e}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \tag{18}$$

This well-defined problem always has a solution, which is unique.

More generally, one can be interested in allocations that are obtained from a maximization of a weighed sum of utilities of all generations. In general, such allocations are not stationary, or homogeneous between generations[5]. However, I show that in the present framework they are stationary if, and only if, different generations' utilities are discounted at a constant rate. I will focus on this case for the main results on decentralization.

Proposition 4. *Let the social planner maximize a weighed sum of all generations' utility, with consumption homogeneous within a given generation. This is equivalent to the following program:*

$$\begin{aligned} &\max_{(c_t,d_t,e_t)_{t\geq 0}} \sum\nolimits_{t\geq 0} (\omega_{t+2}u(c_t) + \omega_{t+1}\beta u(d_t) + \omega_t \beta^2 u(e_t)) \\ &s.\ t.\ \forall t \geq 0, \quad c_t + \frac{d_t}{1+n} + \frac{e_t}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \end{aligned} \tag{19}$$

[5] It is straightforward to show that any such allocation is optimal in the sense of Pareto. However, it is not the whole set of Pareto optimal allocations since I restrict the consumption to be homogeneous *within* each generation.

where $(\omega_t)_{t\geq 0}$ are such that their sum converges and ω_t is the weight of an agent of generation $t-1$ times the size of the generation, N_{t-1}. The solution exists, is unique, and is a stationary allocation $(c_t, d_t, e_t) = (\hat{c}, \hat{d}, \hat{e})$ $\forall t \geq 0$ if, and only if, $\frac{\omega_{t+1}}{\omega_t} = \alpha \in]0,1[$ $\forall t \geq 0$. In this case, the allocation is also obtained from a simplified program:

$$\begin{aligned} \max_{(c,d,e)} \quad & \alpha^2 u(c) + \alpha\beta u(d) + \beta^2 u(e) \\ s.\ t. \quad & c + \frac{d}{1+n} + \frac{e}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2} \end{aligned} \tag{20}$$

The same program results in the Golden Rule allocation if $\alpha = 1$.

Proof. See section 7.5 in the Appendix. □

The notation $(\hat{c}, \hat{d}, \hat{e})$ will be used in the article to denote all the possible stationary optimal allocations that solve program (20). Notation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ is more general and denotes any solution to program (19). The following equality follows immediately from the first-order conditions of program (19):

$$\forall t \geq 0, \quad MRS_{cd}(\hat{c}_t, \hat{d}_{t+1}) = MRS_{de}(\hat{d}_t, \hat{e}_{t+1}) \tag{21}$$

To shorten notation, I use the following shortcuts for any optimal allocation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ and optimal stationary allocation $(\hat{c}, \hat{d}, \hat{e})$:

$$\begin{aligned} M\hat{R}S_t &\equiv MRS_{cd}(\hat{c}_t, \hat{d}_{t+1}) = MRS_{de}(\hat{d}_t, \hat{e}_{t+1}) \\ M\hat{R}S &\equiv MRS_{cd}(\hat{c}, \hat{d}) = MRS_{de}(\hat{d}, \hat{e}) \end{aligned}$$

Lemma 2. *Any stationary optimal allocation $(c_t, d_t, e_t) = (\hat{c}, \hat{d}, \hat{e})$ $\forall t \geq 0$ solving (20) has $M\hat{R}S = \frac{1+n}{\alpha} \geq 1+n$. The last inequality is strict for the Golden Rule allocation.*

Proof. $M\hat{R}S = \frac{1+n}{\alpha}$ follows directly from the first-order conditions of the program (20). Use the values of α from Proposition 4 to conclude. □

Finally, one value of the discount factor makes decentralized equilibrium without policy and the social planner's optimum coincide. Namely, this is the case when the allocation has $M\hat{R}S = R^{LF}$, which means $\alpha = \frac{1+n}{R^{LF}}$ by the previous Lemma. I label this value the *laissez faire* discount factor:

Definition 4 (Laissez-faire discount factor)**.** *For a given economy, the laissez-faire discount factor is $\alpha^{LF} \equiv \frac{1+n}{R^{LF}}$, with R_{LF} defined in Definition 2.*

This value will be used for reference in the decentralization analysis, since the corresponding optimal allocation will be trivially possible to decentralize with any policy.

5.2 Decentralization with full commitment

Three policy instruments can be used for decentralization. These are transfers to the adults $\{\tau_t^1\}_{t\geq 0}$, transfers to the old $\{\tau_t^2\}_{t\geq 0}$ and government debt $\{g_t\}_{t\geq 0}$. The main question of this section is similar to the one of Barro (1974): can the social planner achieve more stationary allocations if transfers are substituted with government debt? I approach this by comparing two types of policy: a balanced-budget one, that uses two transfers and a debt-based one, which uses government debt and transfers to the old. Policies that do not use transfers to the old cannot decentralize the consumption of the initial old generation, so they are not analyzed. On the other hand, policies that use the three instruments trivially decentralize the largest set of allocations, so they too are left out of the analysis.

The notation for the two types of policy to be compared is:

1. Balanced-budget policies $(\bar{\tau}_t^1, \bar{\tau}_t^2)_{t\geq 0}$ with $g_t = 0$ $(\Rightarrow \bar{\tau}_t^2 = -(1+n)\bar{\tau}_t^1)$, $\forall t \geq 0$.

2. Debt-based policies $(\breve{g}_t, \breve{\tau}_t^2)_{t\geq 0}$ with $\tau_t^1 = 0$ $(\Rightarrow \breve{\tau}_t^2 = (1+n)\breve{g}_t - R_t\breve{g}_{t-1})$, $\forall t \geq 0$.

I will refer to the two policies as minimal policies since they use the smallest possible number of instruments.

In line with the seminal result of Barro (1974), transfers or taxes on adults and government debt have similar role in decentralizing allocations in full commitment economies without borrowing constraints. The only difference that may arise between the two policies is whether the constraints on transfers from Assumption 2 are met.

Proposition 5. *An optimal allocation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ can be decentralized with at most one balanced-budget policy $(\bar{\tau}_t^1, \bar{\tau}_t^2)_{t\geq 0}$, where:*

$$\bar{\tau}_0^1 = -\frac{R_0 g_{-1} + \bar{\tau}_0^2}{1+n} = \hat{d}_0 - y_1 - R_0 b_{-1}^0 + (1+n)(\hat{c}_0 - y_0) \tag{22}$$

$$\bar{\tau}_t^1 = -\frac{\bar{\tau}_t^2}{1+n} = \hat{d}_t - y_1 + \hat{MRS}_{t-1} \cdot (\hat{c}_{t-1} - y_0) + (1+n)(\hat{c}_t - y_0), \ \forall t > 0 \tag{23}$$

It can be decentralized with at most one debt-based policy $(\breve{g}_t, \breve{\tau}_t^2)_{t\geq 0}$, where:

$$\breve{g}_t = -\bar{\tau}_t^1 = -\hat{d}_t + y_1 - \hat{MRS}_{t-1} \cdot (\hat{c}_{t-1} - y_0) - (1+n)(\hat{c}_t - y_0), \ \forall t \geq 0 \tag{24}$$

$$\breve{\tau}_0^2 = (1+n)\breve{g}_0 - R_0 g_{-1} \tag{25}$$

$$\breve{\tau}_t^2 = (1+n)\breve{g}_t - \hat{MRS}_{t-1}\breve{g}_{t-1} \quad \forall t > 0 \tag{26}$$

For both policies, the decentralization is possible if and only if the transfers meet the Assumption 2, that is, $\forall t \geq 0$, $\bar{\tau}_t^1 > -y_1$; $\bar{\tau}_t^2 > -y_2$ and $\breve{\tau}_t^2 > -y_2$

Proof. See section 7.6 in the Appendix.

According to the Proposition, both minimal policies are equivalent for decentralization provided that Assumption 2 is respected. This is an equivalence of lump-sum transfers and government debt for an OLG economy without frictions, analogous to the result of

Buiter and Kletzer (1992) for a Diamond economy with production and 2 periods of life, who do not impose constraints on transfers as in Assumption 2. In the present framework, the complete equivalence does not hold for technical reasons: Assumption 2 does not necessarily hold for both policies.

