ESE 524: Detection and Estimation Theory

Recitation 6

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Outline

- Composite hypothesis testing
  - Signal detection under an unknown parameter
  - Signal detection under multiple unknown parameters

Useful Formulas

- Composite testing:

  Θ_0 and Θ_1 form a partition of the parameter space Θ:

  $$\Theta_0 \cup \Theta_1 = \Theta, \;\; \Theta_0 \cap \Theta_1 = \emptyset$$

  we wish to identify which of the following two hypotheses is true:

  $$\begin{aligned} H_0 &: \theta \in \Theta_0, \quad && \text{null hypothesis} \\ H_1 &: \theta \in \Theta_1, \quad && \text{alternative hypothesis} \end{aligned}$$

- Generalized likelihood ratio test:

  $$\Lambda_{\text{GLR}}(\boldsymbol{x}) = \frac{\max_{\theta \in \Theta_1} p(\boldsymbol{x}; \theta)}{\max_{\theta \in \Theta_0} p(\boldsymbol{x}; \theta)} > \gamma$$

Useful Formulas (Cont.)

Consider testing

$$H_0 : \boldsymbol{h}(\boldsymbol{\theta}) = \mathbf{0} \quad \text{vs.} \quad H_1 : \boldsymbol{h}(\boldsymbol{\theta}) \neq \mathbf{0}$$

- Wald test:

$$T_{\mathrm{W}}(\boldsymbol{x}) = \boldsymbol{h}(\hat{\boldsymbol{\theta}})^T \left[H(\hat{\boldsymbol{\theta}}) \cdot \mathrm{CRB}(\hat{\boldsymbol{\theta}}) \cdot H(\hat{\boldsymbol{\theta}})^T \right]^{-1} \boldsymbol{h}(\hat{\boldsymbol{\theta}}) > \lambda,$$

where $H(\boldsymbol{\theta}) = \partial \boldsymbol{h}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^T$, $\mathrm{CRB}(\boldsymbol{\theta}) = \mathcal{I}(\boldsymbol{\theta})^{-1}$, and $\hat{\boldsymbol{\theta}}$ is an unrestricted ML estimator of $\boldsymbol{\theta}$ (under H_1). Then

$$T_{\mathrm{W}}(\boldsymbol{x}) \sim \chi_r^2 \quad \text{under} \;\; H_0$$

Useful Formulas (Cont.)

- Rao test:

$$T_{\mathrm{R}}(\boldsymbol{x}) = \boldsymbol{s}(\tilde{\boldsymbol{\theta}})^T \mathrm{CRB}(\tilde{\boldsymbol{\theta}}) \boldsymbol{s}(\tilde{\boldsymbol{\theta}}),$$

where $\boldsymbol{s}(\boldsymbol{\theta}) = \frac{\partial \log p(\boldsymbol{x};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\theta}}$ is the restricted estimate of $\boldsymbol{\theta}$ (under H_0). Then

$$T_{\mathrm{R}}(\boldsymbol{x}) \sim \chi^2_r \quad \text{under} \;\; H_0$$

Composite Hypothesis Examples

- In the first set of examples, we will explore various aspects of composite hypothesis testing.
- For mathematical simplicity, we will use very simple signals, namely sinusoidal signals.
- Our goal is to determine whether a signal is present in a time series or not.

Sinusoid Composite Hypothesis Examples

- $H_0 : \boldsymbol{x}[n] = \boldsymbol{w}[n]$
- $H_1 : \boldsymbol{x}[n] = A\cos(2\pi f_0(n - n_0) + \phi) + \boldsymbol{w}[n]$

- We want to estimate the cases where:
  - A is unknown
  - A, ϕ are unknown
  - A, ϕ,f_0 are unknown
  - All parameters unknown
- Section 7.6 in Kay's detection theory

Example 1: Unknown Amplitude

- The generalized likelihood ratio test for the unknown amplitude is:

$$\frac{\exp(\frac{-1}{2\sigma^2}\Sigma_{n=0}^{N-1}(x[n]^2-2x[n]\hat{A}\cos(2\pi f_0(n-n_0)+\phi)+\hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2))}{\exp(\frac{-1}{2\sigma^2}\Sigma_{n=0}^{N-1}x[n]^2)} \underset{H_0}{\gtrless} \lambda$$

- $\hat{A}$ is the maximum likelihood estimate of A based on $\boldsymbol{x}$:

$$\hat{A} = \arg\max_{A} \Sigma_{n=0}^{N-1}(x[n]^2 - x[n]\hat{A}\cos(2\pi f_0(n-n_0)+\phi) + \hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2))$$

- Take the derivative with respect to A:

$$\Sigma_{n=0}^{N-1} -2x[n]\cos[2\pi f_0(n-n_0)+\phi] + 2A\cos[2\pi f_0(n-n_0)+\phi]^2 = 0$$

- This means that $\hat{A} = \frac{\Sigma_{n=0}^{N-1}x[n]\cos(2\pi f_0(n-n_0)+\phi)}{\Sigma_{n=0}^{N-1}\cos(2\pi f_0(n-n_0)+\phi)^2}$

Unknown Amplitude (Cont.)

