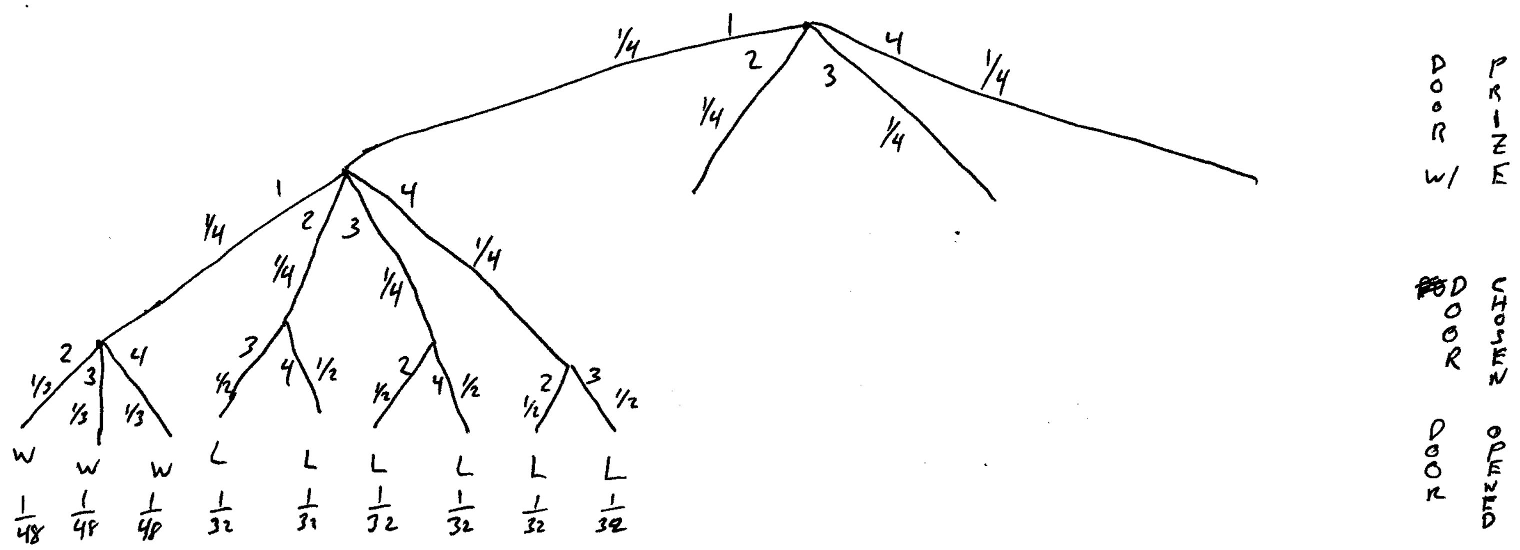
1 The Four-Door Deal

Suppose that *Let's Make a Deal* is played according to different rules. Now there are <u>four</u> doors, with a prize hidden behind one of them. The contestant is allowed to pick a door. The host must then reveal a different door that has no prize behind it. The contestant is allowed to stay with his or her original door or to pick one of the other two that are still closed. If the contestant chooses the door concealing the prize in this second stage, then he or she wins.

1. Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that he wins the prize?

   The tree diagram is awkwardly large. This often happens; in fact, sometimes you'll encounter *infinite* tree diagrams! Try to draw enough of the diagram so that you understand the structure of the remainder.