Proposition 5 implies that transfers and government debt that decentralize stationary allocations are constant, except for period 0. For the balanced-budget case, I denote such stationary policies $(\bar{\tau}_0^1, \bar{\tau}_0^2, \bar{\tau}^1, \bar{\tau}^2)$, meaning that $\bar{\tau}_t^1 = \bar{\tau}^1; \bar{\tau}_t^2 = \bar{\tau}^2 \ \ \forall t > 0$. For the debt-based case, the notation is $(\breve{g}_0, \breve{\tau}_0^2, \breve{g}, \breve{\tau}^2)$. The following Lemma relates the discount factor of the social planner in problem (20) to the sign of transfers and government debt in the two minimal policies:

Lemma 3. *Let a balanced-budget policy $(\bar{\tau}_0^1, \bar{\tau}_0^2, \bar{\tau}^1, \bar{\tau}^2)$ and a debt-based policy $(\breve{g}_0, \breve{\tau}_0^2, \breve{g}, \breve{\tau}^2)$ decentralize a stationary optimal allocation $(\hat{c}, \hat{d}, \hat{e})$ with full commitment. Then $\alpha < \alpha^{LF} \Leftrightarrow \bar{\tau}^1 < 0$ $(\Leftrightarrow \breve{g} > 0)$, where α is the discount factor in problem (20) associated with $(\hat{c}, \hat{d}, \hat{e})$ and α^{LF} follows Definition 4.*

Proof. By Definition 4, with $\alpha = \alpha^{LF}$, $\bar{\tau}^1 = \bar{\tau}^2 = \breve{g} = \breve{\tau}^2 = 0$. By Proposition 1, $M\hat{R}S > R^{LF} \Leftrightarrow \bar{\tau}^1 < 0 \Leftrightarrow \breve{g} > 0$, where the last equivalence is given by $\breve{g} = -\bar{\tau}^1$ (Proposition 5). Then, $\alpha = (1+n)/M\hat{R}S$ (Lemma 2) can be used to conclude. □

I impose a restriction on the value of the discount factor of the social planner to achieve two goals: (1) empirically plausible optimal policies with $\breve{g} > 0$ and (2) equivalence of the two minimal policies under full commitment:

Assumption 3. *In the social planner problem (20), $\alpha \in (\alpha^{min}, \alpha^{LF})$, where α^{min} is the largest value that makes transfers of associated minimal policies have either $\bar{\tau}^1 \leq -y_1$ or $\breve{\tau}^2 \leq -y_2$.*

This assumption is sufficient for equivalence of balanced-budget and debt-based policies in case of full commitment, as in Buiter and Kletzer (1992). Indeed, it guarantees that Assumption 2 is never violated, which is the only reason for decentralization to fail under full commitment. We are now in a position to rank these policies in case of limited commitment.

5.3 Decentralization with limited commitment

The Definition 3 of equilibrium with limited commitment differs from its full commitment counterpart only by the presence of borrowing constraints (7), (12) and individual rationality (10) for adults of generation 0. However, borrowing constraints never bind when optimal allocations are decentralized in the present setting. This is implied by the programs (19), (20) that produce these allocations. Intuitively, borrowing constraints that bind lead to a static inefficiency of the allocation: the sum of utilities could be improved by transferring resources from savers to borrowers. Such allocations cannot maximize a weighed sum of utilities, as do optimal allocations. It follows that an optimal allocation can be decentralized with limited commitment only if the borrowing constraints do not bind in the corresponding equilibrium. Equivalently, decentralization under limited commitment can happen only if decentralization of the same allocation can happen with full commitment.

This is summarized in the following Lemma that focuses on decentralization of allocations where the young are net borrowers[6], i.e. allocations with $\hat{c}_t > y_0,\ \forall t \geq 0$:

Lemma 4. *An optimal allocation* $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ *with* $\hat{c}_t > y_0,\ \forall t \geq 0$ *and initial conditions* $b^0_{-1} \leq 0,\ b^1_{-1} \geq 0$ *is decentralized under limited commitment by a minimal policy (either balanced-budget or debt-based one) if, and only if, two conditions hold:*

1. *The minimal policy decentralizes* $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ *with full commitment*
2. *The individual rationality constraint* (IR1) *holds for each period*

Proof. See section 7.7 in the Appendix.

The Proposition allows to define a procedure for ranking balanced-budget and debt-based policies under limited commitment. If both policies can decentralize a given allocation with full commitment, which is guaranteed by Assumption 3 on the value of α, it is sufficient to compare the individual rationality conditions implied by the two policies. As the left-hand side of (IR1) is determined by the allocation and is the same for any policy, only the right-hand sides need to be compared across policies. If one policy produces smaller right-hand side of (IR1) than another, the condition is more likely to hold for the policy. This ranking can be stated as follows: a set of optimal allocations that one policy can decentralize is a subset of the set of optimal allocations decentralized by another policy.

5.3.1 Golden Rule allocation

I first consider the simplest case of Golden Rule allocation under limited commitment. This allocation is admitted by Assumption 3 only if $\alpha^{LF} > 1$ ($R^{LF} < 1 + n$), which will be assumed for this subsection and relaxed below. To simplify exposition, I also ignore incentive compatibility of the initial generations, which will be included in the more general analysis below.

Under full commitment, the Golden Rule allocation can be decentralized with government debt and no transfers. By Proposition 5, $\breve{\tau}^2_t = 0$ if $\hat{MRS}_t = 1 + n$, which is the case for the Golden Rule. At the same time, the balanced-budget policy has nonzero transfers as long as $\alpha < \alpha^{LF}$ and it must have $\bar{\tau}^1 < 0$, by Assumption 3 and Lemma 3. Then, by Proposition 2, the endogenous borrowing constraint must be tighter under the balanced budget policy than under the debt-based one. This means that the allocation

[6] If this is not true, endogenous borrowing constraints do not matter for decentralization. However, the case where the young are net savers confirms the main result of this section – debt-based policies being more adapted for decentralization that balanced-budget ones. See proof of Proposition 6.

can be decentralized with a debt-based policy if it can be with a balanced-budget one, but the reverse is not true. The following subsection generalizes this result to all stationary optimal allocations admitted by Assumption 3 and the next one gives parametric examples of economies where the Golden Rule can be decentralized with the government debt, but not with transfers.

5.3.2 General stationary optimal allocations

I now turn to general optimal stationary allocations that solve social planner problem (20). In order to include generations $1, 2\ldots$ in the analysis, I make additional assumptions for the initial conditions. Given a stationary optimal allocation $(\hat{c}, \hat{d}, \hat{e})$, I restrict the initial asset income (debt repayment) of adults of generation 0 not to be larger (not to be smaller) as that of subsequent generations: $R_0 b_{-1}^0 \leq \hat{MRS} \cdot (y_0 - \hat{c})$. Intuitively, if this condition is verified as equality, generation 0 has the same budget constraint as all the other generations, and if it is verified as inequality, the debt of the initial adults is relatively large, so that τ_0^1 must be small enough for individual rationality to hold.

Proposition 6. *Consider a stationary optimal allocation* $(\hat{c}, \hat{d}, \hat{e})$ *with* $\hat{MRS} \cdot (y_0 - \hat{c}) \geq R_0 b_{-1}^0$*, which is solution to the social planner problem* (20) *that meets Assumption 3. If this allocation can be decentralized with a balanced-budget policy under limited commitment, it can also be done with a debt-based policy, but the reverse is not true.*

Proof. See section 7.8 in the Appendix.

To understand this result, recall the comparative statics exercise of Proposition 2. Moving from a balanced-budget policy to a debt-based one means going in the direction of larger τ^1 and smaller τ^2: the taxes on adults are null for the latter policy, and the old are being taxed instead of receiving transfers. Therefore, the endogenous borrowing constraints for the young are more tight under the balanced-budget policy.

The following section provides a parametric example of a set of economies where Golden Rule allocations can be decentralized with debt-based policies but not with balanced-budget ones.

5.4 Example: Golden Rule decentralization with limited commitment

Consider an economy with $n = 0, \beta = 1$ and:

$$0 < y_2 < y_0 < y_1 \tag{27}$$

which satisfies the Assumption 1. Suppose that the social planner aims to decentralize the Golden Rule allocation with limited commitment, which means $\alpha = \alpha^{LF}$ in problem (20) and Assumption 3 is satisfied. In this case, the FOC of the program (18) lead to $u'(c) =$

$u'(d) = u'(e)$, so the solution is:

$$\hat{c} = \hat{d} = \hat{e} = \frac{1}{3}(y_0 + y_1 + y_2) \tag{28}$$

For the borrowing constraints on the young to be relevant, I also assume that they are net borrowers, so:

$$\begin{aligned} \hat{c} > y_0 \;\Leftrightarrow \frac{1}{3}(y_0 + y_1 + y_2) \;\Leftrightarrow \\ \Leftrightarrow y_0 < (y_1 + y_2)/2 \end{aligned} \tag{29}$$

Consider first the decentralization for generations $1, 2, 3, \ldots$; I will revisit initial generations at the end of the section. By Lemma 4, a policy that can decentralizes the allocation under limited commitment necessarily does it under full commitment. The following two minimal policies can potentially do the decentralization:

$$\bar{\tau}^1 = \hat{c} - y_1 - (\hat{MRS} + 1 + n)(y_0 - \hat{c}) = y_2 - y_0 < 0 \tag{30}$$

$$\bar{\tau}^2 = -(1 + n)\tau^1 = y_0 - y_2 > 0 \tag{31}$$

$$\breve{g} = y_0 - y_2 > 0 \tag{32}$$

$$\breve{\tau}^2 = (1 + n - \hat{MRS})\breve{g} = 0 \tag{33}$$

where Lemma 2 was used to obtain $\hat{MRS} = 1 + n = 1$.