- Using the fact that

  $$\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0(n-n_0)+\phi) = \hat{A}\Sigma_{n=0}^{N-1}\cos(2\pi f_0(n-n_0)+\phi)^2$$

  we can rewrite the likelihood ratio:

  $$-\Sigma_{n=0}^{N-1} - 2\hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2 + \hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2 \underset{H_0}{\gtrless} \log\lambda'$$

  $$\hat{A}^2 \underset{H_0}{\gtrless} \frac{\lambda'}{\Sigma_{n=0}^{N-1}\cos(2\pi f_0(n-n_0)+\phi)^2}$$

- This means that our final test statistic is:

  $$(\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0(n-n_0)+\phi))^2 \underset{H_0}{\gtrless} \lambda'$$

Performance of Only Amplitude Detector

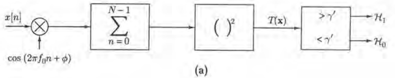

Figure 1: Block Diagram of Detector when Only amplitude is unknown

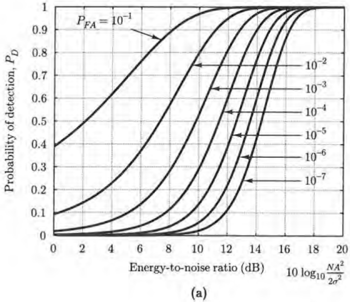

Figure 2: Performance of Detector when Only amplitude is unknown

Example 2: Amplitude and Phase Unknown

- When the amplitude and the phase are unknown, we have to reduce to the case where either $A > 0$ or $A < 0$, otherwise shifting the phase by π means we have two functions that can produce the same signal.
- Then we have to find $\hat{A}$ and $\hat{\phi}$ from:

$$\arg\max_{A,\phi} \Sigma_{n=0}^{N-1}(x[n]^2 - x[n]\hat{A}\cos(2\pi f_0(n-n_0)+\phi) + \hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2))$$

- Now we have to set the gradient equal to $(0,0)$:

$$\nabla \log(p(x;A,\phi)) = \begin{bmatrix} \Sigma_{n=0}^{N-1} - 2x[n]\cos(2\pi f_0(n-n_0)+\phi) + 2A\cos(2\pi f_0(n-n_0)+\phi) \\ \Sigma_{n=0}^{N-1} 2x[n]A\sin(2\pi f_0(n-n_0)+\phi) - A^2\sin(2\pi f_0(n-n_0)+\phi) \end{bmatrix}$$

Amplitude and Phase Unknown (Cont.)

- Using some trig identities we can approximate the solutions with :

$$\hat{A} = \sqrt{(\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0 n))^2 + (\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\sin(2\pi f_0 n))^2}$$

$$\hat{\phi} = \arctan(\frac{\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)}{\frac{2}{N}\Sigma_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)})$$

- Now the Likelihood ratio is:

$$-1/2\sigma^2 \Sigma_{n=0}^{N-1} - 2\hat{A}^2\cos(2\pi f_0(n-n_0)+\hat{\phi})^2 + \hat{A}^2\cos(2\pi f_0(n-n_0)+\hat{\phi})^2 \underset{H_0}{\gtrless} \log\lambda$$

- Using the substitution $\hat{\alpha_1} = \hat{A}\cos(\hat{\phi})$ and $\hat{\alpha_2} = -\hat{A}sin(\hat{\phi})$ and some more trigonometry we can simplify this expression into:

$$\frac{N}{4\sigma^2}(\hat{\alpha}_1^2 + \hat{\alpha}_2^2) \underset{H_0}{\gtrless} \log\lambda$$

Amplitude and Phase Unknown (Cont.)

- But it turns out that:
$\hat{\alpha}_1^2 + \hat{\alpha}_2^2 = \frac{2^2}{N^2\sigma^2}((\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)^2 + (\Sigma_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)^2)$

- so the test becomes:

$$\frac{1}{N\sigma^2}((\Sigma_{n=0}^{N-1} x[n]\cos(2\pi f_0 n)^2 + (\Sigma_{n=0}^{N-1} x[n]\sin(2\pi f_0 n)^2) = \frac{I(f_0)}{\sigma^2} \underset{H_0}{\gtrless} \log\lambda$$

- This is either the sum of two correlators similar to the unknown amplitude case, or something called the periodogram, which estimates $|X(f)|^2$, where $X(f)$ is the Fourier transform of $x[n]$.

