Total time 1:33:54

6.042/18.062J Mathematics for Computer Science — October 27, 2010
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Midterm

Name: ______________________________

- This quiz is **closed book**, but you may have one 8.5 × 11″ sheet with notes in your own handwriting on both sides.
- Calculators are not allowed.
- You may assume all of the results presented in class.
- Please show your work. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- Be neat and write legibly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- If you get stuck on a problem, move on to others. The problems are not arranged in order of difficulty.
- The exam ends at 9:30 PM.

Problem	Points	Grade	Grader
1	10	10	
2	10	10	
3	20	20	
4	15	4	
5	20	20	
6	25	25	
7	10	10	
8	10	10	
Total	120	109	

$\boxed{109/120 = 90.8\%}$

10:25
+ 8:45

Problem 1. [10 points] $\boxed{10}$

Consider these two propositions:

$$P: \ (A \vee B) \Rightarrow C$$

$$Q: \ (\neg C \Rightarrow \neg A) \vee (\neg C \Rightarrow \neg B)$$

Which of the following best describes the relationship between P and Q? Please circle exactly one answer.

1. P and Q are equivalent
2. $P \Rightarrow Q$ ← THIS ONE (not enough info to say that $Q \Rightarrow P$; we know $\neg C$ but not C)
3. $Q \Rightarrow P$
4. ~~All of the above~~ ~~THIS ONE~~
5. None of the above

Draw a truth table to illustrate your reasoning. You can use as many columns as you need.

A	B	$A \vee B$	$\neg A \vee \neg B$	$\neg(A \vee B)$	$\neg A$	$\neg B$
T	T	T	F	F	F	F
T	F	T	T	F	F	T
F	T	T	T	F	T	F
F	F	F	T	T	T	T

5:00

Problem 2. [10 points] $\boxed{10}$

Let $G_0 = 1$, $G_1 = 3$, $G_2 = 9$, and define

$$G_n = G_{n-1} + 3G_{n-2} + 3G_{n-3} \tag{1}$$

for $n \geq 3$. Show by induction that $G_n \leq 3^n$ for all $n \geq 0$.

<u>By induction</u>

Base : $\left\{ \begin{array}{l} G_0 = 1 \\ G_1 = 3 \\ G_2 = 9 \end{array} \right\}$ $G_3 = 9 + 3 \cdot 3 + 3 \cdot 1 = 21 \leq 3^3$

Predicate : $G_n \leq G_{n-1} + 3G_{n-2} + 3G_{n-3}$

assume $G_n \leq 3^n$, $G_{n-1} \leq 3^{n-1}$, ...

$$\begin{aligned} G_{n+1} &= G_n + 3G_{n-1} + 3G_{n-2} \\ &\leq 3^n + 3 \cdot 3^{n-1} + 3 \cdot 3^{n-2} \\ &\leq 2 \cdot 3^n + \tfrac{1}{3} 3^n \\ &\leq (2 + \tfrac{1}{3}) 3^n \\ &\leq 3 \cdot 3^n \\ &\leq 3^{n+1} \quad \square \end{aligned}$$

9:48

Problem 3. [20 points] $\boxed{20}$

In the game of Squares and Circles, the players (you and your computer) start with a shared finite collection of shapes: some circles and some squares. Players take turns making moves. On each move, a player chooses any two shapes from the collection. These two are replaced with a single one according to the following rule:

A pair of identical shapes is replaced with a square. A pair of different shapes is replaced with a circle.

At the end of the game, when only one shape remains, you are a winner if the remaining shape is a circle. Otherwise, your computer wins.

(a) [5 pts] Prove that the game will end.

Game start : n shapes.

Move 1 : 2 removed, 1 added $\Rightarrow n-2+1 = n-1$ shapes

Move 2 : . . . . . . $\Rightarrow n-2$ shapes

Move i : $n-i$ shapes remaining

For finite n, $i=n-1$ is the ~~penulti~~ last move of the game, because # shapes remaining $= n-(n-1) = 1$.

shapes is monotonically decreasing in increments of 1 and must therefore reach 1.

(b) [15 pts] Prove that you will win if and only if the number of circles initially is odd.

Let starting # circles be C_0.

Two possibilities:

1. Two identical shapes removed.
   a. Two circles: $C_1 = C_0 - 2$
   b. Two squares: $C_1 = C_0$

2. Two different shapes removed:
   $C_1 = C_0 - 1 + 1 = C_0$ (one removed, one added)

True of any move, not just move 1.

ΔC for any move $= 0$ or 2.

Circle parity is invariant.

assume starting C_0 is even.

Base case: C_0 even

Assume C_n even. $C_{n+1} = C_n - 2$ or C_n, both even.

