6.042/18.062J Mathematics for Computer Science — December 17, 2008

Tom Leighton and Marten van Dijk

Final Exam

Name: ______________________________

- This final is **closed book**, but you may have three 8.5″ × 11″ sheets with notes in your own handwriting on both sides.
- You may not use a calculator or a Python interpreter, and while exercising skill with a PostScript interpreter would highly impress at least one member of the course staff, you are not allowed to use that either. You *may* work with any of the following: a slide rule, an abacus, a Curta, Napier's bones, any original version of the Antikythera mechanism, an Enigma machine, and/or the difference engine. You are also permitted to use a perpetual motion machine as a source of energy, and an antigravity device to elevate yourself above the rest of the class.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- Please resist the urge to roll on the floor laughing out loud.
- You have three hours to complete the exam.

Problem	1	2	3	4	5	6	7	8	9	10	Total
Points	25	25	25	15	15	15	25	15	20	20	200
Score	25	20	7	15	15	15	25	15	20	20	177
Grader											

$\frac{177}{200} \approx 88.5\%$

Time: 2:29

21:05
21:19
0:14

Problem 1. [25 points] The Final Breakdown 25/25

Suppose the 6.042 final consists of:

- 36 true/false questions worth 1 point each.
- 1 induction problem worth 15 points.
- 1 giant problem that combines everything from the semester, worth 49 points.

Grading goes as follows:

- The TAs choose to grade the easy true/false questions. For each individual point, they flip a fair coin. If it comes up heads, the student gets the point.
- Marten and Brooke split the task of grading the induction problem.
  - With 1/3 probability, Marten grades the problem. His grading policy is as follows: Either he gets exasperated by the improper use of math symbols and gives 0 points (which happens with 2/5 probability), or he finds the answer satisfactory and gives 15 points (which happens with 3/5 probability).
  - With 2/3 probability, Brooke grades the problem. Her grading policy is as follows: She selects a random integer point value from the range from 0 to 15, inclusive, with uniform probability.
- Finally, Tom grades the giant problem. He rolls two fair **seven**-sided dice (which have values from 1 to 7, inclusive), takes their product, and subtracts it from 49 to determine the score. (Example: Tom rolls a 3 and a 4. The score is then $49 - 3 \cdot 4 = 37$.)

Assume all random choices during the grading process are mutually independent.

The problem parts start on the next page. Show your work to receive partial credit.

21:05
21:11

(a) [7 pts] What is the expected score on the exam? 7/7

T/F : $E[X_1 + \cdots X_{36}] = 36\,E[X_1] = 18$

I : M: $E[M] = \frac{2}{5}\cdot 0 + \frac{3}{5}\cdot 15 = 9$

$E[B] = \frac{1}{16}(0+1+\cdots+15) = \frac{1}{16}\,\frac{15(15+1)}{2} = 7.5$

$E[I] = \frac{1}{3}\cdot 9 + \frac{2}{3}\cdot\frac{15}{2} = 3+5 = 8$

G : $E[X_1X_2] = E[X_1]E[X_2] = 4\cdot 4 = 16$

$E[G] = E[49 - X_1X_2] = 49-16 = 33$

$E[\text{Score}] = 18 + 8 + 33 = \underline{\underline{59}}$

21:11
21:12

(b) [5 pts] What is the variance on the 36 true/false questions? 5/5

$Var(T/F) = np(1-p) = 36\left(\frac{1}{4}\right) = \underline{\underline{9}}$

21:12
21:15

(c) [5 pts] What is the variance on the induction score, given that Marten graded the problem? 5/5

$$Var(I \mid M) = E[(M-9)^2] = \tfrac{2}{5}(0-9)^2 + \tfrac{3}{5}(15-9)^2$$

$$= \frac{162}{5} + \frac{108}{5} = \frac{270}{5} = \underline{54}$$

21:15
21:18

(d) [3 pts] Argue why the Markov bound can be used to determine an upper bound on the probability that the score on the exam is ≥ 80. You do not need to compute the actual bound. 3/3

all the possible scores are > 0.

21:16
21:19

(e) [5 pts] Use the Chebyshev bound to determine an upper bound on the probability that the score on the true/false questions is ≥ 24. 5/5

$$P(|S - E[S]| \geq a) \leq \frac{Var(S)}{a^2}$$

$$P(|S-18| \geq a) \leq \frac{9}{a^2}$$

$$P(S \geq a+18) = P(S \geq 24) \leq \frac{9}{36} = \underline{\frac{1}{4}}$$

21:31
21:~~3~~41
0:10

Problem 2. [25 points] Woodchucks Chucking Wood

20/25

All woodchucks can chuck wood, but only some can do it well.

