Finding morphisms with gradient descent

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Representations of finite groups

Let $|G| < \infty$ and $V = \mathbb{C}^n$.

Definition

$\rho\colon G \to \mathrm{GL}(V)$ is *irreducalbe*, if there is no $0 < W < V$, such that $G \cdot W \leq W$.

Theorem

Every representation $\rho\colon G \to GL(V)$ is equal to $\bigoplus_{i \in I} \rho_i$, where every ρ_i is irreducable.

Modeling representations

Take any model

$$\begin{aligned} \hat{\rho} \colon G &\to \mathbb{C}^{n \times n} \\ g &\mapsto \begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & & \vdots \\ g_{n,1} & \cdots & g_{n,n} \end{bmatrix} = \phi_g \end{aligned}$$

and keep changing its parameters $\phi = \{\phi_g \mid g \in G\}$ until $\hat{\rho}$ becomes a representation.

Just on generators

Let $G =< S|R >$. Define $\hat{\rho}\colon G \to \mathbb{C}^{n\times n}$ as

$$\hat{\rho}(s) = \begin{bmatrix} s_{1,1} & \cdots & s_{1,n} \\ \vdots & & \vdots \\ s_{n,1} & \cdots & s_{n,n} \end{bmatrix} = \phi_s$$

on generators $s \in S$ and $\hat{\rho}(s_1 s_2 \cdots s_m) = \hat{\rho}(s_1)\hat{\rho}(s_2)\cdots\hat{\rho}(s_m)$ elsewhere.

Relation loss

Let $G =< S|R >$. $\hat{\rho}$ is a homomorphism iff

$$\hat{\rho}(r) - I = 0$$

for every $r \in R$.

$$\mathcal{L}_{\text{rel}}(\hat{\rho}) = \frac{1}{|R|} \sum_{r \in R} ||\hat{\rho}(r) - I||_F^2$$

Theorem

$\hat{\rho}$ is homomorphism iff $\mathcal{L}_{\text{rel}}(\hat{\rho}) = 0$

Gradient descent

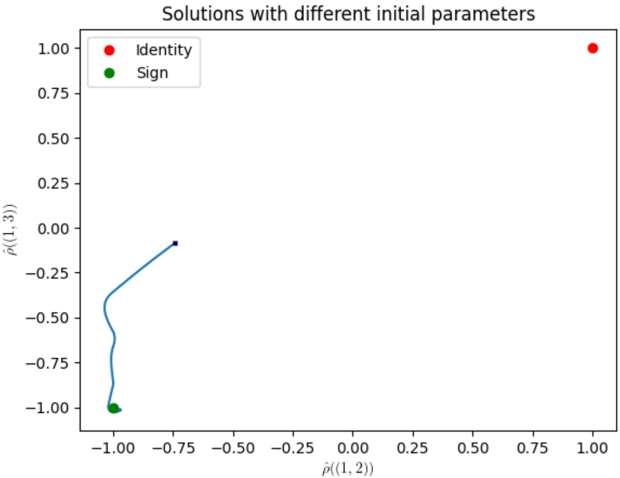
Optimisation using

$$\phi_{i+1} = \phi_i - \eta \nabla \mathcal{L}(\phi_i)$$

Gradient flow equation

$$\frac{d\phi}{dt} = -\nabla \mathcal{L}(\phi), \quad \phi(0) = \phi_0$$

One dimensional representations of S_3



Path of model $\hat{\rho}\colon S_3 =< (1,2),(1,3) > \to \mathbb{R}$

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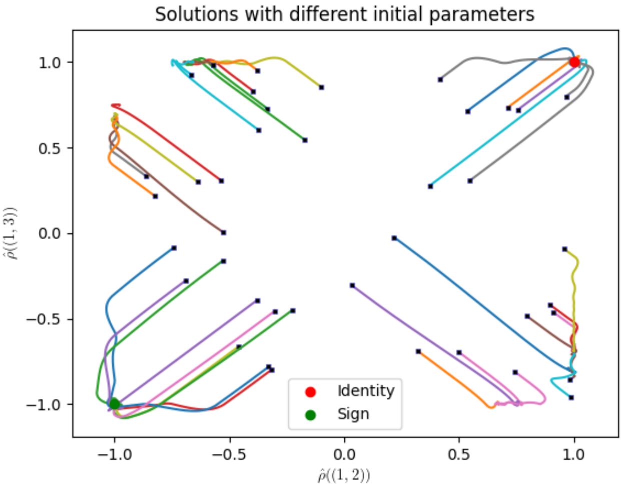


Path of model $\hat{\rho}\colon S_3 =< (1,2),(1,3) > \to \mathbb{R}$ for different initial values.

