

Final

Name: _____

- This quiz is **closed book**, but you may have two $8.5'' \times 11''$ sheets with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 4:30 PM.

Problem	Points	Grade	Grader	Problem	Points	Grade	Grader
1	20			8	20		
2	12			9	11		
3	10			10	10		
4	10			11	10		
5	15			12	32		
6	10			13	10		
7	10			total	180		

Time: 2:07

*There were no solutions available,
so this exam remains ungraded.
I feel very good about it, though.*

Problem 1. [20 points]

We define the following recurrence for $n \geq 0$:

$$T_{n+2} = T_{n+1} + 2T_n$$

where $T_0 = T_1 = 1$.

(a) [8 pts] Prove by induction that T_n is odd for $n \geq 0$. You do not need to solve the recurrence for this.

By induction

Base case: $T_0 = 1$ is odd, $T_1 = 1$ is odd

Inductive step: assume T_{n+1} and T_n are odd

$2T_n$ is even.

An even number plus an odd number is odd.

$T_{n+2} = T_{n+1} + 2T_n$ is odd.

□

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(b) [12 pts] Prove by induction that $\gcd(T_{n+1}, T_n) = 1$ for $n \geq 0$. You may assume that T_n is odd for all n . You do not need to solve the recurrence for this.

Base case: $\gcd(T_1, T_0) = 1$

Inductive step: assume $\gcd(T_{n+1}, T_n) = 1$

$$\gcd(T_{n+2}, T_{n+1}) = \gcd(T_{n+1} + 2T_n, T_{n+1})$$

$$= \gcd(T_{n+1}, (-T_{n+1} + (T_{n+1} + 2T_n))) \quad \leftarrow \text{as in the Pulverizer}$$

$$= \gcd(T_{n+1}, 2T_n) = \cancel{2} \quad \text{since } T_{n+1} \text{ is odd}$$

$$= \gcd(T_{n+1}, T_n) \quad \rightarrow$$

$$= 1 \quad \square$$

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0:04

Problem 2. [12 points]

Find a closed-form solution to the following recurrence:

$$x_0 = 4$$

$$x_1 = 23$$

$$x_n = 11x_{n-1} - 30x_{n-2} \text{ for } n \geq 2.$$

$$x_n + 11x_{n-1} + 30x_{n-2} = 0$$

$$\text{Let } x_n = r^n$$

$$r^2 - 11r + 30 = 0 \Rightarrow r_1 = 5, r_2 = 6$$

$$x_n = A(5)^n + B(6)^n$$

$$x_0 = 4 = A + B \Rightarrow \underline{B = 4 - A}$$

$$x_1 = A(5) + (4 - A)(6) = 23$$

$$= 5A - 6A + 24 = 23 \Rightarrow A = 1, B = 3$$

$$\Rightarrow \boxed{x_n = 5^n + 3(6)^n}$$

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0:06

Problem 3. [10 points] Note: in this question, you may use "choose" notation or factorials in your answers for both (a) and (b). In the card game of bridge, you are dealt a hand of 13 cards from the standard 52-card deck.

(a) [5 pts] A balanced hand is one in which a player has roughly the same number of cards in each suit. How many different hands are there where the player has 4 cards in one suit and 3 cards in each of the other suits?

There are $\binom{52}{13}$ possible 13-card hands. \leftarrow not relevant

There are $\binom{4}{1}\binom{13}{4}$ ways of choosing 4 cards from one suit

There are $\binom{3}{3}\binom{13}{3}^3$ ways of choosing 3 cards from remaining 3 suits

There are $\binom{4}{1}\binom{13}{4}\binom{13}{3}^3$ balanced hands.

(b) [5 pts] Not surprisingly, a non-balanced hand is one in which a player has more cards in some suits than others. Hands that are very desired are ones where over half the cards are in one suit. How many different hands are there where there are exactly 7 cards in one suit?