- 1/3 of all woodchucks like to chuck wood.
- 2/3 of all woodchucks can chuck wood well.
- 1/2 of those that like chucking wood can do it well.
- The expected amount of wood chucked by a woodchuck (randomly chosen with uniform probability) is 7 kg/day.
- The expected amount of wood chucked by a woodchuck that likes chucking wood but can't do it well is 1 kg/day.
- A woodchuck that does not like chucking wood does not chuck any wood at all, regardless of its wood-chucking skillz or lack thereof.

21:31
21:35

(a) [10 pts] What is the probability that a woodchuck (randomly chosen with uniform probability) likes chucking wood, given that it can do it well?

10/10

$$P(likes \mid well) = \frac{P(likes)\,P(well \mid likes)}{P(well)}$$

$P(likes) = 1/3$

$P(well \mid likes) = 1/2$

$$p(well) = p(well \mid likes)\,p(likes) + p(well \mid \neg likes)\,p(\neg likes)$$

$$= \frac{1}{2} \cdot \frac{1}{3} +$$

$$= 2/3$$

$$P(likes \mid well) = \frac{(1/3)(1/2)}{(2/3)} = \frac{1/6}{2/3} = \frac{1}{2} \cdot \frac{3}{6} = \underline{\underline{\frac{1}{4}}}$$

21:35
21:41

(b) [15 pts] On average, how much wood would a woodchuck chuck if the woodchuck could chuck wood well?

10/15

$E[\text{wood}] = 7$

$E[\text{wood} \mid \text{well}] =$

$E[\text{wood} \mid \text{likes}, \neg\text{well}] = 1$

$E[\text{wood} \mid \neg\text{likes}] = 0$

$E[\text{wood}] = E[\text{wood} \mid \neg\text{well}]\, p(\neg\text{well}) + E[\text{wood} \mid \text{well}]\, p(\text{well})$

X 1/6

$7 = 1 \cdot \frac{1}{3} + E[\text{wood} \mid \text{well}] \cdot \frac{2}{3}$

$\frac{20}{3} \cdot \frac{3}{2} = E[\text{wood} \mid \text{well}] = 10$

X 10.25

21:42
22:06
0:24

Problem 3. [25 points] Cardsharing☆Revolution

7/25

Three 6.042 students—Kirari, Noelle, and Cobeni—are playing a game of Tan Tan Taan!. During each round of Tan Tan Taan!, each player is dealt 4 cards of their own, and one additional card is shared among all players, so that each player has 5 cards that they can use (the 4 cards of their own along with the single shared card). Cards are uniformly distributed from a 52-card deck. If you get four of a kind (for example, four aces or four 2's), you can continue playing in the next round. If you don't get four of a kind, you must quit and return to doing your 6.042 homework. Cards from round to round are mutually independent. This game is so fun that even if two of the three players must quit and return to their 6.042 homework, the third player will continue playing alone as long as they are able to.

(a) [5 pts] What is the probability that Kirari has four aces in the first round? 3/5

$$P(4A) = \cancel{\frac{4}{52}\frac{3}{52}\frac{2}{52}\frac{1}{52}\frac{48}{52}}$$

$$= \frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\cdot\frac{1}{49}\cdot\frac{48}{48}\;\cancel{\frac{51}{4}} = \frac{48\cdot\cancel{4!}}{\binom{52}{5}}$$

OFF BY FACTOR OF 4! IN BOTH CASES

(b) [5 pts] What is the probability that Kirari doesn't get four of a kind in the first round (and must quit playing)? 2/5

$$P(4) = \frac{1}{\binom{52}{5}}\left(\overset{13}{\cancel{52\cdot 3\cdot 2}}\cdot 1\right)\cdot 48 = \frac{6}{51\cdot 50\cdot 49} = \frac{1}{17\cdot 25\cdot 49}$$

$$P(\neg 4) = 1 - P(4) = 1 - \frac{\overset{13}{\cancel{52\cdot 3\cdot 2}}\cdot 1\cdot 48}{\binom{52}{5}}$$

(c) [5 pts] What is the expected number of rounds that Kirari will play?