Irreducibility loss

Theorem

ρ is irreducible iff $|\chi_\rho| = \frac{1}{|G|}\sum_{g\in G} \mathrm{tr}(\rho(g))\mathrm{tr}(\rho(g^{-1})) = 1.$

$$\mathcal{L}_{\mathsf{irr}}(\hat{\rho}) = (|\chi_{\hat{\rho}}| - 1)^2$$

Unitary loss

Theorem

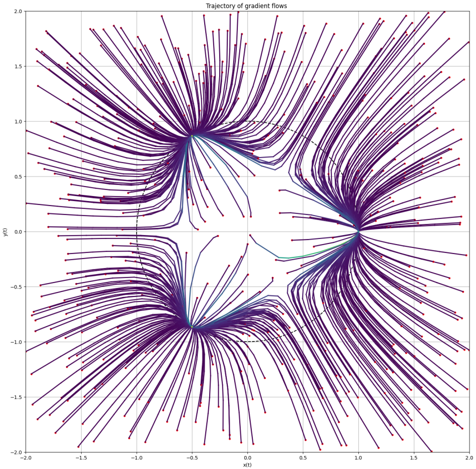
Every representation $\rho \colon G \to GL_n$ *of finite group is equivalent to unitary representation* $\rho_{\text{unitary}} \colon G \to U_n$.

$$\mathcal{L}_{\text{unitary}} = \frac{1}{|S|} \sum_{s \in S} ||\hat{\rho}(s)\hat{\rho}(s)^H - I||$$

$$\mathcal{L} = \mathcal{L}_{\text{rel}} + \mathcal{L}_{\text{irr}} + \mathcal{L}_{\text{unitary}}$$

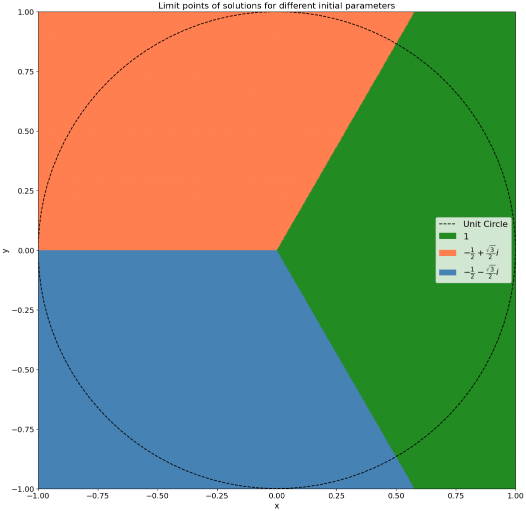
Cyclic groups

$C_n =< z \mid z^n = 1 >$, model $\hat{\rho}(z) = x + iy$ with parameters x, y.



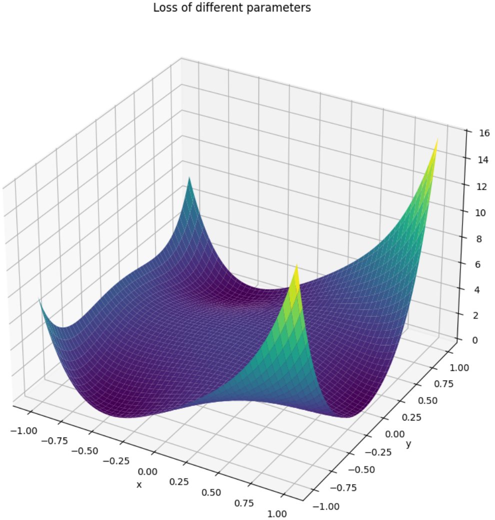
Solutions with different initial values.

