Total time: 1:17:20

$110/125 = 88\%$

Overall, not bad for not being too careful and not keeping a crib sheet!

10

10:30 **Problem 1. [10 points]** In problem set 1 you showed that the nand operator by itself can be used to write equivalent expressions for all other Boolean logical operators. We call such an operator *universal.* Another universal operator is nor, defined such that $P \text{nor} Q \Leftrightarrow \neg(P \vee Q)$.

Show how to express $P \wedge Q$ in terms of: nor, P, Q, and grouping parentheses.

P	Q	$P \wedge Q$	P nor Q
T	T	T	F
T	F	F	F
F	T	F	F
F	F	F	T

P nor $P = \neg P$

✓

$$\boxed{P \wedge Q = (P \text{ nor } P) \text{ nor } (Q \text{ nor } Q)}$$

P	Q	$\neg P$ nor Q	P nor $\neg Q$	P nor Q	$\neg P$ nor $\neg Q$ ✓
T	T	F	F	F	T
T	F	T	F	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Problem 2. [15 points] We define the sequence of numbers

15

3:07

$$a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 3, \\ a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4} & \text{if } n \geq 4. \end{cases}$$

Prove that $a_n \equiv 1 \pmod 3$ for all $n \geq 0$.

By induction

base case:
$a_0 = 1 \equiv 1 \bmod 3$
$a_1 = 1 \equiv 1 \bmod 3$
$a_2 = 1 \equiv 1 \bmod 3$
$a_3 = 1 \equiv 1 \bmod 3$

Inductive step: $a_n = \underbrace{a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}}$

Hypothesis: all $\equiv 1 \bmod 3$

$\Rightarrow a_n \equiv (1 \bmod 3) + (1 \bmod 3) + (1 \bmod 3) + (1 \bmod 3) \equiv 4 \bmod 3 \equiv 1 \bmod 3$

□

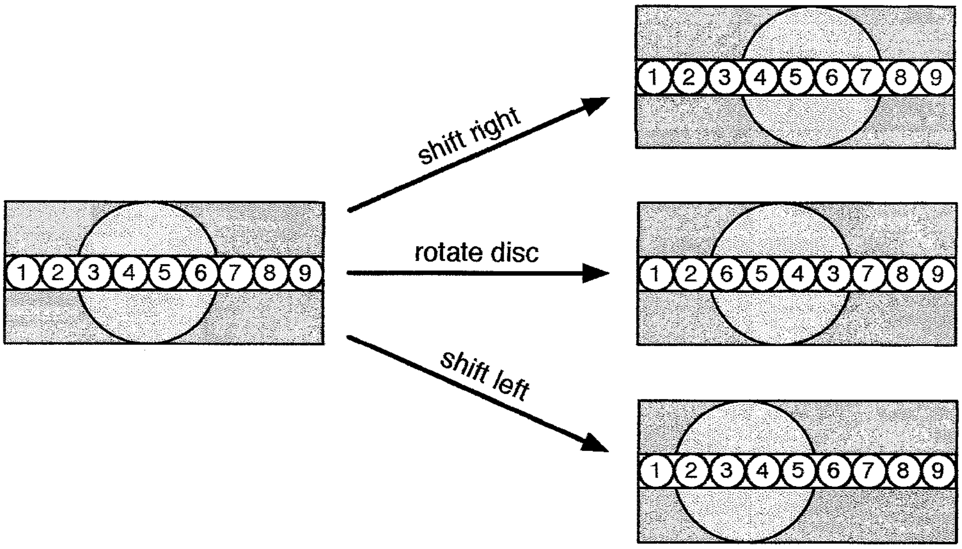
25:18 $\boxed{18}$

Problem 3. [20 points] The Slipped Disc Puzzle™ consists of a track holding 9 circular tiles. In the middle is a disc that can slide left and right and rotate 180° to change the positions of *exactly* four tiles. As shown below, there are three ways to manipulate the puzzle:

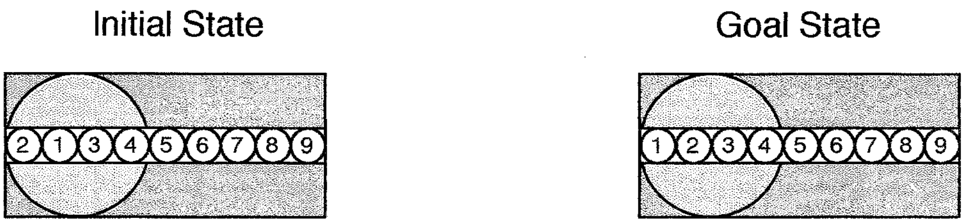
Shift Right: The center disc is moved one unit to the right (if there is space)

Rotate Disc: The four tiles in the center disc are reversed

Shift Left: The center disc is moved one unit to the left (if there is space)



Prove that if the puzzle starts in an initial state with all but tiles 1 and 2 in their natural order, then it is impossible to reach a goal state where all the tiles are in their natural order. The initial and goal states are shown below:



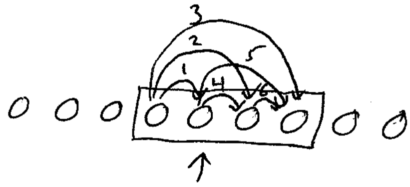
Write your proof on the next page...

