Total Time: 1:59

$\frac{85}{100} = 85\%$

18:49

0:12

Problem 1. [13 points] Give an inductive proof that the Fibonacci numbers F_n and F_{n+1} are relatively prime for all $n \geq 0$. The Fibonacci numbers are defined as follows: 13/13

$$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2} \quad (\text{for } n \geq 2)$$

By induction

$F_n = F_{n-1} + F_{n-2}$ $\qquad$ $F_2 = 1 + 0 = 1$

$F_{n+1} = F_n + F_{n-1}$ $\qquad$ $F_3 = 1 + 1 = 2$

Base case: F_2 and F_3 are relatively prime

Inductive step: ~~Suppose F_n and F_{n+1} have a common factor α (that they are not relatively prime).~~

~~Then $\left(\frac{F_{n+1}}{\alpha}\right)$ is a whole number, and~~

~~$\frac{F_n + F_{n-1}}{\alpha} = \frac{F_n}{\alpha} + \frac{F_{n-1}}{\alpha}$ is also an integer.~~

assume F_n and F_{n-1} are relatively prime, so they have no common factor α.

If F_{n+1} and F_n have a common factor α, then $\frac{F_{n+1}}{\alpha}$ is an integer, $\frac{F_n + F_{n-1}}{\alpha}$ is an integer, and $\frac{F_n}{\alpha}$ is an integer. This implies that $\frac{F_{n-1}}{\alpha}$ is an integer. But, by the inductive hypothesis, F_n and F_{n-1} have no common factor α.

=> Contradiction

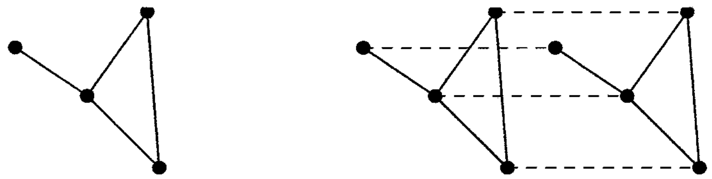
=> F_n and F_{n+1} are relatively prime for all $n \geq 0$

18:58

$\boxed{0:08}$

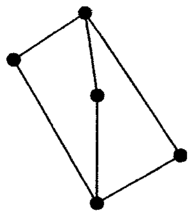
Problem 2. [15 points] The *double* of a graph G consists of two copies of G with edges joining corresponding vertices. For example, a graph appears below on the left and its double appears on the right.

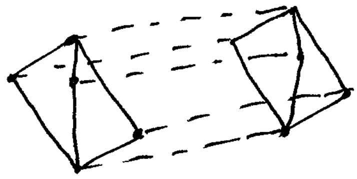
$\frac{15}{15}$



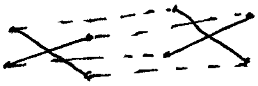
Some edges in the graph on the right are dashed to clarify its structure.

(a) Draw the double of the graph shown below.



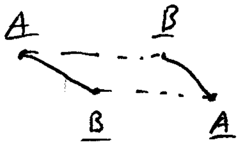


(b) Suppose that G_1 is a bipartite graph, G_2 is the double of G_1, G_3 is the double of G_2, and so forth. Use induction on n to prove that G_n is bipartite for all $n \geq 1$.



By induction:

Base case:



The double of a bipartite graph with sides labeled $\underline{A}$ and $\underline{B}$ has corresponding edges that belong to opposite sides.

Inductive step: G_n is bipartite, with sides labeled $\underline{A}$ and $\underline{B}$. G_{n+1} contains ~~the~~ G_n and its mirror image. All $\underline{A}$ from G_n are connected only to $\underline{B}$ in the mirror, and vice versa. All $\underline{A}$ in the mirror are connected only to $\underline{B}$ in G_n.

$\Rightarrow$ G_{n+1} is bipartite. $\square$

for all $n \geq 1$

18:58
19:03
$\boxed{0:05}$

Problem 3. [12 points] *Finalphobia* is a rare disease in which the victim has the delusion that he or she is being subjected to an intense mathematical examination.

$\frac{12}{12}$

- A person selected uniformly at random has finalphobia with probability 1/100.
- A person with finalphobia has shaky hands with probability 9/10.
- A person without finalphobia has shaky hands with probability 1/20.

Note that the solutions used a tree diagram, which is equivalent.

What is the probablility that a person selected uniformly at random has finalphobia, given that he or she has shaky hands?