The policies above also decentralize the Golden Rule with limited commitment for generations $1, 2, 3, \ldots$ if they result in the condition (IR1) being respected, as seen in Lemma 4. Furthermore, $\bar{\tau}^1 < 0 \Leftrightarrow \alpha < \alpha^{LF}$ by Lemma 3, so the assumptions of Proposition 6 are verified and (IR1) is respected under a debt-based policy if they are so under the balanced-budget one, while the reverse is not true. As for (IR1), by substituting the values of transfers of the two policies, the constraint is:

$$u\left(\frac{1}{3}(y_0 + y_1 + y_2)\right) \geq \frac{1}{2}u(y_1 + y_2 - y_0) + \frac{1}{2}u(y_0) \qquad \text{for } \tau^1 = \bar{\tau}^1,\ \tau^2 = \bar{\tau}^2 \tag{34}$$

$$u\left(\frac{1}{3}(y_0 + y_1 + y_2)\right) \geq \frac{1}{2}u(y_1) + \frac{1}{2}u(y_2) \qquad \text{for } \tau^1 = 0,\ \tau^2 = \breve{\tau}^2 \tag{35}$$

According to Proposition 6, (34) is more strict than (35); Figure 1 illustrates it for one example of parameter values, while also showing that the result applies to any parameter values that respect (27) and (29).

When is government debt necessary for decentralization? One parametrization that makes decentralization possible with $(\breve{g}, \breve{\tau}^2)$ and impossible with $(\bar{\tau}^1, \bar{\tau}^2)$ is $y_0 = (y_1 + y_2)/2 - \varepsilon$, with ε small and positive. Indeed, in this case (IR1) for the two policies writes:

$$u\left(\frac{y_1 + y_2}{2} - \frac{\varepsilon}{3}\right) \geq \frac{1}{2}u\left(\frac{y_1 + y_2}{2} + \varepsilon\right) + \frac{1}{2}u\left(\frac{y_1 + y_2}{2} - \varepsilon\right) \text{ for } \tau^1 = \bar{\tau}^1,\ \tau^2 = \bar{\tau}^2 \tag{36}$$

$$u\left(\frac{y_1 + y_2}{2} - \frac{\varepsilon}{3}\right) \geq \frac{1}{2}u(y_1) + \frac{1}{2}u(y_2) \qquad \text{for } \tau^1 = 0,\ \tau^2 = \breve{\tau}^2 \tag{37}$$

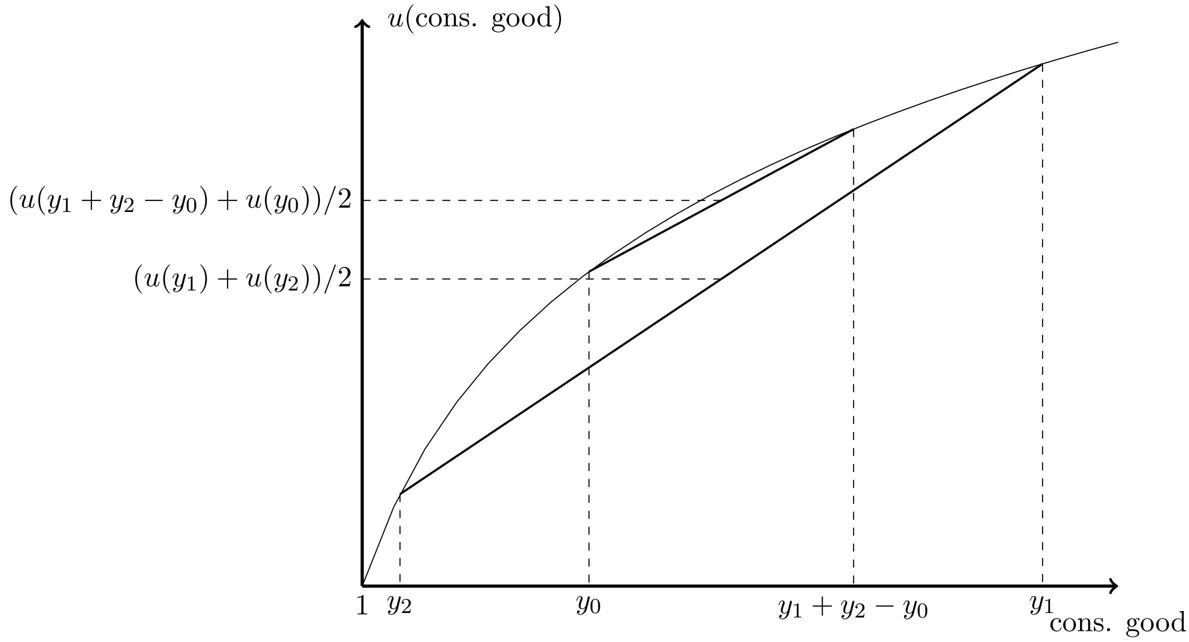

Figure 1: Example with $u(\cdot) = 3\ln(\cdot)$; $y_2 = 1.5$, $y_0 = 4$, $y_1 = 10$. The relationship $(u(y_1 + y_2 - y_0) + u(y_0))/2 > (u(y_1) + u(y_2))/2$ holds for any parameters satisfying the constraints assumed in the example: $0 < y_2 < y_0 < y_1$; $y_0 < (y_1 + y_2)/2$; $\sigma \geq 1$.

In case $\varepsilon = 0$, (36) would be verified as equality. With a marginal increase of ε by $\Delta\varepsilon$, the change of the left hand side is $-\frac{1}{3}u'\left(\frac{y_1+y_2}{2}\right)\Delta\varepsilon < 0$ in first-order approximation. The same approximation for the right-hand side is null. Therefore, (36) does not hold. At the same time, (37) does hold as inequality for $\varepsilon = 0$, by strict concavity of u. The difference of the right hand side and the left hand side is not marginal if IES is finite. Therefore, for a small ε, (37) is verified.

Finally, the above analysis is true for generations $-1, 0$ if $b^1_{-1} \geq 0$ and $R_0 b^0_{-1} \leq \hat{MRS} \cdot (y_0 - \hat{c}) = y_0 - \hat{c}$. The first inequality is necessary for any economy to have an equilibrium with limited commitment. The second one, used in Proposition 6, ensures that (IR1) holds for generation 0 if it holds for the subsequent ones. Indeed, from Proposition 5, $\bar{\tau}^1_0 = 2\hat{c} - y_1 - y_0 - R_0 b^0_{-1}$. If $R_0 b^0_{-1} \leq y_0 - \hat{c}$, then $\bar{\tau}^1_0 \geq \tau^1$, so the right hand side of (IR1) is more strict for this generation than for subsequent ones:

$$u(\hat{d}) + \beta u(\hat{e}) > u(y_1 + \bar{\tau}^1_0) + \beta u(y_2 + \bar{\tau}^2) > u(y_1 + \bar{\tau}^1) + \beta u(y_2 + \bar{\tau}^2)$$

The constraint (IR1) under the debt-based policy is the same for all generations as adults of $t = 0$ do not receive transfers. As a result, the constraint (IR1) in $t = 0$ is more strict under a balanced-budget policy that under a debt-based one.

5.5 Decentralization and autarky

The main result of section 5.3 is that, under mild conditions, it is easier for the social planner to make a given optimal allocation equilibrium with a debt-based minimal policy than with a balanced-budget one. However, Section 4.2 has shown that the equilibrium targeted by the social planner is not necessarily the only equilibrium of the economy under a given policy. In economies with $b^0_{-1} = 0$, autarky can be a second equilibrium steady state, which will prevail if all agents have corresponding expectations. The analysis of Section 4.2 implies that decentralization with a balanced-budget policy allows for autarky as an equilibrium, while a debt-based policy rules it out.

Autarky is ruled out for the reasons discussed in Proposition 6: the use of government debt leads to individual rationality constraints — equivalently, the borrowing constraints — less strict than under balanced-budget policies. This is sufficient to rule out autarky in this setting, since null borrowing limits are necessary for autarky to be an equilibrium.

5.6 Comparison to simple borrowing constraints

In this final section, I compare my results to simpler frameworks with exogenous borrowing constraints. In particular, I study two versions of the constraint commonly found in the literature. In the first version, the lower limit on the assets of the young is an exogenous constant. In the second version, it is a fraction of the present value of the agent's future income, as in Jappelli and Pagano (1994). The results of the previous two sections do not hold in both cases: minimal policies are equivalent for decentralization under such constraints.

The first form of an exogenous borrowing constraint is the following:

$$b^0_{t-1} \geq \bar{b} \; \forall t \geq 0 \tag{38}$$

where $\bar{b} \leq 0$ is an exogenous parameter. Using the budget constraint, one obtains $y_0 - c_{t-1} \geq \bar{b}, \; \forall \; t \geq 0$. To examine the implications for decentralization of optimal allocations, note that $y_0 - \hat{c}_{t-1} \geq \bar{b}, \; \forall \; t \geq 0$ is only a constraint on the values of the optimal allocation and of exogenous parameters. It follows that the choice between two minimal policies has no influence on whether the constraint is verified.

Now assume the constraint is as in Jappelli and Pagano (1994):

$$b^0_{t-1} \geq -\phi \cdot \left(\frac{y_1 + \tau^1_t}{R_t} + \frac{y_2 + \tau^2_{t+1}}{R_t R_{t+1}} \right) \; \forall t \geq 0 \tag{39}$$

where the exogenous constant $\phi \in [0, 1]$ can be interpreted as the fraction of lifetime after-transfer income that lenders can confiscate if the agent does not repay her debts. Although the transfers on the right hand side of (39) are different for a balanced-budget and a debt-based policy, the resulting present value of the lifetime income after transfers is the same.