$W: \frac{3}{48} = \frac{1}{16}$ $L: \frac{6}{32} = \frac{3}{16}$

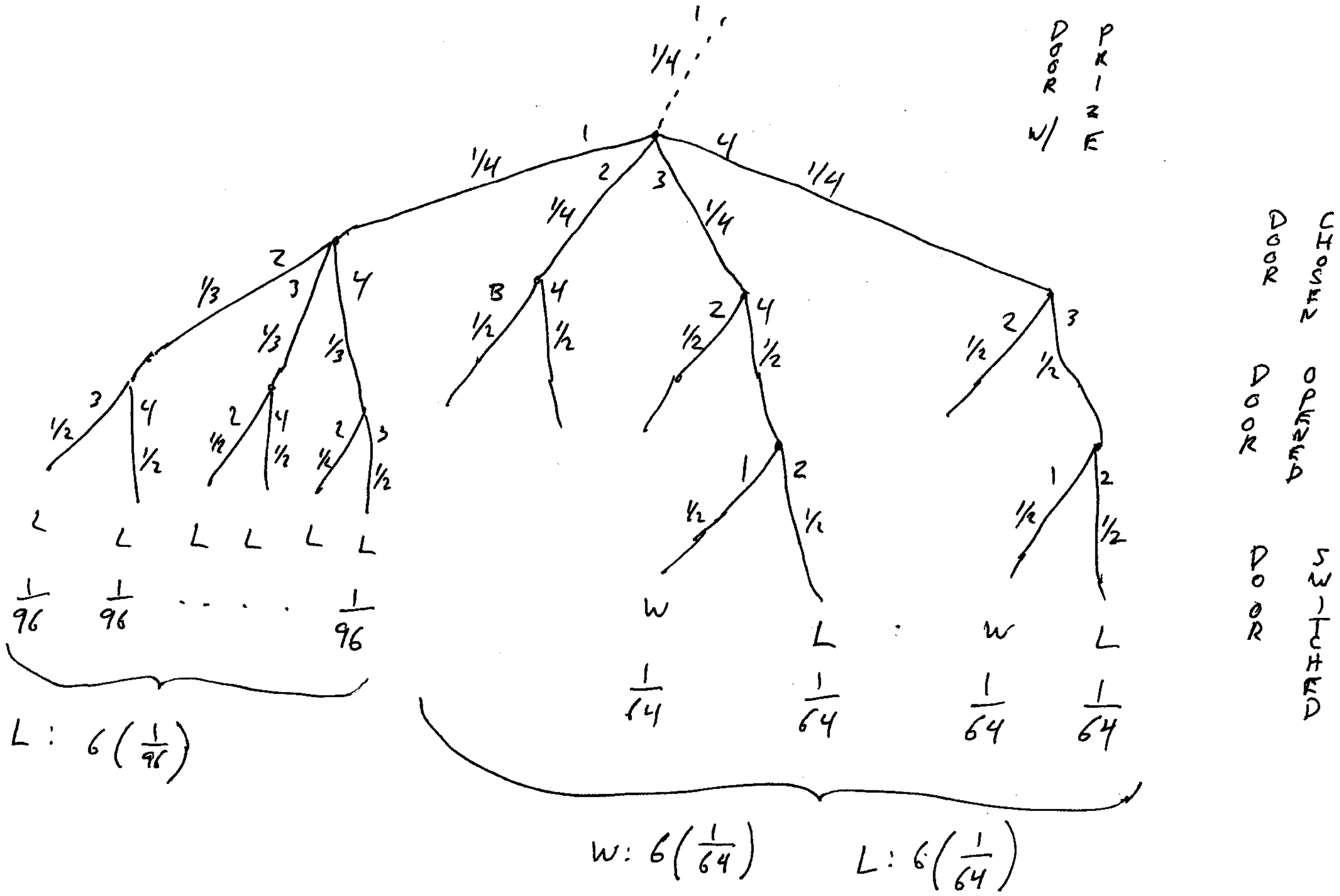
Same for other branches of the tree.

$$\boxed{\begin{aligned} P(W) &= 4\left(\tfrac{1}{16}\right) = \tfrac{1}{4} \\ P(L) &= 4\left(\tfrac{3}{16}\right) = \tfrac{3}{4} \end{aligned}}$$

In hindsight, the last branch for which door was opened was unnecessary since no choice was made based on it.

2. Contestant Zelda, an alien abduction researcher from Helena, Montana, switches to one of the remaining two doors with equal probability. What is the probability that she wins the prize?

Just draw leftmost branch:



$$P_{B1}(W) = 6\left(\tfrac{1}{64}\right) = \tfrac{3}{32}$$

$$P_{B1}(L) = 6\left(\tfrac{1}{64}\right) + 6\left(\tfrac{1}{96}\right) = \tfrac{18+12}{192} = \tfrac{30}{192}$$

There are four symmetrical branches

$$\boxed{\begin{aligned} P(W) &= 4\left(\tfrac{3}{32}\right) = \tfrac{3}{8} \\ P(L) &= 4\left(\tfrac{30}{192}\right) = \tfrac{5}{8} \end{aligned}}$$

Intuition can be wrong even after getting the answer right!