Dynamics



Limit points of different initial values

Loss function



Plot of $\mathcal{L}$ for $x, y \in [-1, 1]$

Dihedral Groups

$$D_{2n} = \langle r, s \mid r^n = s^2 = (rs)^2 = 1 \rangle$$

Theorem

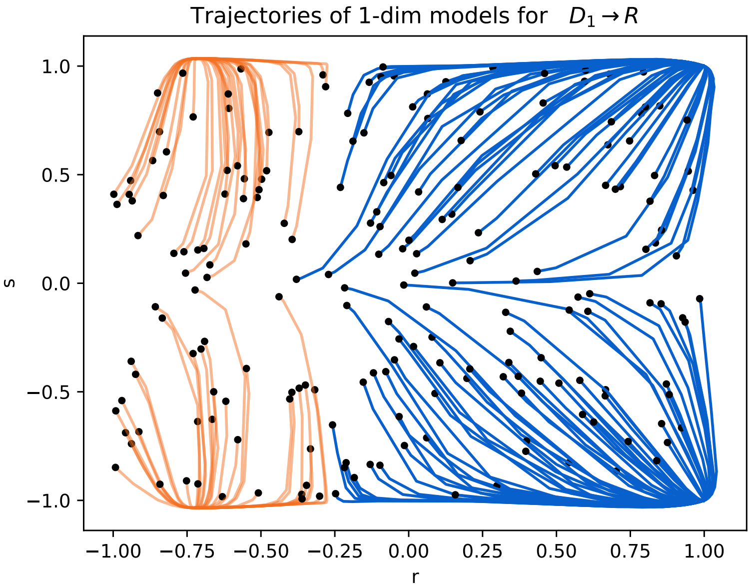
The irreducible representations of the dihedral group D_{2n} are:

$$\rho_k : \; s \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad r \mapsto \begin{bmatrix} \cos\left(\frac{2\pi k}{n}\right) & -\sin\left(\frac{2\pi k}{n}\right) \\ \sin\left(\frac{2\pi k}{n}\right) & \cos\left(\frac{2\pi k}{n}\right) \end{bmatrix}$$

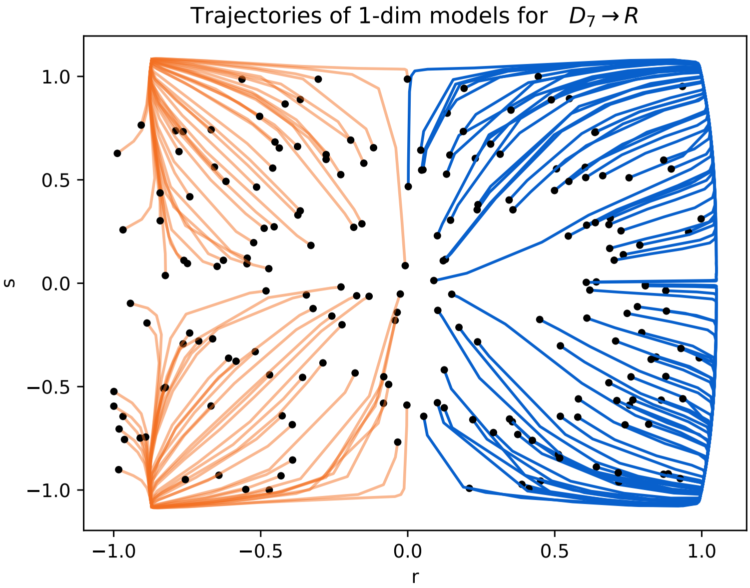
and

$$\chi_{\varepsilon,\delta} : \quad s \mapsto \varepsilon, \quad r \mapsto \delta$$

where $0 \leq k < \frac{n}{2}$, $\varepsilon \in \{-1, 1\}$, *and* $\delta \in \begin{cases} \{-1, 1\} & \text{if } n \text{ is even,} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$



Showing 200 out of 5000 sampled trajectories



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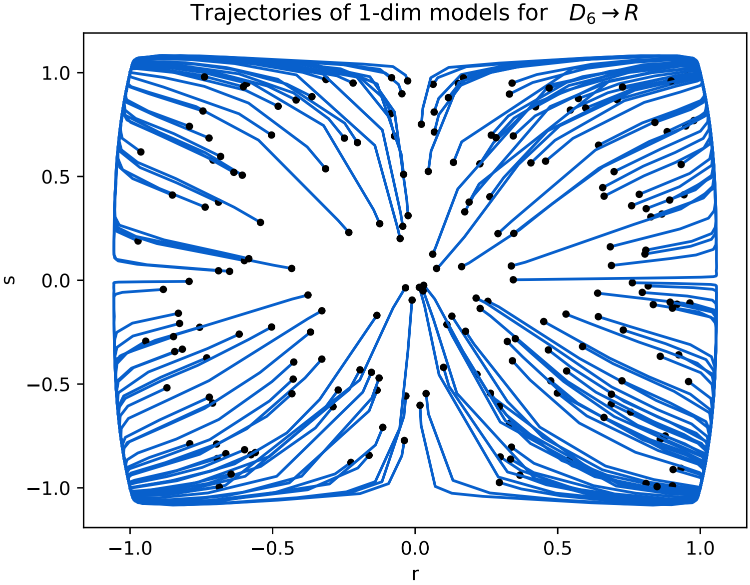