Indeed, when a given optimal allocation is decentralized, the present value of the income after transfers is determined by the present value of the optimal allocation:

$$\hat{c}_{t-1} + \frac{\hat{d}_t}{M\hat{R}S_{t-1}} + \frac{\hat{e}_{t+1}}{M\hat{R}S_{t-1}M\hat{R}S_t} = y_0 + \frac{y_1 + \tau_t^1}{M\hat{R}S_{t-1}} + \frac{y_2 + \tau_{t+1}^2}{M\hat{R}S_{t-1}M\hat{R}S_t}$$
$$\Rightarrow -\phi \cdot \left(\frac{y_1 + \tau_t^1}{R_t} + \frac{y_2 + \tau_{t+1}^2}{R_t R_{t+1}} \right) = -\phi \cdot \left(\hat{c}_{t-1} + \frac{\hat{d}_t}{M\hat{R}S_{t-1}} + \frac{\hat{e}_{t+1}}{M\hat{R}S_{t-1}M\hat{R}S_t} - y_0 \right)$$

One concludes again that the borrowing constraint (39) is verified or not for a given optimal allocation, independently of the policy instruments used for decentralization. However, a less standard version of the latter constraint with only period t income entering the right-hand side of (39) would not result in the equivalence of the two minimal policies.

6 Conclusion

This paper provides a novel argument for the use of government debt for redistribution of resources between generations. Permanent government debt roll-over, coupled with lower taxation of adult workers, discourages consumers from personal bankruptcy. Endogenous borrowing constraints become less tight in response, so optimal allocations become feasible in equilibrium. I show this in a decentralization problem of a social planner that can use either a balanced-budget or debt-based policy with the same number of instruments. Under mild conditions, the set of allocations decentralized with a balanced-budget policy is a subset of those decentralized with a debt-based policy.

Furthermore, the use of public debt alleviates the problem of equilibrium multiplicity. If government debt is used, a suboptimal, autarkic equilibrium does not co-exist with the one targeted by the social planner, whereas such multiplicity is always present if government budgets are balanced in every period.

This article uses a simple structure for tractability: endowment economy with no uncertainty, identical agents within and across generations, three periods of life, no bequest motive, and so on. However, the mechanism identified in the model can generalize to larger, quantitative life-cycle models and produce results that differ from the recent models (Rohrs and Winter, 2015; Antunes and Ercolani, 2020) analyzing government debt and credit constraints in infinite-horizon economies. Furthermore, the findings can potentially enrich strands of literature that have used compact dynamic models to study long-run phenomena. One such potential avenue is long-run inequality analysis: Bhattacharya et al. (2016) explored the role of endogenous borrowing constraints and Borissov and Kalk (2020) studied public debt in its relation to long-run inequality, but the role of the two factors has not been explored jointly.

Finally, the distortion of incentives for personal bankruptcy by lump-sum taxation calls for revisiting classical questions in public finance. What is that optimal use of government debt as a device for smoothing tax distortions (Barro, 1979)? How does is an unexpected increase in government spending optimally financed? Is fiscal policy time consistent? Answers to these questions, as well as the results of the present article, are needed to understand how the personal bankruptcy incentives are to be taken into account by fiscal policies.

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7 Appendix

7.1 Proof of Proposition 1

Some additional notation is needed for the proof. The results of the utility maximisation problem of a generation t agent are the following functions of interest rates and transfers:

$$c_{t-1} = \sigma(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1}); \quad d_t = \delta(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1}); \; e_{t+1} = \varepsilon(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1})$$

then, $b^0_{t-1} = \beta^0(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1}) = y_0 - \sigma(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1})$

$$b^1_t = \beta^1(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1}) = 1/R_{t+1}(\varepsilon(R_t, R_{t+1}, \tau^1_t, \tau^2_{t+1}) - \tau^2_{t+1} - y_2)$$

The following functions are the demands for consumption and saving of an agent facing a stationary interest rate R and stationary balanced-budget transfers $(\tau^1, -(1+n)\tau^1)$:

$$c^s(R, \tau^1) := \sigma(R, R, \tau^1, -(1+n)\tau^1); \qquad b^0(R, \tau^1) := \beta^0(R, R, \tau^1, -(1+n)\tau^1)$$
$$d^s(R, \tau^1) := \delta(R, R, \tau^1, -(1+n)\tau^1); \qquad b^1(R, \tau^1) := \beta^1(R, R, \tau^1, -(1+n)\tau^1)$$
$$e^s(R, \tau^1) := \varepsilon(R, R, \tau^1, -(1+n)\tau^1)$$

Denote by W_t the present value in t of intertemporal wealth of an adult of period t: $W_t \equiv R_t y_0 + y_1 + \tau^1_t + \frac{y_2 + \tau^2_{t+1}}{R_{t+1}}$. The following lemma gives a simplified description of demand functions:

Lemma 5. *The demands for c_{t-1}, d_t, e_{t+1} can be described by $R_t c_{t-1} = s^c(R_t, R_{t+1})W_t; d_t = s^d(R_t, R_{t+1})W_t; \frac{1}{R_{t+1}} e_{t+1} = s^e(R_t, R_{t+1})W_t$, where $s^c(R_t, R_{t+1}) + s^d(R_t, R_{t+1}) + s^e(R_t, R_{t+1}) = 1$. For the case $u(\cdot) = \ln(\cdot)$, the expressions of $s^c(R_t, R_{t+1}), s^d(R_t, R_{t+1}), s^e(R_t, R_{t+1})$ hold true if one substitutes $\sigma = 1$ and the three functions are degenerate, i.e. constant with respect to their arguments.*

Proof. The FOC of the consumer with a CIES utility function and $\sigma > 1$ give: $d_t = (\beta R_t)^\sigma c_{t-1}$; $e_{t+2} = (R_{t+1} R_t)^\sigma \beta^{2\sigma} c_{t-1}$. Substituting in the intertemporal budget constraint, one gets $R_t\ c_{t-1} = R_t / \left(R_t + (\beta R_t)^\sigma + (R_t)^\sigma R_{t+1}^{\sigma-1} \beta^{2\sigma}\right) W_t$. It follows that :

$$s^c(R_t, R_{t+1}) = R_t / \left(R_t + \beta^\sigma R_t^\sigma + \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1}\right)$$
$$s^d(R_t, R_{t+1}) = \beta^\sigma R_t^\sigma / \left(R_t + \beta^\sigma R_t^\sigma + \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1}\right)$$
$$s^e(R_t, R_{t+1}) = \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1} / \left(R_t + \beta^\sigma R_t^\sigma + \beta^{2\sigma} R_t^\sigma R_{t+1}^{\sigma-1}\right)$$

and $s^c(R_t, R_{t+1}) + s^d(R_t, R_{t+1}) + s^e(R_t, R_{t+1}) = 1$ is verified. In the case $u(\cdot) = \ln(\cdot)$, one obtains $R_t c_{t-1} = \frac{1}{1+\beta+\beta^2} W_t$; $d_t = \frac{\beta}{1+\beta+\beta^2} W_t$; $e_{t+1}/R_{t+1} = \frac{\beta^2}{1+\beta+\beta^2} W_t$, which is consistent with substituting $\sigma = 1$ in the above formulas. □

By taking derivatives of the formulas of the Lemma, one can show that the consumption demands satisfy:

$$\partial\sigma(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_t < 0,\ \partial\sigma(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_{t+1} < 0$$
$$\partial\delta(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_t > 0,\ \partial\delta(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_{t+1} < 0$$
$$\partial\varepsilon(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_t > 0,\ \partial\varepsilon(R_t, R_{t+1}, \tau_t^1, \tau_{t+1}^2)/\partial R_{t+1} > 0$$

Consumption at different dates is substitutes: if one sets the price of d_t to 1 the price of c_{t-1} to R_t and the price of e_{t+1} to $1/R_{t+1}$, each period's consumption is decreasing in its own price and increasing in the relative prices of other periods' consumption.

A steady state without government debt is given by $(1+n)b^0(R,\tau^1) + b^1(R,\tau^1) = 0$.

The existence of at least one steady state can now be shown. First, the steady-state asset position of the private sector is a continuous function of R for $R > 0$, since the demands derived in Lemma 5 are continuous on this domain. Then, consider $R \to 0+$. According to the formulas of Lemma 5, $c^s \to +\infty$ under such a stationary interest rate, so $b^0 < 0$. Moreover, $e^s \to 0$, so $b^1 < 0$ since $y_2 + \tau^2 > 0$ by Assumption 2. One gets $(1+n)b^0(R,\tau^1) + b^1(R,\tau^1) < 0$ for $R \to 0$. On the other hand, if $R \to +\infty$, then the opposite is true: $c^s \to 0$ and $e^s \to +\infty$, so $(1+n)b^0(R,\tau^1) + b^1(R,\tau^1) > 0$. Therefore, at least one steady state with $R > 0$ exists.