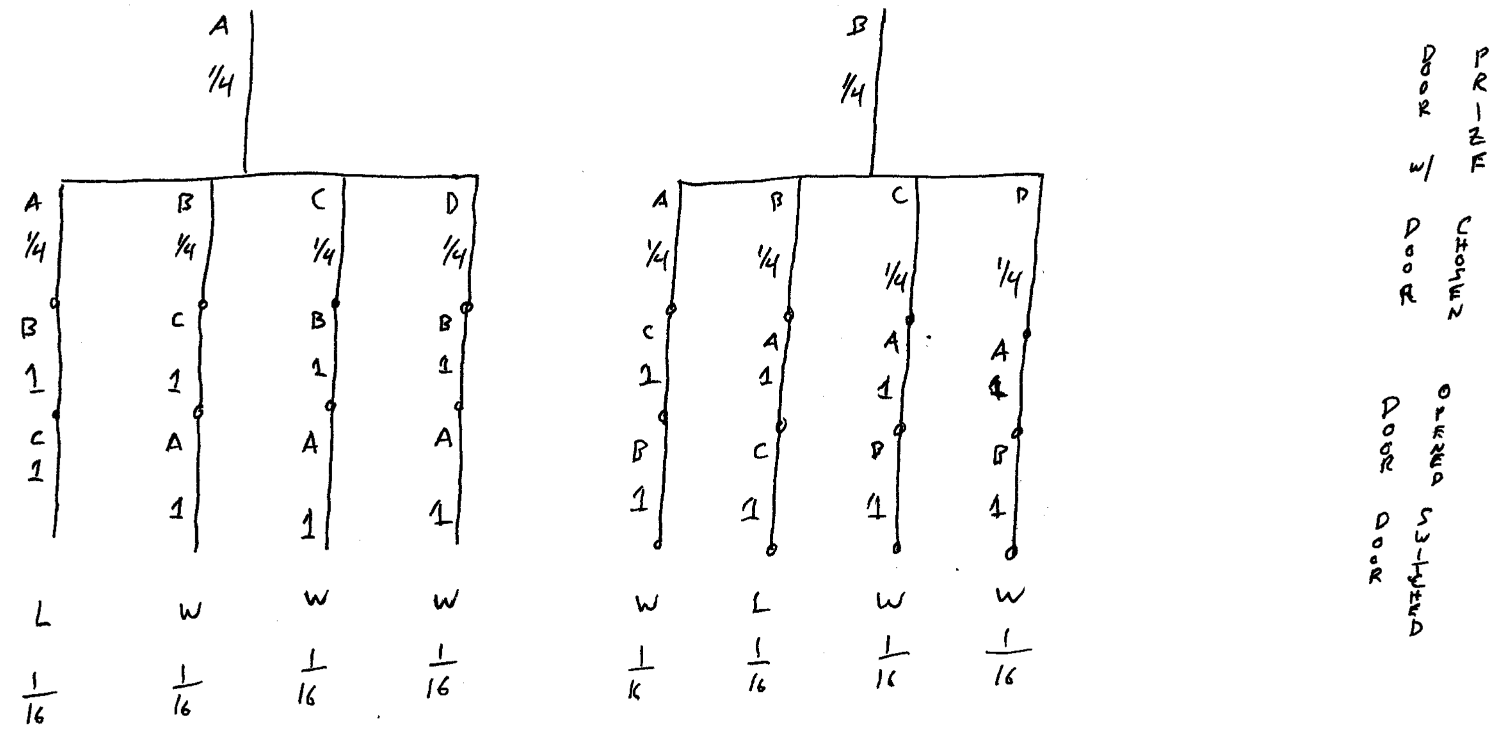
~~Switching is a bad strategy here.~~

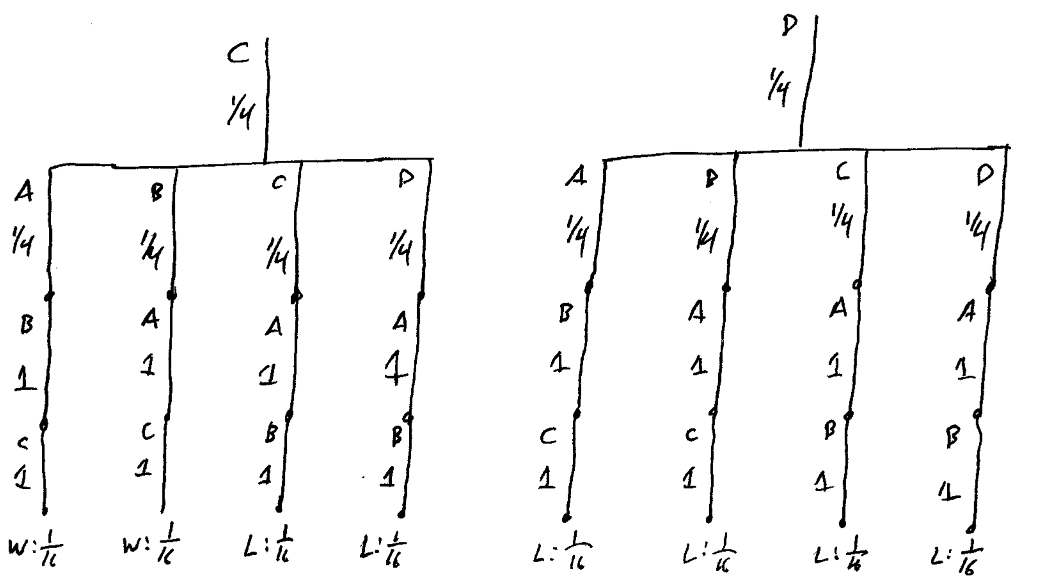
~~If you originally chose wrong, then in switching there's a 50% chance of winning. If you originally chose right, there's a 100% chance of losing.~~

2 Earliest Door

Let's consider another variation of the four-doors problem. Say the doors are labeled A, B, C, and D. Suppose that Carol always opens the *earliest* door possible (the door whose label is earliest in the alphabet) with