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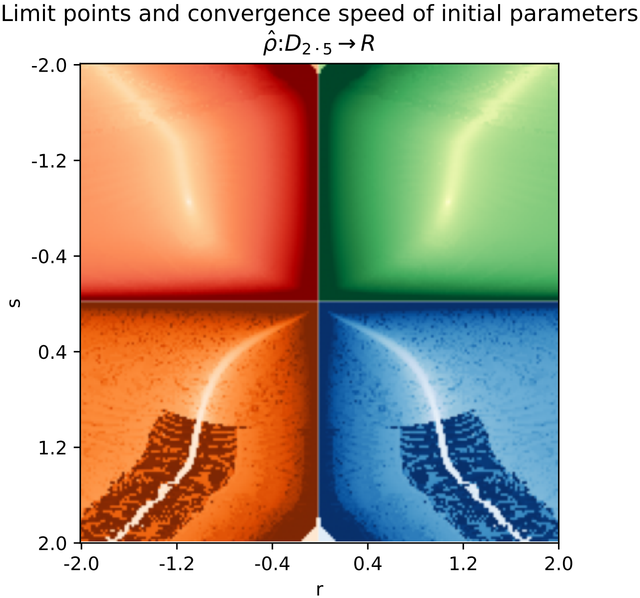
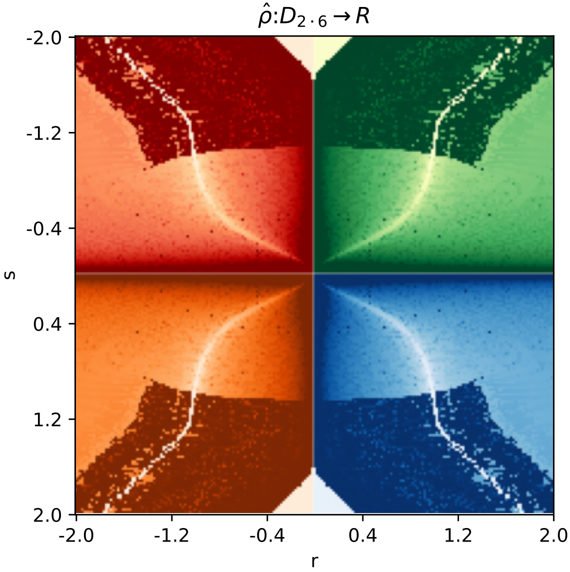


Each colormap encodes one limit point. Faster the convergence, lighter the color.