2/5

~~X_i = indicator for does get 4 of a kind~~

Geometric distribution, $p(k) = P(4)^{k-1} P(\neg 4)$

$E[k] = 1 \quad x = 1/p \approx 1$

(d) [10 pts] What is the probability that all three can play a second round?

0/10

Probability that two were dealt 4-of-a-kind, and the third either got that or used the card in the middle.

Player 1: $52 \cdot 3 \cdot 2 \cdot 1$ choices, $4!$ permutations

Player 2: $48 \cdot 3 \cdot 2 \cdot 1$ choices, $4!$ permutations

Player 3: $44 \cdot 3 \cdot 2 \cdot 1 \cdot 40$ choices, $5!$ permutations

$3!$ permutations of the players

$\binom{52}{13}$ ways of dealing the cards

$$P(\text{all } 3) = \frac{5!(4!)^2\, 3!\; 52 \cdot 48 \cdot 44 \cdot 40 \cdot (3 \cdot 2 \cdot 1)^3}{\binom{52}{13}}$$

22:06
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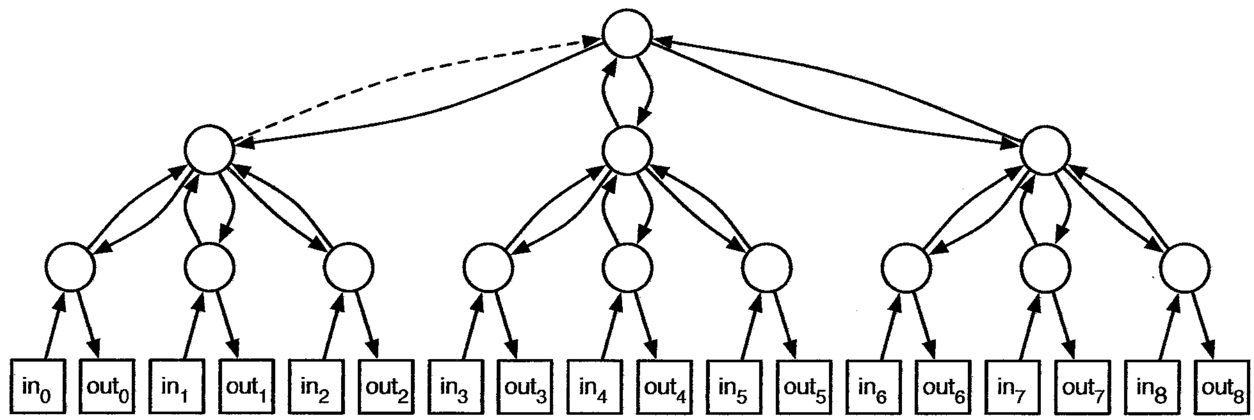
0:13

Problem 4. [15 points] Packet Racket! 15/15

Consider the complete ternary-tree network with 9 inputs and 9 outputs shown below where packets are routed randomly. The route each packet takes is the shortest path between input and output. Let I_0, I_1, and I_2 be indicator random variables for the events that a packet originating at in_0, in_1, and in_2, respectively, crosses the dashed edge in the figure. Let $T = I_0 + I_1 + I_2$ be a random variable for the number of packets passing through the dashed edge.



(a) [10 pts] Suppose that each input sends a single packet to an output selected uniformly at random; the packet destinations are mutually independent. (Note that outputs may receive packets from multiple inputs including their corresponding input.) 10/10

What are the expectation and variance of T?

9 possible outputs; must cross dashed for 6 of them

$$I_0, I_1, I_2 = \begin{cases} 1 & w.p.\ 6/9 \\ 0 & w.p.\ 3/9 \end{cases}$$

$$E[T] = E[I_0 + I_1 + I_2] = \frac{6}{9}\cdot 3 = \underline{2}$$

$$Var(T) = 3\cdot Var(I_0) = 3\cdot\left[\frac{6}{9}\cdot\left(1-\frac{6}{9}\right)^2 + \frac{3}{9}\left(0-\frac{6}{9}\right)^2\right]$$

$$= \frac{1}{3}\left[6\cdot\left(\frac{1}{3}\right)^2 + 3\cdot\left(\frac{2}{3}\right)^2\right] = 2\cdot\left(\frac{1}{9}\right) + \frac{4}{9} = \frac{6}{9} = \underline{\frac{2}{3}}$$

(b) [5 pts] Now consider the situation where a permutation of inputs to outputs is chosen uniformly at random; each input sends a packet to a distinct output. What is the expected value of T? Briefly justify your answer.