the restriction that she can neither reveal the prize nor open the door that the player picked.

This gives contestant Mergatroid— an engineering student from Cambridge, MA— just a little more information about the location of the prize. Suppose that Mergatroid always switches to the earliest door, excluding his initial pick and the one Carol opened. What is the probability that he wins the prize?





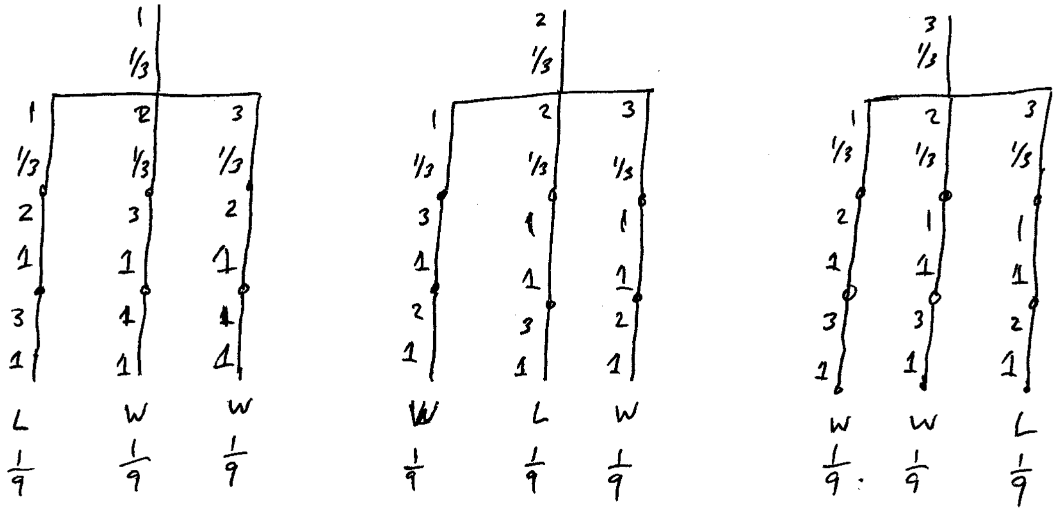
$$P(W) = 8\left(\tfrac{1}{16}\right) = \tfrac{1}{2}$$