Limit points and convergence speed of initial parameters



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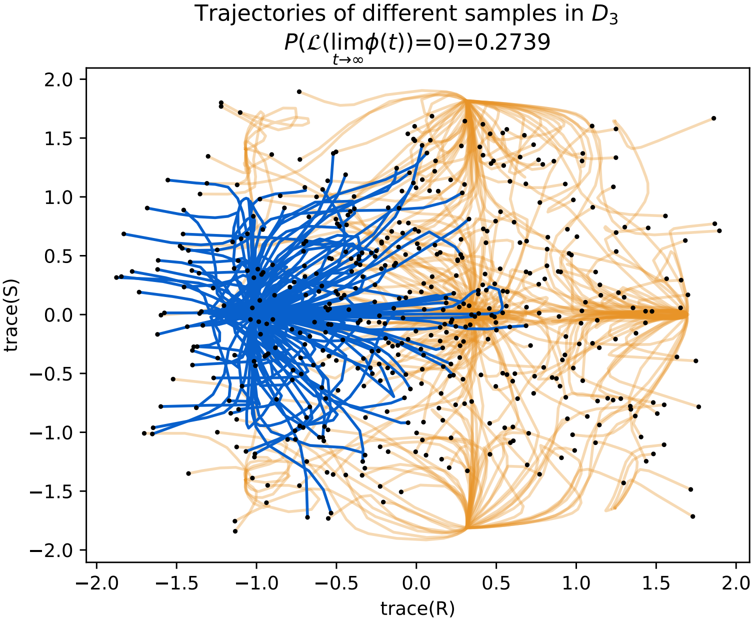
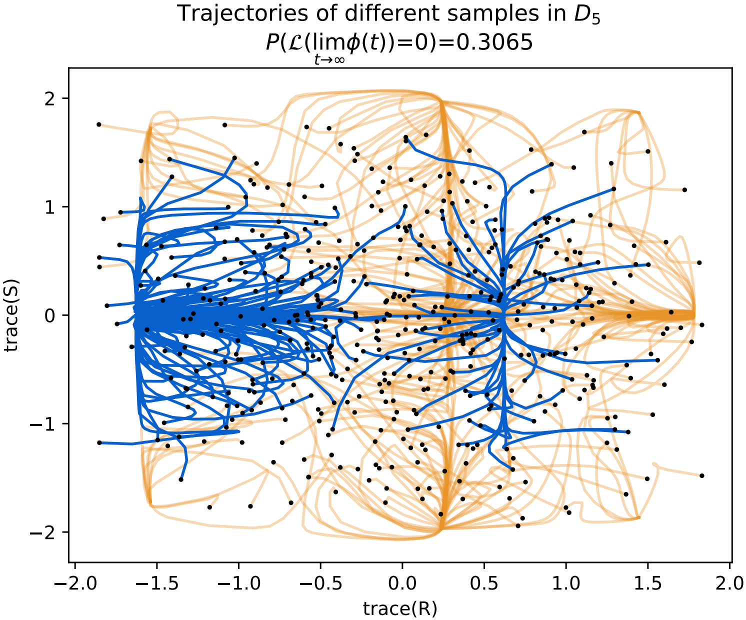


Sample size: 10^4. $\mathrm{tr}(R), \mathrm{tr}(S) \in (-2, 2)$.



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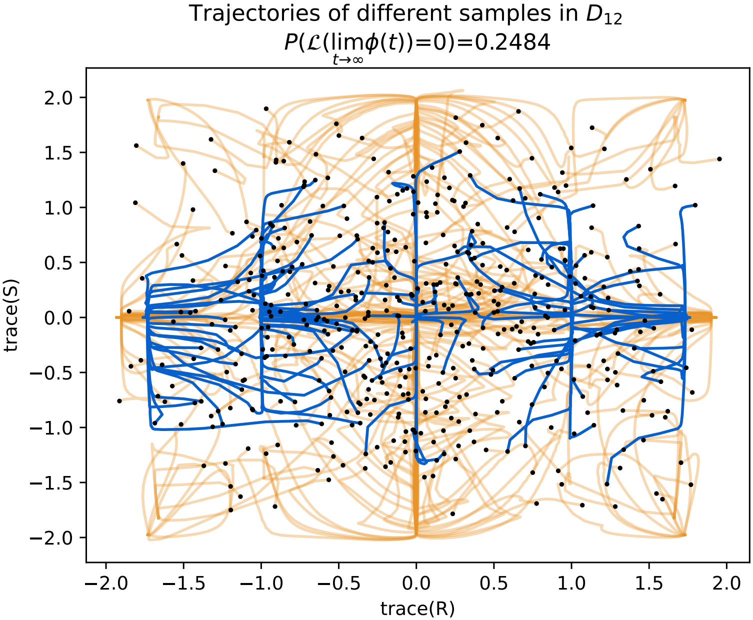

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Probability for success

Group	$P\left(\mathcal{L}(\lim_{t\to\infty}\phi(t)) = 0\right)$
D_3	0.274
D_4	0.158
D_5	0.306
D_6	0.21
D_7	0.311
D_8	0.235
D_9	0.33
D_{10}	0.241
D_{11}	0.327
D_{12}	0.248
D_{13}	0.319

Group actions

Map generators of $G =< S|R >$ to random maps on $[n]$.