Showing uniqueness of the steady state is then equivalent to showing that the steady-state asset position of the private sector is increasing in the steady-state interest rate $R > 0$. To show this, use the substitutability shown above: $\frac{\partial\sigma}{\partial R_t} < 0, \frac{\partial\sigma}{\partial R_{t+1}} < 0$ implies $\frac{\partial c^s}{\partial R} = \frac{\partial\sigma}{\partial R_t} + \frac{\partial\sigma}{\partial R_{t+1}} < 0$. Then, $\frac{\partial b^0}{R} = -\frac{\partial c^s}{\partial R} > 0$. To show the signs of derivatives of b^1, first write $b_t^1 = y_1 + \tau_t^1 + R_t b_{t-1}^0 - d_t$. It follows that $\frac{\partial\beta^1}{\partial R_{t+1}} = R_t \frac{\partial\beta^0}{\partial R_{t+1}} - \frac{\partial\delta}{\partial R_{t+1}} > 0$. Then write $b_t^1 = \frac{1}{R_{t+1}}(e_{t+1} + (1+n)\tau_{t+1}^1 - y_2)$. Then $\frac{\partial\beta^1}{\partial R_t} = \frac{1}{R_{t+1}}\frac{\partial\varepsilon}{\partial R_t} > 0$ as $\frac{\partial\varepsilon}{\partial R_t} > 0$ by substitutability. The signs of the two derivatives of β^1 imply $\frac{\partial b^1}{\partial R} > 0$. We finally get the sign of the derivative of the total asset position, $(1+n)\frac{\partial b^0}{\partial R} + \frac{\partial b^1}{\partial R} > 0$. This proves uniqueness of the steady state.

To show the relationship between τ^1 and R, note that the steady-state asset position of the private sector is continuous in τ^1. The steady state interest can then be given by an implicit function of τ^1:

$$R^s :\ ]-y_1, y_2/(1+n)[\to \mathbb{R};\ (1+n)b^0(R^s(\tau^1),\tau^1) + b^1(R^s(\tau^1),\tau^1) = 0 \tag{40}$$

where the domain is given by Assumption 2.

The claim that the steady state interest rate decreases in τ^1 can then be summarized as $\frac{dR^s}{d\tau^1} < 0$. By the implicit function theorem, this derivative exists and is given by

$$\frac{dR^s}{d\tau^1} = \frac{(1+n)\frac{\partial b^0}{\partial\tau^1} + \frac{\partial b^1}{\partial\tau^1}}{-(1+n)\frac{\partial b^0}{\partial R} - \frac{\partial b^1}{\partial R}} \tag{41}$$

The remaining proof relies on the signs and the magnitudes of $\frac{\partial b^0}{\partial R}; \frac{\partial b^0}{\partial\tau^1}; \frac{\partial b^1}{\partial R}; \frac{\partial b^1}{\partial\tau^1}$.

As shown above, substitutability leads to $\frac{\partial b^0}{\partial R} > 0, \frac{\partial b^1}{\partial R} > 0$. One obtains the negative sign of $-(1+n)\frac{\partial b^0}{\partial R} - \frac{\partial b^1}{\partial R}$, the denominator of $\frac{dR^s}{d\tau^1}$.

To get the sign of $(1+n)\frac{\partial b^0}{\partial \tau^1} + \frac{\partial b^1}{\partial \tau^1}$, use definitions of $\beta^0, \beta^1, b^0, b^1$ the budget constraints and Lemma 5 to get:

$$\frac{\partial b^0}{\partial \tau^1} = -\frac{\partial c^s}{\partial \tau^1}; \qquad \frac{\partial b^1}{\partial \tau^1} = \frac{1}{R}\left(\frac{\partial e^s}{\partial \tau^1} + 1 + n\right)$$
$$\frac{\partial c^s}{\partial \tau^1} = \frac{s^c(R,R)}{R}\left(1 - \frac{1+n}{R}\right); \quad \frac{\partial e^s}{\partial \tau^1} = s^e(R,R)(R-(1+n))$$

To see the sign of $(1+n)\frac{\partial b^0}{\partial \tau^1} + \frac{\partial b^1}{\partial \tau^1}$, two different cases must be studied depending on the sign of $R-(1+n)$:

Case $R \leq 1+n$: $\frac{\partial b^0}{\partial \tau^1} = -\frac{\partial c^s}{\partial \tau^1} = -s^c(R,R)\frac{R-(1+n)}{R^2} \geq 0$ and $\frac{\partial b^1}{\partial \tau^1} > 0$ (the latter is true in both cases), so $(1+n)\frac{\partial b^0}{\partial \tau^1} + \frac{\partial b^1}{\partial \tau^1} > 0$.

Case $R > 1+n$: $(1+n)\frac{\partial b^0}{\partial \tau^1} + \frac{\partial b^1}{\partial \tau^1} = (1+n)\left(\frac{1}{R} - \frac{\partial c^s}{\partial \tau^1}\right) + \frac{1}{R}\frac{\partial e^s}{\partial \tau^1}$ and since $s^c(R,R) < 1$, one gets $\frac{\partial c^s}{\partial \tau^1} < \frac{R-(1+n)}{R^2}$. This leads to

$$\begin{aligned}(1+n)\frac{\partial b^0}{\partial \tau^1} + \frac{\partial b^1}{\partial \tau^1} &> (1+n)\left(\frac{1}{R} - \frac{R-(1+n)}{R^2}\right) + \frac{1}{R}\frac{\partial e^s}{\partial \tau^1}\\ &= \frac{1}{R}\left((1+n)\frac{1+n}{R} + \frac{\partial e^s}{\partial \tau^1}\right) > 0\end{aligned}$$

where the last inequality is due to $\frac{\partial e^s}{\partial \tau^1} = s^e(R,R)(R-(1+n)) > 0$. One obtains $(1+n)\frac{\partial b^0}{\partial \tau^1} + \frac{\partial b^1}{\partial \tau^1} > 0$. The numerator of $\frac{dR^s}{d\tau^1}$ is shown to be positive and the denominator is shown to be negative. As a result, $\frac{dR^s}{d\tau^1} < 0$.

7.2 Proof of Lemma 1

The function f exists since V is continuous (by the Maximum theorem), monotonic in the second argument and has a sufficiently wide range. Indeed, V is monotonic in second argument since an increase of b^0_{t-1} expands the budget set; this makes indirect utility larger because the utility function is monotonic. Then, note that $V(R_{t+1}, 0, \tau^1_t, \tau^2_{t+1}) \geq u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1})$: the choice of the agent is at least weakly preferred to the endowment. At the same time, denoting $\zeta \equiv -y_1 - \tau^1_t - (y_2 + \tau^2_{t+1})/R_{t+1}$, one obtains $\lim_{R_t b^0_{t-1} \to \zeta+} V(R_{t+1}, R_t b^0_{t-1}, \tau^1_t, \tau^2_{t+1}) < u(y_1 + \tau^1_t) + \beta u(y_2 + \tau^2_{t+1})$ since the budget set collapses in this limit. This proves the existence of f. Continuous differentiability of the function follows from continuous differentiability of the utility function and the implicit function theorem. Since $\zeta < 0$ by Assumption 2, $f \leq 0$ also follows from the above arguments.

To show when $f(R_{t+1}, \tau^1_t, \tau^2_{t+1}) = 0$, first note that if the adults could borrow (as in the full commitment environment), then $b^0_{t-1} \leq 0$ and $R_{t+1} \leq MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ would imply $b^1_t \leq 0$. Indeed, $R_{t+1} = MRS_{de}(y_1 + \tau^1_t, y_2 + \tau^2_{t+1})$ is the condition for an adult with $b^0_{t-1} = 0$ to demand $b^1_t = 0$, so if $b^0_{t-1} < 0$, the demand for e_{t+1} decreases by normality of old-age consumption, leading to $b_t^1 < 0$. When the no-borrowing constraint (7) is taken into account, $b_t^1 = 0$ for $R_{t+1} \leq MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$. One then can conjecture $f(R_{t+1}, \tau_t^1, \tau_{t+1}^2) = 0$ for such R_{t+1}. To verify this, substitute $b_{t-1}^0 = 0$ in the adult agent's problem to obtain the demands for adult and old age consumption under such interest rates equal to $d_t = y_1 + \tau_t^1$; $e_{t+1} = y_2 + \tau_{t+1}^2$. This implies $V(R_{t+1}, 0, \tau_t^1, \tau_{t+1}^2) = u(y_1 + \tau_t^1) + \beta u(y_2 + \tau_{t+1}^2)$ for $R_{t+1} \leq MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$, or equivalently $f(R_{t+1}, \tau_t^1, \tau_{t+1}^2) = 0$.