$$P(L) = 8\left(\tfrac{1}{16}\right) = \tfrac{1}{2}$$

3 The 3 doors version revisited

3.1 Carol picks the smallest door

Suppose we are in the original game show with 3 doors. In our original analysis we assumed Carol picked the door randomly. In this case suppose Carol picks the smallest door, while still making sure of both i) it contains a goat and ii) it is not the contestant's first choice. The contestant follows the switching strategy. What is the probability the contestant wins?

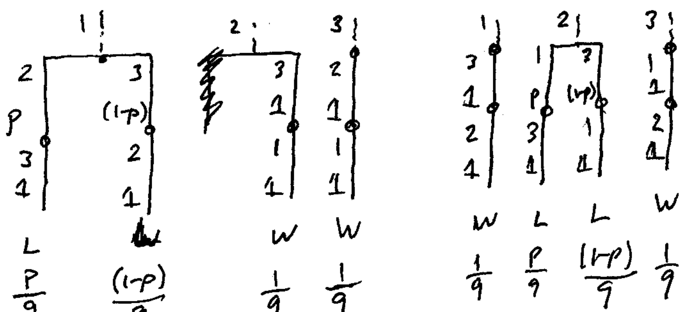


$$\boxed{P(W) = 6\left(\tfrac{1}{9}\right) = \tfrac{2}{3}}$$

3.2 Carol picks the smallest door with probability p

This time, when Carol has a choice she chooses the smallest possible door with probability p and the other remaining door with probability $1 - p$. The contestant still follows the switching strategy. What is the probability the contestant wins, in terms of p?

Just show the bottommost portions of the tree.



and so on. The only time there is a choice of two doors for Carol to open, both choices lead to L.

$$\boxed{\begin{aligned} P(L) &= 3\left(\tfrac{p}{9}\right) + 3\left(\tfrac{1-p}{9}\right) = \tfrac{3}{9} = \tfrac{1}{3} \\ P(W) &= \tfrac{2}{3} \end{aligned}}$$

There is no change

This problem actually referred to Carol's strategy in part 3.2, where I took it to be referring to 3.1. The solutions were more precise than mine, assigning a probability to my decision whether to switch or stay; I instead worked out the probability of winning directly. Hence I only considered the corner cases q=0 and q=1 of the solutions.

3.3 Optimal strategery

So far we assumed the contestant always switches. We also know from lecture another strategy: the contestant always sticks to her original choice. We determined that the probability of winning with the "always stay" strategy is simple to calculate from the probability of winning with the "always switch" strategy, and that switching was better.

What if the contestant decides whether to switch or not on a case by case basis? That is, suppose the contestant makes a decision of whether to switch or to stay based on 1) Her original choice, and 2) Carol's choice of door. Suppose the doors are labelled A, B and C. Show "always switching" is optimal. (Hint: a strategy can be seen as a mapping that assigns a pair (D_1, D_2) of observations to a decision: switch to D_3 or stay in D_1. The strategy needs to be defined for all pairs (A, B), $(A, C)\cdots$. You can optimize the reaction for each observation individually.)

assume that we are back to the case of Carol choosing the smallest door. If Carol chooses randomly, we have already seen that switching is optimal. Use the tree from 3.1.

Branch	D_1	D_2	Winning Strategy
1	A	B	Stay
2	A	B	—
3	A	B	Switch
1	A	C	~~Stay~~
2	A	C	Switch
3	A	C	—
1	B	A	—
2	B	A	Stay
3	B	A	Switch

Branch	D_1	D_2	Winning Strategy
1	B	C	Switch
2	B	C	Stay
3	B	C	—
1	C	A	—
2	C	A	Switch
3	C	A	Stay
1	C	B	Switch
2	C	B	—
3	C	B	Stay

Each pair (D_1, D_2) maps to both decisions depending on the prize location. With 2/3 chance, you chose the wrong door... so switch!