Using Bayes' rule:

has finalphobia : F
has shaky hands : S

want $P(F \mid S)$

$$P(F \mid S) = \frac{P(F)\,P(S \mid F)}{P(S)}$$

$$P(F) = \frac{1}{100}$$

$$P(S \mid F) = \frac{9}{10}$$

$$P(S) = P(F)\,P(S \mid F) + P(\neg F)\,P(S \mid \neg F)$$

$$= \left(\frac{1}{100}\right)\left(\frac{9}{10}\right) + \left(\frac{99}{100}\right)\left(\frac{1}{20}\right)$$

$$= \frac{9}{1000} + \frac{99}{2000} = \frac{117}{2000}$$

$$= \frac{\left(\frac{9}{10}\right)\left(\frac{1}{100}\right)}{\frac{117}{2000}}$$

$$= \frac{9/1000}{117/2000} = \frac{18}{117}$$

19:03

19:08

$\boxed{0:05}$

Problem 4. [12 points] Suppose that you roll five 6-sided dice that are fair and mutually independent. For the problems below, answers alone are sufficient, but we can award partial credit only if you show your work. Also, you do not need to simplify your answers; you may leave factorials, binomial coefficients, and arithmetic expressions unevaluated. $\frac{10}{12}$

✓**(a)** What is the probability that all five dice show different values?
Example: $(1,2,3,4,5)$ is a roll of this type, but $(1,1,2,3,4)$ is not.

Total options: 6^5

$$P(\text{all different}) = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{6^5}$$

✓**(b)** What is the probability that two dice show the same value and the remaining three dice all show different values?

Example: $(6,1,6,2,3)$ is a roll of this type, but $(1,1,2,2,3)$ and $(4,4,4,5,6)$ are not.

$\binom{5}{2}$ ways of choosing the dice, and 6 values they could take

3 remaining dice choose from 5 values

$$P = \frac{\binom{5}{2}\cdot 6\cdot 5\cdot 4\cdot 3}{6^5}$$

X **(c)** What is the probability that two dice show one value, two different dice show a second value, and the remaining die shows a third value?

Example: $(6,1,2,1,2)$ is a roll of this type, but $(4,4,4,4,5)$ and $(5,5,5,6,6)$ are not.

didn't account for the double-counting of the two shared values, off by factor of 2

as before: $\binom{5}{2}$ for first share, 6 options

$\binom{3}{2}$ for next share, 5 options

4 options for last die

$$P = \frac{\binom{5}{2}\cdot 6\cdot\binom{3}{2}\cdot 5\cdot 4}{6^5}$$

19:08
19:17
$\boxed{0:09}$

Problem 5. [12 points] An electronic toy displays a 4×4 grid of colored squares. At all times, four are red, four are green, four are blue, and four are yellow. For example, here is one possible configuration:

$\frac{12}{12}$

R	B	Y	R
Y	B	G	G
B	R	R	G
B	G	Y	Y

① ② ③ ④ ⑤

For parts (a) and (b) below, you need not simplify your answers.

✓ **(a)** How many such configurations are possible?

$$\binom{16}{4,4,4,4}$$

✓ **(b)** Below the display, there are five buttons numbered 1, 2, 3, 4, and 5. The player may press a sequence of buttons; however, the same button can not be pressed twice in a row. How many different sequences of n button-presses are possible?

First press, 5 options; every other press, 4 options

$$5 \cdot 4^{n-1}$$

✓(c) Each button press scrambles the colored squares in a complicated, but nonrandom way. Prove that there exist two *different* sequences of 32 button presses that both produce the *same* configuration, if the puzzle is initially in the state shown above. (Hint: $4^{32} = 16^{16} > 16!$)

Need only show that there are more possible sequences than possible configurations.

Show :

$$\binom{16}{4,4,4,4} < 5 \cdot 4^{31}$$

$$\frac{16!}{4!4!4!4!} < 5 \cdot 4^{31}$$

$$\frac{16!}{4!} < 5 \cdot 4^{32}$$

math mistake ! doesn't change answer.

$$\underbrace{\underbrace{\left(\frac{1}{5 \cdot 4!}\right)}_{<1} 16!}_{<16!} < 16^{16}$$

Since $16! < 16^{16}$

$$\frac{1}{5 \cdot 4!} 16! < 16^{16}$$

and # of configurations is less than the number of possible sequences.

19:17
19:28
0:11

Problem 6. [12 points] MIT students sometimes delay laundry for a few days. Assume all random values described below are mutually independent. 11/12

✓**(a)** A *busy* student must complete 3 problem sets before doing laundry. Each problem set requires 1 day with probability 2/3 and 2 days with probability 1/3. Let B be the number of days a busy student delays laundry. What is $\text{Ex}(B)$?

Example: If the first problem set requires 1 day and the second and third problem sets each require 2 days, then the student delays for $B = 5$ days.

$$B = T_1 + T_2 + T_3$$

$$E[T_1] = 1 \cdot \tfrac{2}{3} + 2 \cdot \tfrac{1}{3} = 4/3$$

$$E[B] = E[T_1] + E[T_2] + E[T_3] = 3 \cdot \tfrac{4}{3} = \underline{4 \text{ days}}$$

bad formulas, right method.

✗ **(b)** A *relaxed* student rolls a fair, 6-sided die in the morning. If he rolls a 1, then he does his laundry immediately (with zero days of delay). Otherwise, he delays for one day and repeats the experiment the following morning. Let R be the number of days a relaxed student delays laundry. What is $\text{Ex}(R)$?