$$\hat{\rho}\colon S \to \mathcal{P}(\mathrm{fun}([n],[n]))$$

$$s \mapsto P_s = \begin{bmatrix} P(s(1)=1) & P(s(1)=2) & \cdots & P(s(1)=n) \\ P(s(2)=1) & P(s(2)=2) & \cdots & P(s(2)=n) \\ \vdots & \vdots & & \vdots \\ P(s(n)=1) & P(s(n)=2) & \cdots & P(s(n)=n) \end{bmatrix}$$

For $f \in \mathrm{fun}([n],[n])$, define

$$P(s = f) = \prod_{i=1}^{n} P(s(i) = f(i)) = \prod_{i=1}^{n} s_{i,f(i)}$$

Construction details

For $g = s_1 s_2 \cdots s_m \in G$ define

$$P_g = P_{s_1} P_{s_2} \cdots P_{s_m} = \begin{bmatrix} P(g(1) = 1) & \cdots & P(g(1) = n) \\ \vdots & \vdots & \vdots \\ P(g(n) = 1) & \cdots & P(g(n) = n) \end{bmatrix}$$

Get stochastic matrices using $\operatorname{softmax}(v)_i = \frac{e^{v_i}}{\sum_{j=1}^{n} e^{v_j}}$

$$P \colon \mathbb{R}^{n \times n} \to \mathcal{S}_n$$
$$\begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} \mapsto \begin{bmatrix} \operatorname{softmax}(a_1^T) \\ \vdots \\ \operatorname{softmax}(a_n^T) \end{bmatrix}$$

Make relations probable

For $r \in R$, we want $P(r = \mathrm{id}) = 1$

$$0 = \log(P(r = \mathrm{id})) = \log(\prod_{i=1}^{n} P(r(i) = i)) = \mathrm{tr}(\log(P_r))$$

$$\mathcal{L}_{\text{rel}} = -\frac{1}{|R|} \sum_{r \in R} \mathrm{tr}(\log(P_r))$$

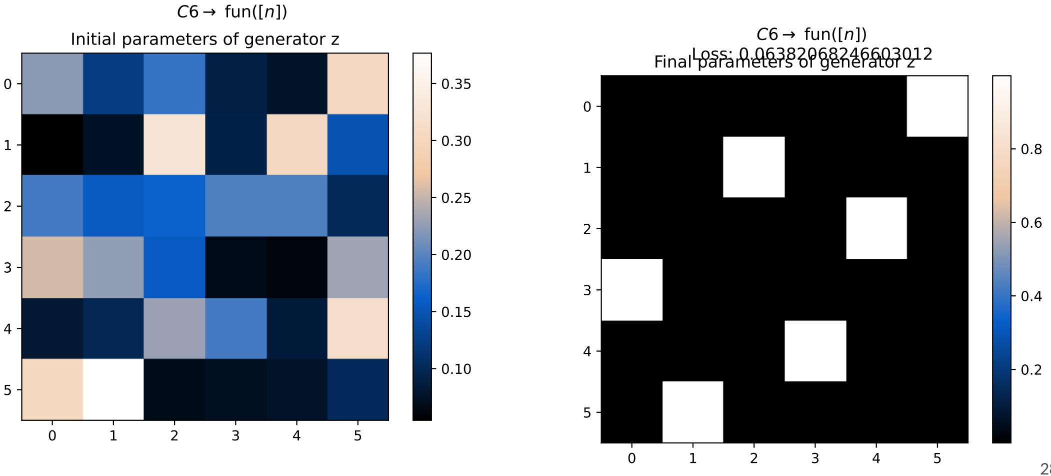
Make bijections probable

$$P(s \in S_n) = \sum_{\sigma \in S_n} P(s = \sigma) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} s_{i,\sigma(i)} = \text{Perm}(P_s).$$

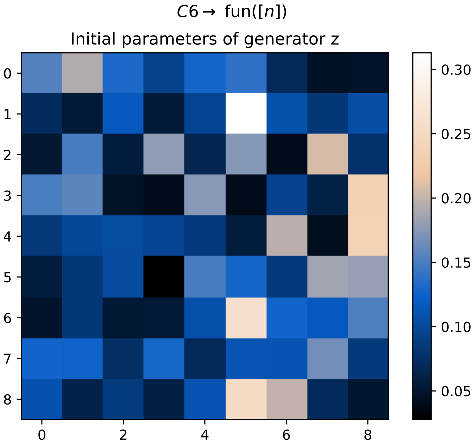
$$P(s \in S_n) = 1 \iff P_S \text{ is permutation matrix} \iff P_S \in U_n$$