Perfect problem to be solved by finding an invariant.

For any invariant, a shift left or right changes nothing.

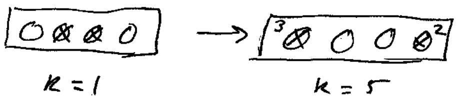
Try: # of tiles out of order (that have a following tile w/ a lower value). Better: Σ of all following tiles w/ lower value.

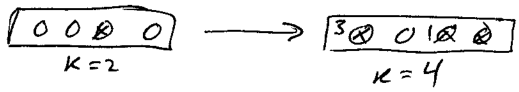
room for problem 3...



Rotation here does not change relationship w/ tiles to the left or right, regardless of position.

There are a total of 6 relationships among the tiles whose ~~odds~~ order will be flipped. Each relationship will reverse when that happens. Ex: all out of ~~order~~ order : 6, 5, 4, 3 $\Rightarrow k = 6$; after flip, 3, 4, 5, 6, $k = 0$.





If $k_i = 0$, $k_f = 6$; if $k_i = 1$, $k_f = 5$; if $k_i = 2$, $k_f = 4$
If $k_i = 3$, $k_f = 3$.

Therefore the sum of all relationships for any set of 4 tiles is switched only with respect to the above relations.

For $k_f = 0$ on a final flip, the $k_i = 6$ before that flip; but that in turn requires ~~t~~ that the tiles be in order. With $k_i = 1$ to start with, $k_f = 0$ will never be possible.

I didn't illuminate that the number of "inversions" must always be odd from an odd starting point; but it is implicit in my argument.

10

5:16 **Problem 4. [10 points]** Find the multiplicative inverse of 17 modulo 72 in the range $\{0, 1, \ldots, 71\}$.

$4 \cdot 17 = 68$

Pulverizer:

$\cancel{5 \cdot 17 = 85}$

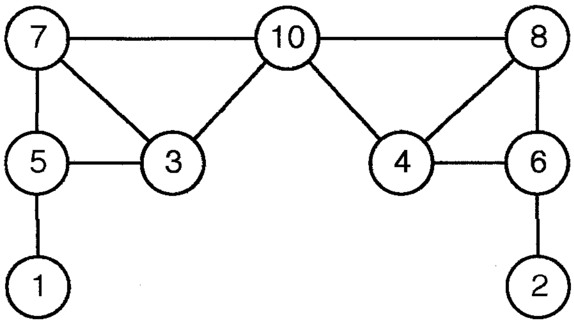
17	72
17	4
119	288
170	
289	

$\cancel{5(17)}$ $\gcd(72, 17)$

x	y	Rem(x, y)	
72	17	4	$72 - 4 \cdot 17$
17	4	1	$17 - 4 \cdot 4 = 17 - 4(72 - 4 \cdot 17)$
			$= \cancel{5} \cdot 17 - 4 \cdot 72$ (17)
4	1	0	$4 \cdot 4 \cdot 1$

$$\boxed{17 = 17^{-1} \bmod 72}$$

8:55 **Problem 5. [15 points]** Consider a graph representing the main campus buildings at MIT. $\boxed{13}$



(a) [3 pts] Is this graph bipartite? Provide a brief argument for your answer. $\boxed{2}$

No. There are cycles in this graph, and no bipartite graph can have cycles.

Cycles are ok, but odd cycles are not ok. Missed this subtlety.

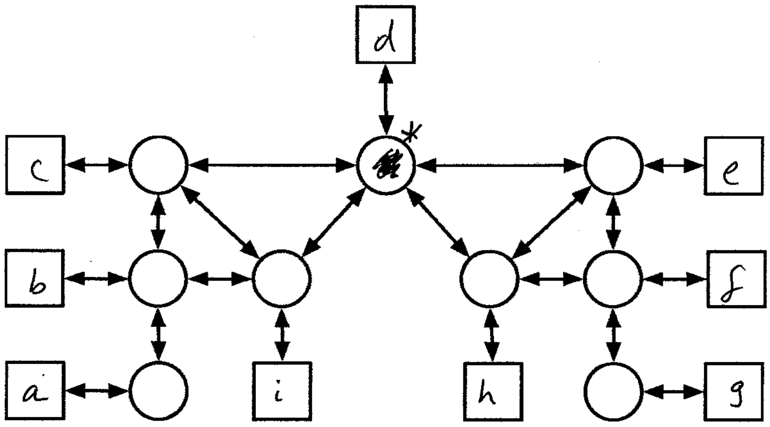
(b) [4 pts] Does this graph have an Euler circuit? Provide a brief argument for your answer. $\boxed{3}$

No. There are multiple = 8 nodes with odd degree; if there were an ~~Eulerian~~ Euler circuit, only the first and last nodes could have odd degree.