5/5

$\binom{9}{3}$ different permutations $= 9 \cdot 8 \cdot 7$

$T = 0$: $\binom{3}{3} = 1$ permutation $= 1$

$T = 1$: $\binom{3}{2} \cdot 6$ permutations $= 18$

$T = 2$: $\binom{3}{1} \cdot \binom{6}{2}$ permutations $= 90$

$T = 3$: $\binom{3}{0} \cdot \binom{6}{3}$ " $= 120$

$$E[T] = 0 \cdot \frac{1}{9 \cdot 8 \cdot 7} + 1 \cdot \frac{18}{9 \cdot 8 \cdot 7} + 2 \cdot \frac{90}{9 \cdot 8 \cdot 7} + 3 \cdot \frac{120}{9 \cdot 8 \cdot 7}$$

$$= 0 + \frac{1}{4 \cdot 7} + \frac{10}{4 \cdot 7} + \frac{20}{4 \cdot 7}$$

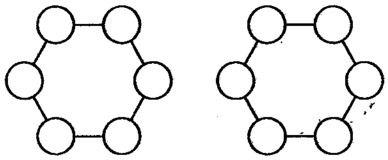
$$= \frac{31}{28}$$

DIFFERENT THAN THE OFFICIAL ANSWER; BUT THE QUESTION WAS EXCEEDINGLY POORLY WORDED.

22:19
22:27
0:08

Problem 5. [15 points] Connected or Not? That Is the Question 15/15

Suppose we have a simple, undirected graph G with $2n$ vertices and $2n$ edges, where $n \geq 3$. The graph consists of two disjoint cycles with n edges each. For example, if $n = 6$, the graph would look like this:



(a) [5 pts] A pair of vertices u and v from G is selected uniformly at random from the pairs of distinct vertices with no edge between them. A new graph G' is constructed to be the same as G, except that there is an edge between u and v. What is the probability that G' is connected? 5/5

Connected: $2n$ edges for first choice n edges for second choice

all possible: $2n$ edges for first choice; $2n-3$ edges for second

$$P(G' \text{ connected}) = \frac{2n \cdot n}{2n \cdot (2n-3)} = \frac{n}{2n-3}$$

(b) [10 pts] k pairs of vertices from G are selected uniformly at random from the pairs of distinct vertices with no edge between them. Repetition is allowed; it is possible, for example, that the same pair appears multiple times in the set of k pairs. A new graph G'' is constructed to be the same as G, except that there are k new edges: the edges that correspond to the k selected pairs. What is the probability that G'' is **not** connected? 10/10

(*Hint: For $k = 1$, the sum of your answers to part (a) and part (b) should equal* 1.)

$$P(\text{not connected}, k=1) = 1 - \frac{n}{2n-3} = \frac{n-3}{2n-3}$$

$$P(\text{not connected}, G'') = \left(\frac{n-3}{2n-3}\right)^k$$

22:27
22:31
0:04

Problem 6. [15 points] 6.042: The Ultimate Showdown

15/15

There are 100 homework problems in 6.042 throughout the term. Let T_i, $1 \leq i \leq 100$, be the random variable indicating the fraction of a day that is needed by a student to solve the ith problem of 6.042.

The distribution for each T_i is different and unknown. We only know that the T_i are mutually independent and that for all i, $0 \leq T_i \leq 1$ and $\text{Ex}[T_i] = 0.3$.

Let T be the sum of all T_i's; T represents the total number of days needed by a student to complete all homework problems for 6.042. Prove that the probability that T is greater than $30e$ is exceedingly small by deriving the best bound you can on this probability. (*Hint: We do not consider* $1/e$ *to be exceedingly small.*)

Use Chernoff.

$$P(T \geq c E[T]) \leq e^{-zE[T]}, \quad z = c \ln c - c + 1$$

$$c E[T] = c \sum E[T_i] = c \cdot 100 E[T_i] = c \cdot 30 \Rightarrow c = e$$

$$z = e \ln e - e + 1 = 1$$

$$P(T \geq 30e) \leq e^{-1 \cdot 30} = \underline{\underline{e^{-30}}}$$

22:32
22:50
23:08
23:49
0:59

Problem 7. [25 points] Gotta Count 'Em All! 25/25

An unusual species inhabits the forest surrounding Functional City. Each member of the species can take one of three possible forms, called *Schemander, Haskeleon,* and *Camlizard.*

In January of every year, each individual undergoes "evolution"—a process by which the individual splits into two individuals, whose forms depend on the form of the original:

- A Schemander splits into a Schemander and a Haskeleon.
- A Haskeleon splits into a Schemander and a Camlizard.
- A Camlizard splits into a Schemander and a Haskeleon.