Finally, $f(R_{t+1}, \tau_t^1, \tau_{t+1}^2) < 0$ for $R_{t+1} > MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$ since value of f is never positive, as shown above, and f is decreasing in the first argument: $\frac{\partial f}{\partial R_{t+1}} = 0$ for $R_{t+1} = MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$ and $\frac{\partial f}{\partial R_{t+1}} < 0$ for $R_{t+1} > MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$. Indeed, $\frac{\partial f}{\partial R_{t+1}} = -\frac{\partial V}{\partial R_{t+1}} / \frac{\partial V}{\partial (R_t b_{t-1}^0)}$ by implicit function theorem and $\frac{\partial V}{\partial (R_t b_{t-1}^0)} > 0$ is proved above. In addition, for $R_{t+1} = MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$, $\frac{\partial V}{\partial R_{t+1}} = 0$ since the agent makes null savings, as shown above. Finally, for $R_{t+1} > MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$, $\frac{\partial V}{\partial R_{t+1}} > 0$ since the agent is a net saver when adult, which can be proved by contradiction. Indeed, suppose an adult is not a net saver under such interest rate. It can only be the case if she is indebted, since in the case $b_{t-1}^0 = 0$ and $R_{t+1} > MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$ she is a net saver while savings are decreasing in b_{t-1}^0. However, this means that $f(R_{t+1}, \tau_t^1, \tau_{t+1}^2) < 0$, which is impossible because an adult that does not need to make savings must declare bankruptcy on any beginning-of-period debts. Therefore when $R_{t+1} > MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$, adults are net savers and $\frac{\partial V}{\partial R_{t+1}} > 0$, which means $\frac{\partial f}{\partial R_{t+1}} < 0$. This completes the proof.

7.3 Proof of Proposition 2

Define a function for the optimal asset position of an adult of generation t that solves the program (8) as:

$$z(R_{t+1}, R_t b_{t-1}^0, \tau^1, -(1+n)\tau^1) \equiv y_1 + \tau^1 + R_t b_{t-1}^0 - d(R_{t+1}, R_t b_{t-1}^0, \tau^1, -(1+n)\tau^1)$$

The equation (11) that defines $R_t b_{t-1}^c = f(R_{t+1}, \tau^1, -(1+n)\tau^1)$ can then be written:

$$\begin{aligned} & u(y_1 + \tau^1 + R_t b_{t-1}^0 - z(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) \\ & \quad + \beta u(y_2 - (1+n)\tau^1 + R_{t+1} z(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) \\ & \quad = u(y_1 + \tau^1) + \beta u(y_2 - (1+n)\tau^1) \end{aligned} \tag{42}$$

Then, define a function $F(\tau^1, R_t b_{t-1}^c)$ as a difference of the left-hand side and the right-hand side of the equation (42):

$$\begin{aligned} F(\tau^1, R_t b_{t-1}^c) = & f(R_{t+1}, \tau^1, -(1+n)\tau^1) = \\ & u(y_1 + \tau^1 + R_t b_{t-1}^0 - z(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) \\ & + \beta u(y_2 - (1+n)\tau^1 + R_{t+1} z(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) \\ & - (u(y_1 + \tau^1) + \beta u(y_2 - (1+n)\tau^1)) \end{aligned}$$

The expression $F(\tau^1, R_t b_{t-1}^c) = 0$ then defines $R_t b_{t-1}^c$ as an implicit function of τ^1 by the same manner as the more general function $f(R_{t+1}, \tau^1, -(1+n)\tau^1)$ has been defined in Lemma 1. F increases in the second argument by the wealth effect discussed in the proof of Lemma 1, so the proof amounts to showing that $\frac{\partial F}{\partial \tau^1} > (=) \, 0$ for $b_t^1 > (=) \, 0$. This derivative can be written as follows:

$$\begin{aligned}\frac{\partial F}{\partial \tau^1} =& u'(d(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) - \beta(1+n)u'(e(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) \\ &- \left(u'(y_1+\tau^1) - \beta(1+n)u'(y_2-(1+n)\tau^1)\right)\end{aligned}$$

where all the terms of $\frac{\partial F}{\partial \tau^1}$ involving derivatives of z were eliminated by the envelope theorem. Re-arranging terms, one obtains:

$$\begin{aligned}\frac{\partial F}{\partial \tau^1} =& \left[u'(d(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1)) - u'(y_1+\tau^1)\right] + \\ &\beta(1+n)\left[u'(y_2-(1+n)\tau^1) - u'(e(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1))\right]\end{aligned}$$

Given $u'' < 0$, proving the sign of the derivative ($\frac{\partial F}{\partial \tau^1} < (=) \, 0$ if $b_t^1 > (=) \, 0$) amounts to prove the following two statements:

$$d(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1) - y_1 - \tau^1 > (=) \, 0 \qquad \text{if } b_t^1 > (=) \, 0 \tag{43}$$

$$y_2 - (1+n)\tau^1 - e(R_{t+1}, R_t b_{t-1}^c, \tau^1, -(1+n)\tau^1) > (=) \, 0 \qquad \text{if } b_t^1 > (=) \, 0 \tag{44}$$

The left hand side of (44) is equal to $R_{t+1} b_t^1$ by the budget constraints, so the sign is true. Using the budget constraints once again, the left hand side of (43) is $R_t b_{t-1}^c - b_t^1$. It was shown in the proof of Lemma 1 (section 7.2) that $R_t b_{t-1}^c < (=) \, 0$ if $b_t^1 > (=) \, 0$, which proves the sign statement in (43).

One obtains $\frac{\partial F}{\partial \tau^1} < (=) \, 0$ if $b_t^1 > (=) \, 0$, which completes the proof.

7.4 Proof of Proposition 3

As discussed in the proof of Lemma 1, $R_t \leq MRS_{de}(y_1 + \tau_t^1, y_2 + \tau_{t+1}^2)$ implies $b_t^1 = 0$ if $b_{t-1}^0 \leq 0$. At the same time, the demanded level of b_t^0 is $b_{t-1}^0 < 0$ for $b_t^1 = 0$ and $R_t \leq MRS_{cd}(y_0, y_1 + \tau_t^1)$, and the borrowing limit is null according to Lemma 1. Then the young and the adults are constrained to have $b_t^0 = 0, b_t^1 = 0, \ \forall t \geq 0$ under such interest rates, and the asset market clears since $g_t = 0, \ \forall t \geq 0$ by assumption. Autarky then satisfies all the properties of equilibrium from Definition 3.

7.5 Proof of Proposition 4

I first show that the program (19) maximizes a weighed sum of utilities of all the generations while constraining all agents within a generation to have the same consumption.

This weighed sum of utilities is written as:

$$\max_{(c_t,d_t,e_t)_{t\geq 0}} \{\theta_{-1}N_{-1}\beta^2 u(e_0)+\theta_0 N_0\left(\beta u(d_0)+\beta^2 u(e_1)\right)$$
$$+\sum\nolimits_{t\geq 0}\theta_{t+1}N_{t+1}\left(u(c_t)+\beta u(d_{t+1})+\beta^2 u(e_{t+2})\right)\}$$
$$\text{s. t.} \quad c_t+\frac{d_t}{1+n}+\frac{e_t}{(1+n)^2}=y_0+\frac{y_1}{1+n}+\frac{y_2}{(1+n)^2},\ \forall t\geq 0$$

where is θ_t is the weight of one agent of generation t. To obtain a simpler formulation of the same program, define $\omega_t \equiv \theta_{t-1}N_{t-1}$. The objective function can then be rewritten:

$$\omega_0\beta^2 u(e_0)+\omega_1\left(\beta u(d_0)+\beta^2 u(e_1)\right)+\sum\nolimits_{t\geq 0}\omega_{t+2}\left(u(c_t)+\beta u(d_{t+1})+\beta^2 u(e_{t+2})\right)$$
$$=\sum\nolimits_{t\geq 0}(\omega_{t+2}u(c_t)+\omega_{t+1}\beta u(d_t)+\omega_t\beta^2 u(e_t))$$

This results in the sought program (19).

Next, I establish the existence and uniqueness of the solution. According to the resource constraint, aggregate consumption in any period is finite, so the sum of agents' instantaneous utilities is finite, too. Therefore, if $(\omega_t)_{t\geq 0}$ converges, the program is well defined, with a concave objective function and a convex set of feasible allocations. The solution therefore exists and is unique.

Finally, I show the necessary and sufficient condition for a stationary solution and the equivalence to a simplified program (20). The FOC of (19) can be written as:

$$\begin{cases} \dfrac{u'(c_t)}{\beta u'(d_t)} = \dfrac{\omega_{t+1}}{\omega_{t+2}}(1+n) & (45)\\ \dfrac{\beta u'(d_t)}{\beta^2 u'(e_t)} = \dfrac{\omega_t}{\omega_{t+1}}(1+n) & (46)\\ c_t+\dfrac{d_t}{1+n}+\dfrac{e_t}{(1+n)^2} = y_0+\dfrac{y_1}{1+n}+\dfrac{y_2}{(1+n)^2} & (47)\end{cases}$$

The first order conditions are static, so the solution is stationary if, and only if, the weights are such that the system is the same every period. This means $\frac{\omega_{t+1}}{\omega_{t+2}}$ is constant and $\frac{\omega_t}{\omega_{t+1}}$ is constant, the two statements being equivalent. We get $\omega_{t+1}=\alpha\omega_t$, where $\alpha<1$, as the sum of $(\omega_t)_{t\geq 0}$ must converge.