There are $\binom{4}{1}\binom{13}{7}$ ways of choosing 7 cards from one suit

There are $\binom{39}{6}$ ways of choosing the remaining cards.

There are $\binom{4}{1}\binom{13}{7}\binom{39}{6}$ different hands with 7 cards in one suit.

Problem 4. [10 points] Three pairs of twins are sharing a bucket of 24 pieces of chicken from Olive Oyl's. Through the magic of twinship, each person always eats exactly as many pieces of chicken as his or her twin. If each person eats at least one piece of chicken, how many ways can the chicken pieces be distributed amongst this family? For the purposes of this problem, chicken pieces are indistinguishable, and every piece is eaten.

If each person eats at least one piece, there are only $24 - 6 = 18$ pieces in play. There are only 9 "pairs" of chicken pieces in play to split among the 3 pairs.

0 0 0 | 0 0 0 | 0 0 0 \leftarrow 9 zeros, 2 "dividers" to split into 3 divisions.

of 11 elements, 2 must be dividers (or ones)

chicken can be distributed $\binom{11}{2}$ different ways.

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Problem 5. [15 points] A family of seven folk eat a meal at Kansas Flightless Chicken that comes with unlimited free biscuits (the restaurant owner has deep pockets). Being a caring family, they **only** eat biscuits by sharing them with other family members. However, the family obeys the following two rules:

- A biscuit must be shared between **exactly two** people
- If two people share a biscuit, they cannot share another biscuit with each other.

Prove that two members of this family must have shared the same number of biscuits over the course of dinner.

Represent this as a ^{simple, undirected} graph of seven nodes; an edge represents a shared biscuit.

By induction.

Base case. With one edge between any two nodes, those two nodes share the same degree.

Inductive step. Assume the graph is arranged so that two nodes have the same degree.

STOP. BETTER IDEA.

Use the Pigeonhole Principle. One node can have at most ~~six~~ degree six, and at least degree zero. There are therefore seven possible degrees for seven nodes. But, the graph cannot simultaneously have a node of degree 6 and of degree 0; so there are only six possible degrees for seven nodes. At least two must therefore have the same degree. \square

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0:03

Problem 6. [10 points] Prove that for any m, n, k such that $0 \leq k < m < n - k$, the following identity holds:

$$\binom{n}{k} = \sum_{i=0}^k \binom{m}{i} \cdot \binom{n-m}{k-i}$$

Combinatorial proof.

$\binom{n}{k}$ ways of choosing k items from a collection of n

Split the n items into a group of size m and another of size $n-m$. We can get k items by choosing i items from the first group and $k-i$ items from the second group. The total number of ways to get k items this way is

$$\sum_{i=0}^k \binom{m}{i} \binom{n-m}{k-i}$$

$$\Rightarrow \binom{n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n-m}{k-i}$$

because these are just two different ways of doing the same thing.

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Problem 7. [10 points] Let G be a bipartite graph with n nodes and k connected components. You (mutually) independently color each of the nodes of G red or black with equal probabilities. What is the probability that your coloring is a valid 2-coloring of G ? (Hint: the answer does not depend on the number of edges.)

There are 2^n ways of coloring all the nodes either red or black.

For any connected component, there ^{are} ~~is~~ only ^{two} ~~one~~ ways to make a valid 2-coloring. For k connected components, there are 2^k valid 2-colorings.

Assume that individual unconnected nodes are counted among the k connected components (otherwise they are irrelevant.)