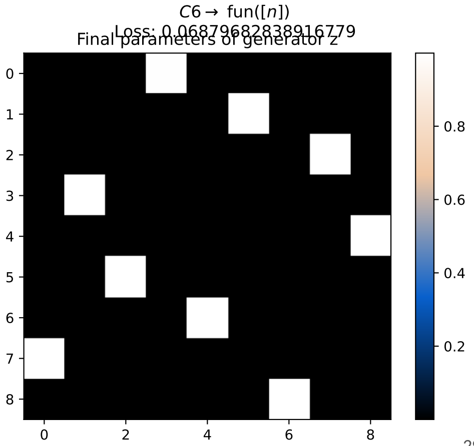
$$\mathcal{L}_{\text{bijective}} = \frac{1}{|S|} \sum_{s \in S} ||P_s P_s^T - I||_F^2 = \mathcal{L}_{\text{unitary}} \circ \mathrm{softmax}$$

Converges to (162354)

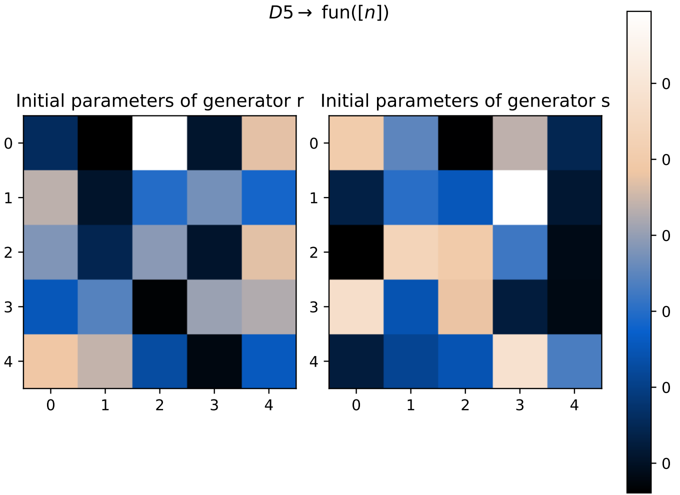


Converges to $(142638)(597)$





Converges to $r \xrightarrow{t\to\infty} (13524), s \xrightarrow{t\to\infty} (14)(23)$



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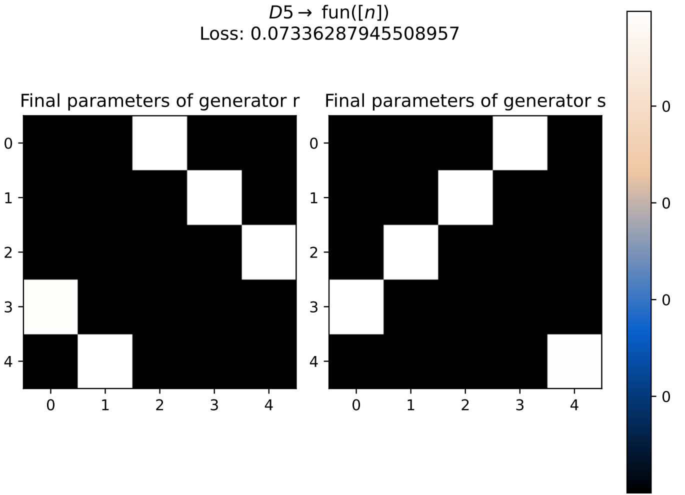


Graph isomorphisms

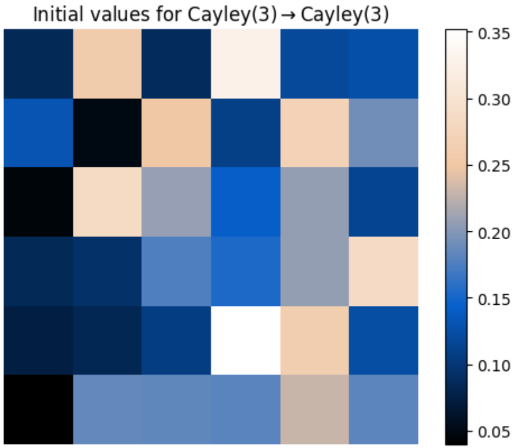
- G_1, G_2 graphs with adjacency matrices $M_1 = \left[m_{i,j}^{(1)}\right]$ and $M_2 = \left[m_{i,j}^{(1)}\right]$
- Random mapping $f : [n] \to [n] \sim P_f = \left[P(f(i) = j)\right]$
- For every $i \sim_1 j$, we want $P(f(i) \sim_2 f(j)) = 1$