ANY nodes of odd degree disallow an Euler circuit.
I forgot that it must begin and end at the same node,
so that node must have even degree as well.

Problem 5 continued...

Now suppose each building has separate mail collection and drop-off boxes and each collection box has a single package destined for a unique drop-off box (i.e. a permutation). We can model this as a permutation routing problem by treating the buildings as switches, attaching an input and output terminal to each of the nine buildings, and treating the existing edges as bidirectional as in the graph below:



(c) [4 pts] Give the diameter of this graph: [4]

$d = 8$ (distance from a to g above)

(d) [4 pts] What is the max congestion of this graph? That is, in the worst case permutation, how many packages would need to pass through a single building? Provide a brief argument for your answer. [4]

Building (*) above is a bottleneck. If packages from (a, b, c) are going to (e, f, g) and (i → h) and (d → d), then * will see congestion 9 (one for each building).

Problem 6. [10 points] $\boxed{10}$

3:30

A tournament graph $G = (V, E)$ is a directed graph such that there is either an edge from u to v or an edge from v to u for *every* distinct pair of nodes u and v. (The nodes represent players and an edge $u \to v$ indicates that player u beats player v.)

Consider the "beats" relation implied by a tournament graph. Indicate whether or not each of the following relational properties hold *for all* tournament graphs and briefly explain your reasoning. You may assume that a player never plays herself.

1. **transitive**

No. Counter-Example:

$A \succ B \succ C$

but $A \not\succ C$

2. **symmetric**

No. Same example: $A \succ B$ but $B \not\succ A$

3. **antisymmetric**

~~No~~ Yes. If $A \succ B$ then $B \not\succ A$ always.

4. **reflexive**

No. Players don't play themselves.

12:56

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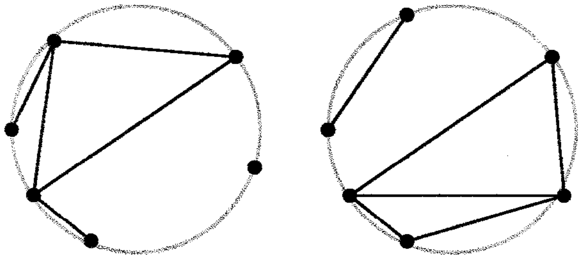
Problem 7. **[20 points]** An outerplanar graph is an undirected graph for which the vertices *can be* placed on a circle in such a way that no edges (drawn as straight lines) cross each other. For example, the complete graph on 4 vertices, K_4, is not outerplanar but any proper subgraph of K_4 with strictly fewer edges is outerplanar. Some examples are provided below:



Prove that any outerplanar graph is 3-colorable. A fact you may use without proof is that any outerplanar graph has a vertex of degree at most 2.

Any edge divides the circle into two areas, and therefore the vertices into two sets.

Any set of edges can only form triangles at densest. Therefore if C_1 is connected to C_2, then both are connected to C_3, no other vertex exists that can be connected to all 3.

If one vertex has degree at most two, then it can be colored C_1 and its neighbors C_2 and C_3. All neighbors of C_2 and C_3 can be colored C_1. The coloring will alternate until all ~~colo~~ nodes are colored.

I didn't actually do a proof here…my thinking was along the right lines but my execution was lacking. Hence the half-credit.

I gave tighter bounds than necessary here

10

4:51 **Problem 8. [10 points]** Give upper and lower bounds for the following expression which differ by at most 1.

$$\sum_{i=1}^{n} \frac{1}{i^3}$$

$I_k = \int_k^n \frac{1}{x^3}dx = \left[-\frac{1}{2}\frac{1}{x^2}\right]_k^n = +\frac{1}{2}\left(\frac{1}{k^2} - \frac{1}{n^2}\right)$

Lower : $\frac{1}{n^3} + 1 + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{n^2}\right)$

Upper : $\frac{1}{8} + 1 + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{n^2}\right)$

Diff : $\frac{1}{8} - \frac{1}{n^3} < 1$

$f(n) + I \leq S_n \leq f(1) + I$

$f(n) + (f(1) + I_2) \leq S_n \leq f(1) + (f(2) + I_2)$

2:48 **Problem 9. [15 points]** Circle every symbol on the left that could correctly appear in the box to its right. For each of the six parts you may need to circle any number of symbols.

12

(a) O Ω Θ o ω $\sim$ $\qquad 6n^2 + 7n - 10 = \boxed{}(n^2)$

(b) O Ω Θ o ω $\sim$ $\qquad 6^n = \boxed{}(n^6)$

X (c) O Ω Θ o ω $\sim$ $\qquad n! = \boxed{}(n^n)$

oops…I was thinking the right thing here but absent-mindedly circled the wrong thing.

(d) O Ω Θ o ω $\sim$ $\qquad \sum_{j=1}^{n} \frac{1}{j} = \boxed{}(\ln n)$

(e) O Ω Θ o ω $\sim$ $\qquad \ln(n^3) = \boxed{}(\ln n)$