Since all probabilities are equal, the probability of a valid 2-coloring is

$$\frac{2^k}{2^n} = 2^{k-n}$$

Problem 8. [20 points] In the far-off land of Spain, two soccer teams, Barcelona and Madrid have been battling each other for centuries to see who is more awesome. They play two games a year: one in the Spring and one in the Fall. The Madrid team has a tendency to fire their coaches very often to try to improve their results. The outcomes of the games are as follows:

1. If Madrid has not fired their coach since the previous game:

- Barcelona wins with probability $\frac{2}{5}$
- Madrid wins with probability $\frac{2}{5}$
- They tie with probability $\frac{1}{5}$

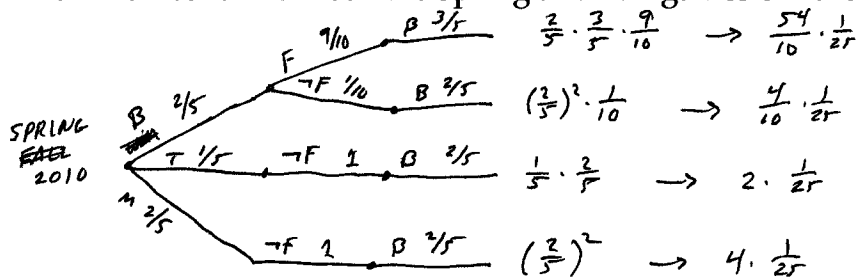
2. If Madrid has fired their coach since the previous game (they haven't yet realized that it's a bad idea):

- Barcelona wins with probability $\frac{3}{5}$
- Madrid wins with probability $\frac{1}{5}$
- They tie with probability $\frac{1}{5}$

Now, Madrid does not fire their coach if they win or tie, but if they lose, they will fire their coach with 90% probability following the loss. For the following two questions, assume that Madrid did not fire their coach before the Spring 2010 game. Here, everything is mutually independent unless otherwise specified. You can use the following timeline to help you visualize the situation.

??

(a) [10 pts] Given that Madrid lost the Fall 2010 game, what is the probability that they fired their coach between the Spring and Fall games of 2010?



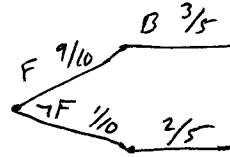
$$P(F) = \frac{54/100}{\frac{4}{100} + \frac{54}{100} + 2 + 4}$$

$$= \frac{54}{118}$$

$$P(F|B) = \frac{P(F) P(B|F)}{P(B)}$$

(b) [10 pts] What is the probability that Madrid fired their coach between the Spring and Fall games in 2010 given that they lost BOTH of the games in 2010?

$$P(B) = \frac{1}{10} \cdot \frac{2}{5} + \frac{9}{10} \cdot \frac{3}{5}$$



$$= \frac{2}{2+18} = \frac{1}{10}$$

$$= \frac{2}{2+27} = \frac{2}{29}$$

$$P(F) = \frac{\frac{9}{10} \cdot \frac{3}{5}}{\frac{9}{10} \cdot \frac{3}{5} + \frac{1}{10} \cdot \frac{2}{5}}$$

$$= \frac{27}{27+2} = \frac{27}{29}$$

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Problem 9. [11 points] Harvard's loan officer is evaluating her portfolio of student loans. There are 4 especially suspicious loans. Loan A is given to an average Harvard student who does average Harvard stuff. It has a probability of $1/10$ of being paid back. Loan B was given to Bill Gates and has a $1/5$ probability of being paid back. Loan C was given to David the evil TA and has a probability of $1/3$ of being paid back. Loan D was given to a 6.042 student and has $1/2$ probability of being paid back. In each of these parts, express any numbers in your answer as ratios of integers.

(a) [3 pts] If the probabilities of being paid back are mutually independent, what is the probability that all four loans are paid back?

$$P(\text{all 4}) = P(A)P(B)P(C)P(D)$$

$$= \frac{1}{10} \cdot \frac{1}{5} \cdot \frac{1}{3} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{300}}}$$

(b) [4 pts] If the probabilities are pairwise independent, what is the most that you can say about the probability that all four loans get paid back?

$$P(\text{all 4}) \leq \underline{\underline{\frac{1}{50}}}$$

(c) [4 pts] If you cannot assume any independence, what is the most you can say about the probability that all four loans are paid back?

$$P(\text{all 4}) \leq \frac{1}{10}$$

Problem 10. [10 points]