The objective function with a constant discount factor α writes:

$$\mathcal{W}=\sum\nolimits_{t\geq 0}\alpha^t(\alpha^2 u(c_t)+\alpha\beta u(d_t)+\beta^2 u(e_t))$$

Knowing that c_t, d_t, e_t are constant, one can solve an equivalent constrained problem with $c_t=c, d_t=d, e_t=e$. Substituting the constant consumption levels in the sum above, a geometric series is obtained:

$$\begin{aligned}\mathcal{W} &= \sum\nolimits_{t\geq 0} \alpha^t(\alpha^2 u(c) + \alpha\beta u(d) + \beta^2 u(e)) \\ &= \frac{1}{1-\alpha}(\alpha^2 u(c) + \alpha\beta u(d) + \beta^2 u(e))\end{aligned}$$

Finally, (20) has $\mathcal{W}^S = (1-\alpha)\mathcal{W}$, so they have the same maximum point.

7.6 Proof of Proposition 5

The FOC of agents' maximisation problem with full commitment gives

$$\forall t \geq 0, \quad M\hat{R}S_t = R_{t+1}$$

It is then sufficient to show that a given sequence of debt and transfers satisfies the budget constraints of all agents and the market asset clearing conditions with interest rates replaced by the MRS.

A general policy $(\tau_1^1, \tau_t^2, g_t)_{t\geq 0}$ decentralizing $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ must then solve the following system:

$$\begin{cases} \hat{c}_{t-1} + b_{t-1}^0 = y_0 & (48) \\ \hat{d}_t + b_t^1 = y_1 + \tau_t^1 + \hat{M}RS_{t-1} b_{t-1}^0 & (49) \\ \hat{e}_{t+1} = y_2 + \tau_{t+1}^2 + \hat{M}RS_t b_t^1 & (50) \\ (1+n) b_t^0 + b_t^1 = g_t & (51) \\ g_t = \dfrac{R_t}{1+n} g_{t-1} + \tau_t^1 + \dfrac{\tau_t^2}{1+n} & (52) \end{cases}$$

The two minimal sets of policy instruments $(\bar{\tau}_t^1, \bar{\tau}_t^2)_{t\geq 0}$ and $(\breve{g}_t, \breve{\tau}_t^2)_{t\geq 0}$ are particular solutions of the above system that have $g_t = 0$ and $\tau_t^1 = 0$, correspondingly.

Balanced-budget policy : First, consider periods $t > 0$. To get the expression for $\bar{\tau}_t^1, t \geq 1$, substitute expressions of asset positions from (48),(49) in (51), then solve the latter for τ_t^1. To check that (50) also holds, substitute $-(1+n)\tau_{t+1}^1$ for τ_{t+1}^2 and use the expression for $\bar{\tau}_t^1$ obtained above. This gives:

$$\hat{e}_{t+1} = y_2 - (1+n)(\hat{c}_t - y_0)\hat{M}RS_t - (1+n)\hat{d}_{t+1} + (1+n)y_1 + (1+n)^2(y_0 - \hat{c}_{t+1}) + \hat{M}RS_t b_t^1$$

while from (51) one has $b_t^1 = -(1+n)b_t^0 = -(1+n)(y_0 - \hat{c}_t)$, so:

$$\begin{aligned}\hat{e}_{t+1} = y_2 - (1+n)(\hat{c}_t - y_0)\hat{M}RS_t - (1+n)\hat{d}_{t+1} + (1+n)y_1 + (1+n)^2(y_0 - \hat{c}_{t+1}) + \hat{M}RS_t b_t^1 \\ \Leftrightarrow \hat{e}_{t+1} = y_2 + (1+n)(y_1 - \hat{d}_{t+1}) + (1+n)^2(y_0 - \hat{c}_{t+1}) \\ \Leftrightarrow \hat{c}_{t+1} + \frac{\hat{d}_{t+1}}{1+n} + \frac{\hat{e}_{t+1}}{(1+n)^2} = y_0 + \frac{y_1}{1+n} + \frac{y_2}{(1+n)^2}\end{aligned}$$

The last equation is the resource constraint for period $t+1$, always verified for an optimal allocation.

The initial transfers $\bar{\tau}_0^1, \bar{\tau}_0^2$ are obtained in analogous manner, using R_0 instead of $\hat{M}RS_{-1}$ and $R_0, b_{-1}^0, g_{-1}, b_{-1}^1$ being exogenous.

Debt-based policy: Denote the asset positions of private agents $(\breve{b}_t^0, \breve{b}_t^1)_{t\geq 0}$ under debt-based policy and $(\bar{b}_t^0, \bar{b}_t^1)_{t\geq 0}$ under the balanced-budget one. Since transfers to the young are absent in both cases, we have $\bar{b}_t^0 = \breve{b}_t^0$. Moreover, since transfer to adults are absent in the debt-based policy, one obtains $\breve{b}_t^1 = \bar{b}_t^1 - \bar{\tau}_t^1$. Asset market equilibrium then leads to $\breve{g}_t = -\bar{\tau}_t^1$. Indeed,

$$\begin{cases} (1+n)\breve{b}_t^0 + \breve{b}_t^1 & = \breve{g}_t \\ (1+n)\bar{b}_t^0 + \bar{b}_t^1 & = 0 \end{cases} \Leftrightarrow \begin{cases} (1+n)\bar{b}_t^0 + \bar{b}_t^1 - \bar{\tau}_t^1 & = \breve{g}_t \\ (1+n)\bar{b}_t^0 + \bar{b}_t^1 & = 0 \end{cases}$$

so $\breve{g}_t = -\bar{\tau}_t^1$. The last equation that should be verified is (50). Using $\breve{\tau}_t^2 = (1+n)\breve{g}_t - \hat{M}RS_{t-1}\breve{g}_{t-1}$ and $\breve{g}_t = -\bar{\tau}_t^1$ in (50), one gets:

$$\begin{aligned} \hat{e}_{t+1} &= y_2 + \hat{M}RS_t\breve{b}_t^1 + \breve{\tau}_{t+1}^2 = y_2 + \hat{M}RS_t(\bar{b}_t^1 - \bar{\tau}_t^1) - (1+n)\bar{\tau}_{t+1}^1 + \hat{M}RS_t\bar{\tau}_t^1 \\ &= y_2 + \hat{M}RS_t\bar{b}_t^1 - (1+n)\bar{\tau}_{t+1}^1 = y_2 + \hat{M}RS_t\bar{b}_t^1 + \bar{\tau}_{t+1}^2, \end{aligned}$$

where the last expression for $\hat{e}_{t+1}$ is true since (50) is respected under the balanced-budget policy.

7.7 Proof of Lemma 4

First, $b_{-1}^1 \geq 0,\ \ b_{-1}^0 \leq 0$ and $\hat{c}_t > y_0,\ \forall t \geq 0$ implies that (IR2) holds for $t \geq 0$. Indeed, $b_t^1 = -(1+n)b_t^0 + g_t > 0$, so (IR2) holds since it is equivalent to $b_t^1 > 0$. The following lemma is used for the analysis of the constraint (IR1).

Lemma 6. *If an optimal allocation $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ with $\hat{c}_t > y_0,\ \forall t \geq 0$ is decentralized under limited commitment with $g_t \geq 0$, the corresponding equilibrium has $b_{t-1}^0 \geq f(R_{t+1}, \tau_t^1, \tau_{t+2}^2)/R_t$ $\forall t > 0$, so the young agents are not constrained in equilibrium.*

Proof. Rewrite the utility maximization program with limited commitment (13) as:

$$\max_{(c_{t-1}, d_t, e_{t+1})>0} \quad u(c_{t-1}) + \beta u(d_t) + \beta^2 u(e_{t+1})$$
$$\text{s.t.} \begin{cases} R_t c_{t-1} + d_t + \frac{e_{t+1}}{R_{t+1}} & = R_t y_0 + y_1 + \tau_t^1 + \frac{y_2 + \tau_{t+1}^2}{R_{t+1}} \\ y_0 - c_{t-1} & \geq f(R_{t+1}, \tau_t^1, \tau_{t+1}^2)/R_t \\ y_1 - d_t + R_t(y_0 - c_{t-1}) + \tau_t^1 & \geq 0 \end{cases} \tag{53}$$

This form is obtained by eliminating b_{t-1}^0, b_t^1 from all the constraints of (13). The last constraint can be omitted for the current context as it is proved above that the constraint is not binding in the decentralized equilibria in question. Denote λ_t the Lagrange multiplier associated with the first constraint and χ_t with the second. Then, the FOC with respect to c_{t-1}, d_t are $u'(c_{t-1}) = (\lambda_t + \chi_t)R_t$ and $\beta u'(d_t) = \lambda_t$. Dividing the FOC by each other, one gets

$$MRS_{cd}(c_{t-1}, d_t) = (1 + \chi_t/\lambda_t)R_t$$

Solving the same problem for an agent of the generation $t-1$, one gets FOC with respect to d_{t-1} and e_t are $\beta u'(d_{t-1}) = \lambda_{t-1}$ and $\beta^2 u'(e_t) = \lambda_{t-1}/R_t$. Dividing the FOC by each other, one gets

$$MRS_{de}(d_{t-1}, e_t) = R_t$$

. From FOC of (19), $MRS_{cd}(\hat{c}_{t-1}, \hat{d}_t) = MRS_{de}(\hat{d}_{t-1}, \hat{e}_t)$ for $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$. This allocation is a solution to the FOC when it is decentralized, so one obtains

$$MRS_{cd}(\hat{c}_{t-1}, \hat{d}_t) = (1 + \chi_t/\lambda_t)MRS_{de}(\hat{d}_{t-1}, \hat{e}_t) = (1 + \chi_t/\lambda_t)MRS_{cd}(\hat{c}_{t-1}, \hat{d}_t),$$

so $\chi_t = 0$ and the second constraint is not binding. This is true for any generation $t \geq 1$. □