C_f (final # of circles) is never odd.

assume C_0 odd.

Base case: C_0 odd

Assume C_n odd. $C_{n+1} = C_n - 2$ or C_n, both odd.

C_f will always be odd. $\square$

7:13

Problem 4. [15 points] $\boxed{4}$

$\boxed{4}$ **(a)** [8 pts] Find a number $x \in \{0, 1, \ldots, 112\}$ such that $11x \equiv 1 \pmod{113}$.

Pulverizer

x	y	rem(x,y)	
113	11	3	$113 - 11 \cdot 10$
11	3	2	$11 - 3 \cdot 3 = 11 - 3 \cdot (113 - 10 \cdot 11)$
3	2	1	$3 - 1 \cdot 2 = (113 - 10 \cdot 11) - 1 \cdot (11 - 3 \cdot (113 - 10 \cdot 11))$

$= \cancel{3 \cdot 113 - 31 \cdot 11}$

$= 4 \cdot 113 - 41 \cdot 11$

$x = 41$

$\times$ $11(41) \equiv -1 \bmod 113$; $x = -41 \equiv 72$

$\cancel{113}$ $\cancel{31}$
3 11
$\cancel{339}$ $\cancel{31}$
310
341

21:00 $\boxed{0}$ **(b)** [7 pts] Find a number $y \in \{0, 1, \ldots, 112\}$ such that $11^{112111} \equiv y \pmod{113}$ (*Hint: Note that 113 is a prime.*)

x	y	gcd(x,y)
11^{112111}	113	1

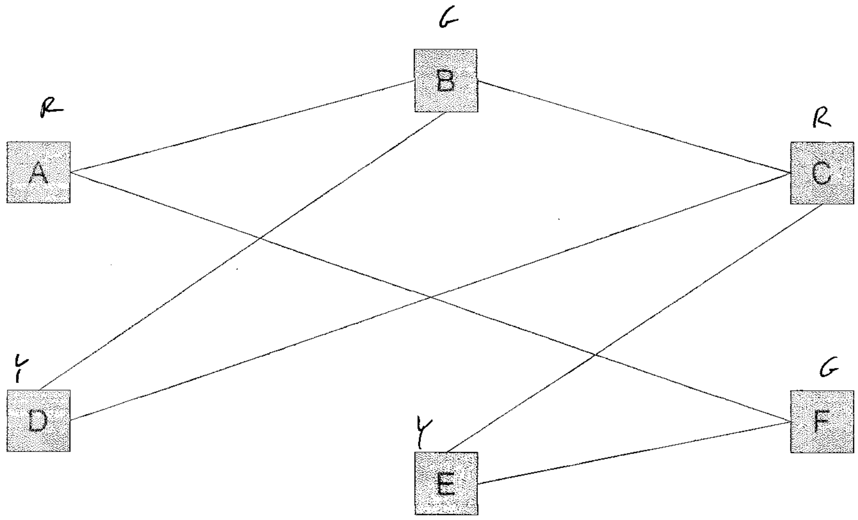
$(113 \neq 11^k$, any k; 11^{112111} has obvious prime factorization$)$

11
121 → 8
88 → 64
26
60
95
28
82

15:00 **Problem 5. [20 points]** $\boxed{20}$

Consider the simple graph G given in figure 1.

Figure 1: Simple graph G



(a) [4 pts] Give the diameter of G.

$d = 3$

(A can get anywhere in 2; so nothing can be greater than getting to A then getting to destination. $1+2=3$) [D, F, for example]

(b) [4 pts] Give a Hamiltonian Cycle on G.

$A \to B \to D \to C \to E \to F \to A$

(c) [4 pts] Give a coloring on G and show that it uses the smallest possible number of colors.

~~At most 4 = 3+1 colorable (Greatest degree 3)~~. (irrelevant)

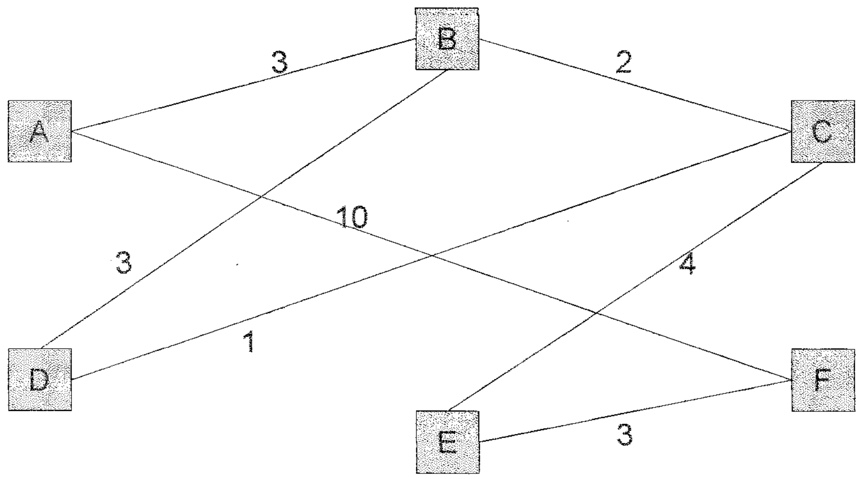
B, C, D form a complete subgraph K_3. 3 colors must be used to color those. So, coloring is either 3 or ~~4~~ greater. A 3-coloring is shown on the diagram, previous page.