Consider a length n vector of integers, $x = (x_1, x_2, \dots, x_n)$ where the entries of the vector are integers in the set $\{1, 2, \dots, n\}$, and the x_i are selected uniformly at random. Note that this means numbers can repeat; x could, for example, be a vector of all 1's. Let A be the number of entries x_i in the vector for which $x_i \leq i$. Calculate $\text{Ex}[A]$ and give it in closed form.

$$P(x_i \leq 1) = \frac{1}{n}$$

$$P(x_i \leq 2) = \frac{2}{n}$$

$$P(x_i \leq i) = \frac{i}{n}$$

Let T_i be the indicator variable $T_i = \begin{cases} 0 & \text{w.p. } 1 - \frac{i}{n} \\ 1 & \text{w.p. } \frac{i}{n} \end{cases}$

$$A = \sum_{i=1}^n T_i$$

$$E[A] = \sum_{i=1}^n E[T_i] = \sum_{i=1}^n \frac{i}{n} = \frac{1}{n} \left[\frac{n(n+1)}{2} \right] = \underline{\underline{\frac{n+1}{2}}}$$

16:18

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0:02

Problem 11. [10 points] Let R be a positive random variable with $\mathbb{E}[R^3] = k$. Prove that $\Pr(R \geq x) \leq k/x^3$ for any $x > 0$.

Start w/ Markov: $P(R \geq x) \leq \frac{\mathbb{E}[R]}{x}$

$$P(R^3 \geq x^3) \leq \frac{\mathbb{E}[R^3]}{x^3} = \frac{k}{x^3}$$

$$\text{but } R^3 \geq x^3 \Rightarrow R \geq x$$

$$\Rightarrow P(R \geq x) \leq \frac{\mathbb{E}[R^3]}{x^3} = \frac{k}{x^3}$$

16:20
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Problem 12. [32 points]

In all parts of this problem, assume that we are using fair, regular dice (six-sided with values 1, 2, 3, 4, 5, 6 appearing equally likely). Furthermore, assume that all dice rolls are mutually independent events.

(a) [4 pts] You roll two dice and look at the sum of the faces that come up. What is the expected value of this sum? Express your answer as a real number.

$$X_1 = \text{die 1}, \quad X_2 = \text{die 2 result}$$

$$E[X_1 + X_2] = E[X_1] + E[X_2] = \frac{21}{6} + \frac{21}{6} = \frac{42}{6} = \frac{21}{3} = \underline{\underline{7}}$$

(b) [7 pts] Assuming that the two dice are independent, calculate the variance of their sum. Express your answer as a real number.

$$\text{Var}(X_1 + X_2) = \text{Var}(X_1) + \text{Var}(X_2) = 2 \text{Var}(X_1)$$

$$E[X_1^2] = \frac{1}{6}(36 + 25 + 16 + 9 + 4 + 1) = \frac{1}{6}(91) = \frac{91}{6}$$

$$\text{Var}(X_1) = E[X_1^2] - (E[X_1])^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{546}{36} - \frac{441}{36} = \frac{105}{36}$$

$$\Rightarrow \text{Var}(X_1 + X_2) = \underline{\underline{\frac{105}{18}}}$$

(c) [7 pts] You repeatedly roll two fair dice and look at the sum. What is the probability that you will roll a sum of 4 before you roll a sum of 7? Express your answer as a real number.

$$P(4) = \frac{3}{36} = \frac{1}{12}$$

$$P(7) = \frac{6}{36} = \frac{1}{6}$$

ignore all non-4 and non-7 rolls. then,

$$P(4 | 4 \text{ or } 7) = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{6}} = \frac{1}{1+2} = \underline{\underline{\frac{1}{3}}}$$

(d) [7 pts] What is the expected number of rolls until you get a sum of 4 or a sum of 7? (For example, if you get 7 on the first roll, the number of rolls is 1.) Express your answer as a real number.