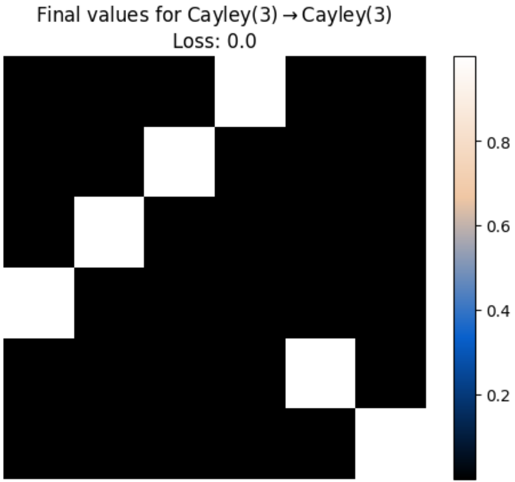
Relation loss

$$P(f(i) \sim_2 f(j)) = \sum_{k=1}^{n} \sum_{h=1}^{n} f_{i,k} f_{j,h} m_{k,h}^{(2)} = (P M_2 P^T)_{i,j}$$

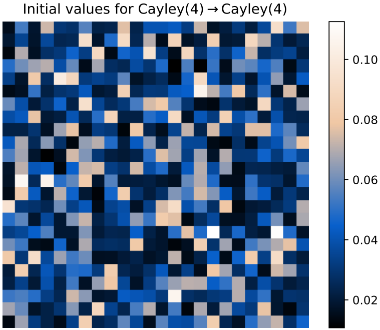
$$\begin{aligned} \mathcal{L}_{\text{rel}} &= - \sum_{i \sim_1 j} P(f(i) \sim_2 f(j)) \\ &= - \sum_{i=1}^{n} \sum_{j=1}^{n} log(f_j^T M_2 f_i) m_{i,j}^{(1)} \\ &= - \operatorname{tr}(\log(P_f M_2 P_f^T) M_1^T) \end{aligned}$$

P_f for f acting on $\mathrm{Cayley}(S_3)$

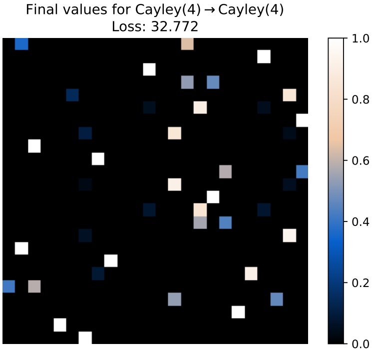




Automorphism of $\mathrm{Cayley}(S_4)$

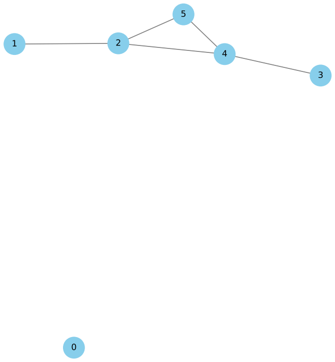


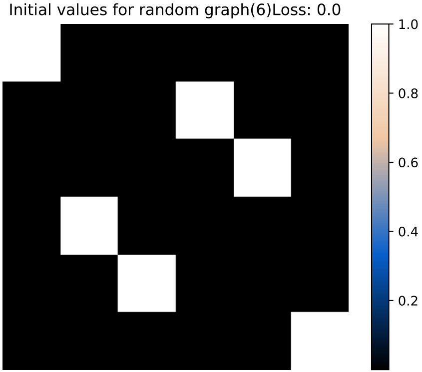
Caption



Caption

Automorphism of random graph





Further usage

- underparameterisation $x \mapsto \begin{bmatrix} \frac{x}{\sqrt{1-x^2}} & -\sqrt{1-x^2} \\ & x \end{bmatrix}$.
- overparameterisation
  $\phi_s = \mathrm{NeuralNetwork}(p), \dim(p) >> \dim(phi_s)$
- adaptive loss (Orthogonality of characters, adaptive dimension) - *bad results*
- paths in graphs, euler walks, ...

Implementation:

`github.com/urhprimozic/gofi`