(d) [4 pts] Does G have an Eulerian cycle? Justify your answer.

No. It has nodes of degree 3, which is odd. An Eulerian cycle requires all nodes to be of even degree.

Now consider graph H, which is like G but with weighted edges, in figure 2:

Figure 2: Weighted graph H



(e) [4 pts] Give a list of edges reflecting the order in which one of the greedy algorithms presented in class (i.e. in lecture, recitation, or the course text) would choose edges when finding an MST on H.

Greedy would begin with the graph as drawn, then remove the highest-weighted edge ~~that still~~ while maintaining graph connectivity.

~~Removing~~

Remove in this order:

1. (A-F) 2. (B-D)

Adding by greedy would go:

1. (D-C) 2. (B-C) 3. (A-B) 4. (C-E) 5. (E-F)

The two resulting graphs are equivalent.

25

1:21
+8:19

Problem 6. [25 points] Let G be a graph with m edges, n vertices, and k components. Prove that G contains at least $m - n + k$ cycles. (Hint: Prove this by induction on the number of edges, m)

By induction.

Base case: $m = 0$ ~~$\Rightarrow n = 0$~~ $\Rightarrow$ n vertices $\Rightarrow k = n$ components

$C = m - n + k = 0$ cycles ✓

Inductive step: $C_m = m - n + k$

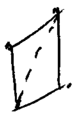
add one edge between two vertices. Two cases.

1. The two vertices were not part of the same component originally. So $m \to m+1$, $k \to k-1$, Since there was no path between vertices to begin with, no cycle has been created and $C_{m+1} = C_m$

$$C_{m+1} = (m+1) - n + (k-1) = m - n + k = C_m$$

2. The two vertices were originally part of the same component. So no ~~two~~ components are removed, $k \to k$. Being part of the same component, there was already a path between the vertices, so a cycle is created!

~~C_{m+1}~~ $C_{m+1} = C_m + 1$



$$C_{m+1} = (m+1) - n + k = (m - n + k) + 1 = C_m + 1$$

$4 - 4 + 1 = 1$

$5 - 4 + 1 = 2$

$\square$

10

2:57 **Problem 7. [10 points]** For the following sum, find an upper and a lower bound that differ by at most 1.

$$\sum_{i=1}^{\infty} \frac{1}{\sqrt{i^3}}$$

$f(1) = \frac{1}{1^{3/2}} = 1 \qquad f(\infty) = 0$

so standard bounds will differ by 1.

Monotonically decreasing function, so

$$f(\infty) + I \leq S \leq f(1) + I$$

$$I \leq S \leq 1 + I$$

$$I = \int_1^\infty \frac{1}{\sqrt{x^3}}\,dx = \int_1^\infty \frac{1}{x^{3/2}}\,dx = \left[-2\frac{1}{x^{1/2}}\right]_1^\infty = -2\left[0-1\right] = 2$$

$$\boxed{2 \leq S \leq 3}$$

10

4:00

Problem 8. [10 points] State whether each of the following claims is True or False and prove your answer.

(a) [2 pts] $x \ln x$ is $O(x)$

$f(x) = x \ln x \qquad g(x) = x$

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim \frac{x \ln(x)}{x} = \ln x \to \infty \not< \infty$$

$\boxed{\text{False}}$

(b) [2 pts] $x/100$ is $o(x)$

$$\lim_{x \to \infty} \frac{x/100}{x} = \frac{1}{100} \neq 0$$

$\boxed{\text{False}}$

(c) [2 pts] x^{n+1} is $\Omega(x^n)$

$$\lim_{x\to\infty} \frac{x^{n+1}}{x^n} = \lim_{x\to\infty} x \to \infty > 0$$

$\boxed{\text{True}}$

(d) [4 pts] $n!$ is $\Theta(n^n)$.

$$\lim_{n\to\infty} \frac{n!}{n^n} \sim \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} \to 0 \not> 0$$

$\boxed{\text{False}}$