We are investigating the distribution of forms within a large population of this species over time. It is known that in June of year 0, the population consisted of a single Schemander. Assume that no individual ever dies and that all individuals successfully undergo evolution exactly once every January.

(a) [3 pts] Let S_n, H_n, and C_n be the number of Schemanders, Haskeleons, and Camlizards, respectively, in June of year n. Express S_n, H_n, and C_n in terms of S_{n-1}, H_{n-1}, and C_{n-1}, for $n > 0$. 3/3

$$S_n = S_{n-1} + H_{n-1} + C_{n-1}$$

$$H_n = S_{n-1} + C_{n-1}$$

$$C_n = H_{n-1}$$

(b) [5 pts] Let $T_n = S_n + H_n + C_n$ be the total number of individuals in June of year n. Use induction to prove that $T_n = 2^n$ for all $n \geq 0$.

5/5

Base case: for $n=0$, $T_n = 1$ ✓

Inductive step. assume $T_{n-1} = 2^{n-1}$

$$T_n = S_n + H_n + C_n$$

$$= S_{n-1} + C_{n-1} + H_{n-1} + S_{n-1} + C_{n-1} + H_{n-1}$$

$$= 2S_{n-1} + 2C_{n-1} + 2H_{n-1} = 2T_{n-1} = 2 \cdot 2^{n-1}$$

$$= 2^n \quad \forall n$$

(c) [2 pts] Show that $H_n = T_{n-1} - H_{n-1}$ for $n > 0$.

2/2

$$H_n = S_{n-1} + C_{n-1} = T_{n-1} - H_{n-1}$$

(d) [15 pts] Give a closed-form expression for H_n. You may ~~use~~, without proof, the fact stated in part (b) and the recurrence given in part (c).

15/15

n: 0 1 2 3 4 5
H_n: 0, 1, 1, 3, 5, 11, ...
H_p - 0 4 4 12

$$H_n = T_{n-1} - H_{n-1} = 2^{n-1} - H_{n-1}$$

$$H_n + H_{n-1} = 2^{n-1}$$

Boundary condition: $H_0 = 0$

~~assume~~ $\cancel{H_{n,p} = c2^{n} + b}$

Homogeneous: $H_n + H_{n-1} = 0$

assume $H_n = \alpha^n$

$$\Rightarrow \alpha + 1 = 0 \Rightarrow \alpha = -1$$

$\cancel{c2^n + c2^{n-1} = 2^{n-1} - 2b}$

$\cancel{2c + c = 1 \Rightarrow c = 1/3}$

$\cancel{H_n + H_{n-1} = 2^{n-1}}$

$\cancel{(-1)^n + \frac{2}{3}2^{n-1} + (-1)^{n-1} + \frac{1}{3}2^{n-1} = 2^{n-1}}$

$\cancel{H_n = (-1)^n + \frac{1}{3}2^n}$

Let $\cancel{H_{pn} = cn2^n \Rightarrow cn2^n + c(n-1)2^{n-1} = 2^{n-1} \Rightarrow 2cn + c(n-1) = 1 \Rightarrow c(n) = \frac{1}{3n+}}$

$\cancel{H_{pn} = cn2^n + nb \Rightarrow cn2^n + nb + c(n-1)2^{n-1} + (n-1)b = 2^{n-1}}$

$$H_{pn} = \frac{1}{3}2^n$$

$$H_n = A(-1)^n + \frac{1}{3}2^n \qquad H_0 = 0 = A(-1)^0 + \frac{1}{3}2^0 = A + \frac{1}{3} \Rightarrow A = -\frac{1}{3}$$

$\cancel{H_n = \frac{1}{3}((-1)^n + 2^n)}$ $\quad H_n = \frac{1}{3}(2^n - (-1)^n)$

22:50
22:54
0:04

Problem 8. [15 points] Asymptotic Awesomeness

For each row in the following table, determine whether there exist functions f and g that satisfy all the properties marked **Yes** and do *not* satisfy the properties marked **No**. You do not have to provide examples.