The main proposition is an "⇔" statement, where the part "⇐" follows from the equivalence of borrowing constraints and individual rationality constraints. Indeed, if a minimal policy decentralizes an allocation under full commitment and both (IR1) and (IR2) hold in $t \geq 0$, then all the conditions for decentralization under limited commitment are verified. To prove the part "⇒", first note that Lemma 6 implies the allocation is decentralized under limited commitment with borrowing constraints not binding for the young. (IR1) is then satisfied in each period for the allocation and the minimal policy in question. A solution to (13) with slack constraints is also a solution to (3), so $(\hat{c}_t, \hat{d}_t, \hat{e}_t)_{t\geq 0}$ fulfills all the conditions for a full commitment equilibrium with the minimal policy in question. Equivalently, this policy decentralizes the allocation with full commitment.

7.8 Proof of Proposition 6

The proposition is trivially verified for allocations with $\hat{c} < y_0$: since $b^1 \geq 0$ by (7), $(1+n)b^0 + b^1 > 0$ and government debt is necessary. The remaining proof is for the case $\hat{c} > y_0$, where the endogenous borrowing constraints apply.

Given the Lemma 4, the proposition being proved is equivalent to (IR1) being more strict under under the balanced-budget policy than under the debt-based one, for a given allocation. I first do it for periods $t > 0$ and then turn to $t = 0$. The constraint (IR1) in periods $t > 0$ under the two policies is:

$$\begin{aligned} u(\hat{d}) + \beta u(\hat{e}) &\geq u(y_1 + \bar{\tau}^1) + \beta u(y_2 + \bar{\tau}^2) \equiv \bar{L} \quad && \text{for balanced-budget} \\ u(\hat{d}) + \beta u(\hat{e}) &\geq u(y_1) + \beta u(y_2 + \check{\tau}^2) \equiv \check{L} && \text{for debt-based} \end{aligned}$$

where the label $\bar{L}$ is used for for lower limit on adults' utility under the first policy and the label $\check{L}$ is used for the same limit under the second policy. The proposition is proved if $\bar{L} > \breve{L}$. The values of transfers $\bar{\tau}^1, \bar{\tau}^2, \breve{\tau}^2$ are given by Proposition 5 since they decentralize the allocation under full commitment, according to Lemma 4. One gets:

$$\bar{\tau}^1 = \hat{d} - y_1 + (M\hat{R}S + 1 + n)(\hat{c} - y_0) < 0 \text{ (by Lemma 3 and Assumption 3)}$$
$$\bar{\tau}^2 = -(1+n)\bar{\tau}^1 > 0$$
$$\breve{\tau}^2 = (M\hat{R}S - 1 - n)\bar{\tau}^1 < 0 \text{ (since } M\hat{R}S > 1 + n \text{ by Lemma 2)}$$

To show that $\bar{L} > \breve{L}$, define a function L as follows:

$$\begin{aligned} L &: \left(\frac{-y_2 - \bar{\tau}^2}{M\hat{R}S}, y_1 - \frac{\bar{\tau}^2}{1+n}\right) \to \mathbb{R} \\ L(x) &= u\left(y_1 - \frac{\bar{\tau}^2}{1+n} - x\right) + \beta u\left(y_2 + \bar{\tau}^2 + M\hat{R}S \cdot x\right) \end{aligned} \tag{54}$$

As u is twice continuously differentiable, so is L. The function describes the utility from adult and old age consumption that an adult agent would have under an asset position $b^0 = 0$, transfers $(-\bar{\tau}^2/(1+n), \bar{\tau}^2)$ and an interest rate $M\hat{R}S$. This no-debt agent would choose consumption on a budget line

$$\mathcal{B}^{ND} = \{(d,e) \in \mathbb{R}^2_{++} : d + e/M\hat{R}S = y_1 - \bar{\tau}^2/(1+n) + (y_2 + \bar{\tau}^2)/M\hat{R}S\}$$

by doing savings of size x. ND in $\mathcal{B}^{ND}$ stands for "no debts". The rest of the proof uses a utility maximisation argument for the hypothetical no-debt agent to show that $\bar{L}$ is a utility level that is higher than $\breve{L}$.

First, according to (54), $\bar{L} = L(0)$ and $\breve{L} = L(-\bar{\tau}^2/(1+n))$, since $\breve{\tau}^2 = (1 - \frac{M\hat{R}S}{1+n})\bar{\tau}^2$. As $-\bar{\tau}^2/(1+n) < 0$ the proof for periods $t > 0$ can be concluded by showing that $L(\cdot)$ is increasing on $(-\bar{\tau}^2/(1+n), 0)$. Since $(\bar{\tau}^1, \bar{\tau}^2)$ decentralize $(\hat{c}, \hat{d}, \hat{e})$ under full commitment in $t > 0$, the consumption levels $(\hat{d}, \hat{e})$ maximize the utility of an adult that has $Rb^0 = M\hat{R}S \cdot (y_0 - \hat{c}) < 0$, transfers $(\bar{\tau}^1, \bar{\tau}^2)$ and interest rate $M\hat{R}S$. I will call this adult indebted to distinguish from the hypothetical, no-debt one mentioned above. The budget line of this agent is

$$\mathcal{B}^{D} = \{(d,e) \in \mathbb{R}^2_{++} : d + e/M\hat{R}S = y_1 + M\hat{R}S \cdot (y_0 - \hat{c}) - \bar{\tau}^2/(1+n) + (y_2 + \bar{\tau}^2)/M\hat{R}S\}$$

If $(\hat{c}, \hat{d}, \hat{e})$ can be decentralized by a balanced-budget policy under limited commitment, it follows that the adult has non-negative optimal savings in equilibrium. On the other hand, the only difference between the budget lines $\mathcal{B}^{ND}$ and $\mathcal{B}^{D}$ is initial assets: 0 in the first case and $(y_0 - \hat{c}) < 0$ in the second. By normality of old-age consumption, we get that the no-debt agent maximizes utility with strictly positive savings. It follows that $\arg\max L > 0$. At the same time, L is concave by concavity of u: $L''(x) = u''(y_1 - \bar{\tau}^2/(1+n) - x) + \beta M\hat{R}S^2 u''(y_2 + \bar{\tau}^2 + M\hat{R}S \cdot x) < 0$. We get that $L'(x) = 0$ in only one point, namely $x = \arg\max L$, and $L'(x) > 0$ for all $x < \arg\max L$. It follows that L is indeed increasing on $(-\bar{\tau}^2/(1+n), 0)$ as $\arg\max L > 0$. This concludes the proof for periods

$t > 0$.

In the initial period $t = 0$, generation -1 is old and generation 0 is adult. Although (IR2) is verified for generations $t \geq 1$ by Lemma 4, it is not always the case for the two initial generations. The constraint (IR2) on generation -1 depends only on the initial condition and is verified under any policy iff $b^1_{-1} > 0$. For generation 0, (IR2) is more strict under the balanced-budget than under the debt-based policy, since $\bar{\tau}^2_1 = \bar{\tau}^2 > \breve{\tau}^2_1 = \breve{\tau}^2$.

The constraint (IR1) for generation 0 is:

$$\begin{aligned}
&u(\hat{d}) + \beta u(\hat{e}) \geq u(y_1 + \bar{\tau}^1_0) + \beta u(y_2 + \bar{\tau}^2) \equiv \bar{L}_0 && \text{for balanced-budget} \\
&u(\hat{d}) + \beta u(\hat{e}) \geq u(y_1) + \beta u(y_2 + \breve{\tau}^2) \equiv \breve{L}_0 && \text{for debt-based}
\end{aligned}$$

which uses the fact that transfers are at their stationary values beginning with $t = 2$. The initial transfer for the balanced-budget policy is $\bar{\tau}^1_0 = \hat{d} - y_1 - R_0 b^0_{-1} + (1+n)(\hat{c} - y_0)$. One can write $\bar{\tau}^1_0 = \bar{\tau}^1 - R_0 b^0_{-1} + \hat{MRS} \cdot (y_0 - \hat{c})$. Then, $\bar{L}_0 \geq \bar{L} \Leftrightarrow R_0 b^0_{-1} \leq \hat{MRS} \cdot (y_0 - \hat{c})$. This is assumed for this Proposition, so one obtains $\bar{L}_0 \geq \bar{L} > \breve{L} = \breve{L}_0$. This means the constraint (IR1) on the generation 0 is more strict under the balanced-budget policy than under the debt-based one. This completes the proof.