$$P(4 \text{ or } 7) = \frac{9}{36} = \frac{1}{4} = p$$

$$P_k(k) = (1-p)^{k-1} p \quad \text{that it will take } k \text{ rolls}$$

$$E[k] = \frac{1}{p} = \underline{\underline{4}}$$

(e) [7 pts] You roll 10 dice. Using the Chernoff Bound, give an upper bound for the probability that 8 or more of them rolled a 1 or a 2? You don't need to calculate the value with a calculator (since you do not have one), but please write it in simplest terms.

$$\text{Chernoff: } P(|T - E[T]| \geq cE[T]) \leq e^{-zE[T]} \quad z = c \ln c + c + 1$$

$$T_i \text{ is indicator for } i^{\text{th}} \text{ die. } T_i = \begin{cases} 1 & \text{w.p. } 1/3 \\ 0 & \text{w.p. } 2/3 \end{cases}$$

$$T = \sum_i T_i \quad E[T] = \sum_{i=1}^{10} E[T_i] = 10 \left(\frac{1}{3} \right) = \frac{10}{3}$$

$$P(T \geq 8) = P(T \geq (c+1)E[T]) \Rightarrow c = \frac{14}{10}$$

$$= P(T - E[T] \geq cE[T]) \leq e^{-z(\frac{10}{3})} \quad \begin{aligned} z &= \left(\frac{14}{10} \right) \ln \left(\frac{14}{10} \right) - \frac{14}{10} + 1 \\ &= 1.4 \ln(1.4) - 0.4 \end{aligned}$$

$$P(T \geq 8) \leq e^{-(1.4 \ln(1.4) - 0.4)(\frac{10}{3})}$$

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17:00

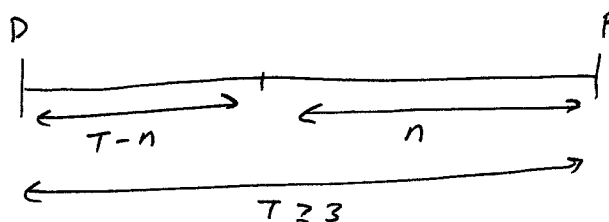
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Problem 13. [10 points]

An MIT student is walking along Mass Ave and is torn between making a trip to the pub and going to her dorm to finish her 6.042 homework. The student is n steps from the pub and $T - n$ steps from the dorm where $T \geq 3$ is the distance between the pub and the dorm, both located on Mass Ave (which we will assume to be a straight line). The student flips a coin at each step to decide which way to go. With probability p , the student takes 2 steps toward the pub and with probability $1 - p$, she takes 1 step toward her dorm. If the student is one or two steps away from the pub and takes two steps toward the pub, then she reaches the pub. If she is one step away from her dorm and takes a step toward her dorm, then she reaches the dorm. Let X_n be the probability that the student reaches the pub before she reaches the dorm. Assume that all coin tosses are mutually independent. Find a recurrence for X_n . (You do not need to solve the recurrence, but your answer should contain sufficient information such that a 6.042 could solve it given enough time.)



~~for $n=1$ and $n=2$, must have sequence $p, (1-p)p^2, (1-p)p^2$~~

~~$X_1 = p$~~
 ~~$X_2 = p$~~

~~$X_n = p X_{n-2} + (1-p) X_{n+1}$~~

~~$X_{T-1} = 0$~~

~~$X_0 = 1$~~

~~for $n=1$, she gets to the pub w.p.~~

$$p + (1-p)p + (1-p)^2 p^2 + (1-p)^3 p^3 + \dots$$

$$= \sum_{i=1}^{T-1} p^i (1-p)^{i-1} + (1-p)^i$$

$$X_n = p X_{n-2} + (1-p) X_{n+1}$$

$$X_0 = 1 \quad X_T = 0$$

written more clearly

$$X_n = p X_{n-2} + (1-p) X_{n+1}$$

$$X_0 = 1$$

$$X_T = 0$$

$$X_{-1} = 1$$

$$1 \leq n \leq T-1$$