15/15

	$f = \Theta(g)$	$f = O(g)$	$f = o(g)$	$f = \Omega(g)$	$f = \omega(g)$	Do f, g exist?
(a)	Yes	Yes	Yes	No	No	No
(b)	No	No	No	Yes	Yes	Yes
(c)	No	No	Yes	No	No	No
(d)	Yes	Yes	No	Yes	No	Yes
(e)	No	Yes	No	No	No	Yes
(f)	No	No	No	No	No	Yes

22:54
22:57
0:03

Problem 9. [20 points] Yet Another Graph Proof

Prove that in a finite directed graph, if every node has at least one outgoing edge, ~~then~~ the graph has a cycle.

20/20

(*Hint: Consider the longest path.*)

Consider the longest path in such a graph that contains no cycle. The terminating node has at least one outgoing edge. If that ~~node~~ edge connects to another node without creating a cycle, then this new path is longer than the original path — a contradiction. So the outgoing edge must connect to a node that has been visited before, creating a cycle. □

22:57
23:07
0:10

20/20

Problem 10. [20 points] Revenge of the Slipped Disc Puzzle™: The Curse of 6.042

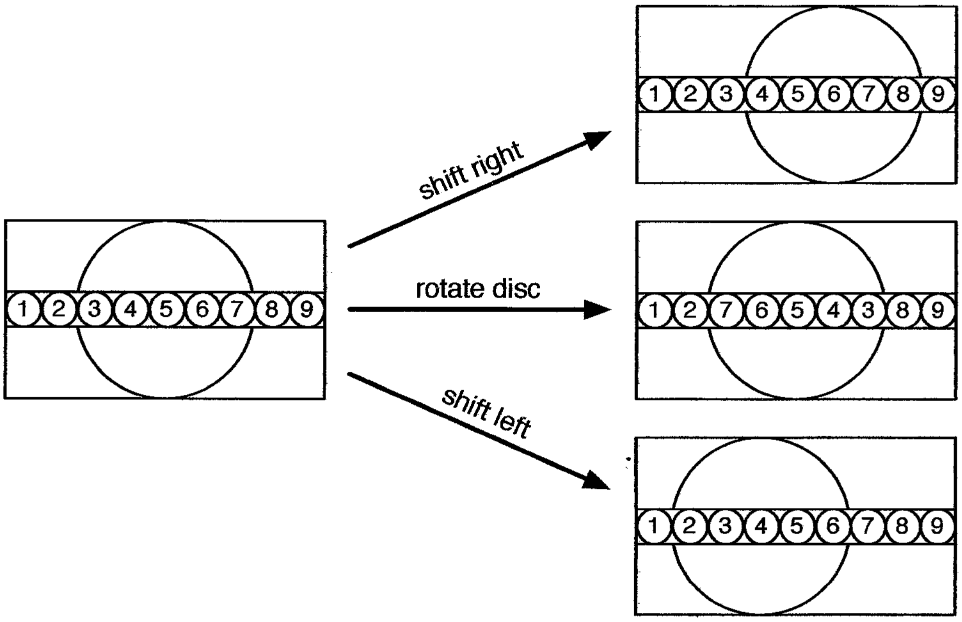
(This problem is similar to the Slipped Disc Puzzle™ of Quiz 1, but here we rotate 5 tiles instead of 4.)

The Super Awesome Extreme zomgroflolwut Spifftastic-to-the-Max Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of *exactly* **five** tiles. As shown below, there are three ways to manipulate the puzzle:

Shift Right: The center disc is moved one unit to the right (if there is space).

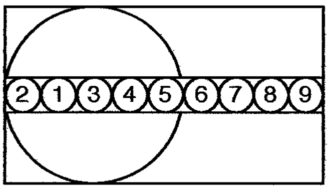
Rotate Disc: The **five** tiles in the center disc are reversed.

Shift Left: The center disc is moved one unit to the left (if there is space).

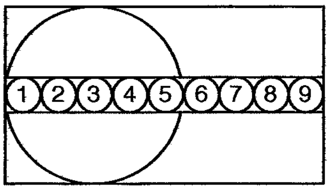


Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:

Initial State



Goal State



Write your proof on the next page...

Room for Problem ...

Consider as an invariant the number of inversions in the order of the numbered tiles.

The relation of the tiles in the disc to those on either side of the disc is not changed by a rotation.

Within the disc, the parity (odd or even) of the number of inversions is unchanged under a rotation.

Therefore, beginning w/ 1 inversion, it is impossible to reach a state w/ 0 inversions. □