Performance of Amplitude and Phase Detector

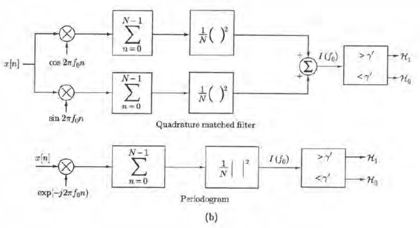

Figure 3: Block Diagram of Detector when Amplitude and Phase are unknown

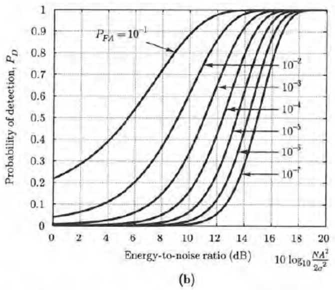

Figure 4: Performance of Detector when Amplitude and Phase are unknown

Example 3: Amplitude, Phase, and Frequency Unknown

- When the Amplitude, Phase, and Frequency are unknown we need to find the maximum of:

  $$\arg\max_{A,\phi} \Sigma_{n=0}^{N-1}(x[n]^2 - x[n]\hat{A}\cos(2\pi f_0(n-n_0)+\phi) + \hat{A}^2\cos(2\pi f_0(n-n_0)+\phi)^2))$$

  across three variables.
- to do this, define

  $$I(f_0) = \frac{2}{N}((\Sigma_{n=0}^{N-1}x[n]\cos(2\pi f_0 n)^2 + (\Sigma_{n=0}^{N-1}x[n]\sin(2\pi f_0 n)^2)$$

  as the periodogram of $x[n]$ as a function of f_0.
- To perform the maximization, note that if we know the best $\hat{f}_0$ we can compute $\hat{A}$ and $\hat{\phi}$ the same way as above.

Amplitude, Phase, and Frequency Unknown (Cont.)

- Also, The likelihood under H_0 is not a function of any of the variables so:

$$\max_{A,f_0,\phi} p(x;A,f_0,\phi) = \frac{\max_{A,f_0,\phi} p(x;A,f_0,\phi)}{p(x;H_0)}$$
$$= \max_{f_0} \frac{p(x;\hat{A},\hat{\phi},f_0)}{p(x;H_0)}$$
$$= \max_{f_0} \log \frac{p(x;\hat{A},\hat{\phi},f_0)}{p(x;H_0)}$$
$$= \max_{f_0} \frac{I(f_0)}{\sigma_2}$$

Performance of Amplitude, Phase, and Frequency Detector

- So the likelihood ratio test becomes:

$$\max_{f_0} \frac{I(f_0)}{\sigma_2} \underset{H_0}{\gtrless} \log \lambda$$

- This is the same as finding the maximum of the FFT output in matlab and comparing it to your threshold.
- In general the performance gets worse as the frequency inreases.

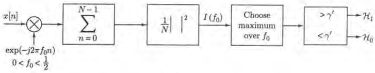

Figure 5: Block Diagram of Detector when Amplitude, Phase, and Frequency are unknown

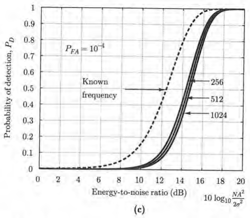

Figure 6: Performance of Detector when

Example 4: Unknown Amplitude, Phase, Frequency, and Arrival Time

- The parameter n_0 is the delay of the signal, also called the "Arrival Time". Assume we have a "long" set of data.
- Estimating the arrival time is to try and find the exact time window $[n_0, n_0 + N - 1]$ steps that the signal is active.
- First, we have to modify our $\hat{A}$ and $\hat{\phi}$ expressions from the previous cases
- Let $\hat{\alpha}_1 = \frac{2}{N} \Sigma_{n=n_0}^{n_0+N-1} x[n] \cos(2\pi \hat{f}_0(n - n_0))$ and $\hat{\alpha}_2 = \frac{2}{N} \Sigma_{n=n_0}^{n_0+N-1} x[n] \sin(2\pi \hat{f}_0(n - n_0))$
- Then, given a frequency and arrival time:

$$\hat{A} = \sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2}$$

$$\hat{\phi} = \arctan(\frac{-\hat{\alpha_1}}{\hat{\alpha_2}})$$

- Our likelihood ratio is still the periodogram, but for a window $[n_0, n_0 + N - 1]$

Unknown Amplitude, Phase, Frequency, and Arrival Time (Cont.)

- If we know the arrival time, we can use the same maximum as in the three paramter case.
- So, starting at $n_0 = 0$ we have to perform a frequency analysis and find the maximum frequency. Then set $n_0 = 1$ and find the same thing. Plot this for every n_0 and find the maximum frequency. This is called the short time periodogram (or short time FFT).
- This can be computed with the spectogram in matlab, and is widely used.

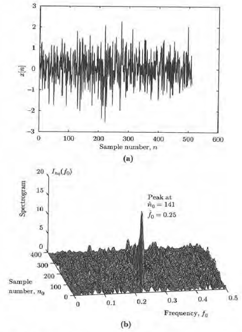

Figure 7: Example of signal and its short time FFT

Conclusions

- The generalized likelihood ratio test is a combination of MLE and the NP test for simple hypotheses.
- As more parameters are unknown, our detection performance generally goes down.
- As more parameters are unknown our MLE is more complicated.