Example: If the student rolls a 2 the first morning, a 5 the second morning, and a 1 the third morning, then he delays for $R = 2$ days.

$$E[R] = 0 \cdot \tfrac{1}{6} + 1 \cdot (\tfrac{5}{6})(\tfrac{1}{6}) + 2 \cdot (\tfrac{5}{6})^2(\tfrac{1}{6}) + \cdots + n(\tfrac{5}{6})^n(\tfrac{1}{6})$$

$$= \sum_{i=0}^{\infty} i(\tfrac{5}{6})^i(\tfrac{1}{6}) = \tfrac{1}{6}\left(\sum_0^{\infty} i(\tfrac{5}{6})^i\right) = \tfrac{1}{6} \cdot \frac{1}{(1-5/6)^2}$$

$$= \frac{1/6}{(1/6)^2} = \underline{6 \text{ days}}$$

The correct answer here is 5 days

Misremembered the formula here!

✓**(c)** Before doing laundry, an ***unlucky*** student must recover from illness for a number of days equal to the product of the numbers rolled on two fair, 6-sided dice. Let U be the expected number of days an unlucky student delays laundry. What is $\text{Ex}(U)$?

Example: If the rolls are 5 and 3, then the student delays for $U = 15$ days.

$$E[U] = E[D_1 D_2] = E[D_1]\,E[D_2] \qquad (D_1,\ D_2 \text{ independent})$$

$$E[D_1] = \frac{1}{6}\cdot(1+2+3+4+5+6) = \frac{21}{6}$$

$$E[U] = \left(\frac{21}{6}\right)^2 = \left(\frac{7}{2}\right)^2 = \underline{\frac{49}{4}\text{ days}}$$

✓**(d)** A student is ***busy*** with probability $1/2$, ***relaxed*** with probability $1/3$, and ***unlucky*** with probability $1/6$. Let D be the number of days the student delays laundry. What is $\text{Ex}(D)$? *Leave your answer in terms of* $\text{Ex}(B)$, $\text{Ex}(R)$, *and* $\text{Ex}(U)$.

$$\underline{E[D] = \frac{1}{2}E[B] + \frac{1}{3}E[R] + \frac{1}{6}E[U]}$$

19:28
19:32
19:35
19:39
21:54
22:20
9:05
9:30
$\boxed{0:49}$

$\frac{2}{12}$

Problem 7. [12 points] I have twelve cards:

× Did it wrong, got wrong answer. Should have used indicator variables and linearity of expectation

1	1	2	2	3	3	4	4	5	5	6	6

I shuffle them and deal them in a row. For example, I might get:

1	2	3	3	4	6	1	4	5	5	2	6

What is the expected number of adjacent pairs with the same value? In the example, there are two adjacent pairs with the same value, the 3's and the 5's.

We can award partial credit only if you show your work.

# of ways to deal the cards: 12!

# of ways to have 6 pairs: $6! \quad = A_6$

~~5 pairs: 0 (having 5, you ma~~

5 pairs: $6 \cdot \binom{7}{2} \cdot 5! - \underbrace{6!}_{A_6} \quad = A_5$

all equally probable

4 pairs: $\binom{6}{2} \cdot \binom{8}{2,2} \cdot 4! - A_5 - A_6 \quad = A_4$

3 pairs: $\binom{6}{3} \cdot \binom{9}{2,2,2} \cdot 3! - A_4 - A_5 - A_6 \quad = A_3$

2 pair: $\binom{6}{4} \cdot \binom{10}{2,2,2,2} \cdot 2! - A_3 - A_4 - A_5 - A_6 = A_2$

$E[A] = \sum i \frac{A_i}{12!}$

1 pair: $\binom{6}{5} \cdot \binom{11}{2,2,2,2,2} \cdot 1! - A_2 - A_3 - A_4 - A_5 - A_6 = A_1$

0 pair: $\binom{12}{2,2,2,2,2,2} - \sum_{1}^{6} A_i \quad = A_0$

See separate sheet for work, $\underline{E[\text{pairs}] = 0.63}$

The correct answer here is 1

$0 \cdot (A_0 - A_1)$

$1 \cdot (A_1 - A_2)$

$2 \cdot (A_2 - A_3)$

$3 \cdot (A_3 - A_4)$

$4 \cdot (A_4 - A_5)$

$5 \cdot (A_5 - A_6)$

$6 \cdot A_6$

$1 \cdot A_1$

$(2-1)A_2$

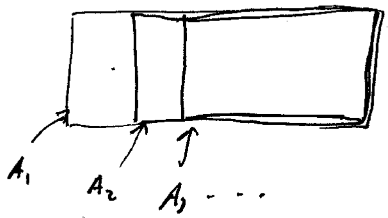
$(3-2)A_3$

$(4-3)A_4$

$(5-4)A_5$

$(6-5)A_6$

$\Rightarrow \sum A_i$



$A_6 = 6!\binom{6}{0}$

$A_5 = 7!\binom{6}{1}$

$A_4 = 8!\binom{6}{2}$

$A_3 = 9!\binom{6}{3}$

$A_2 = 10!\binom{6}{4}$

$A_1 = 11!\binom{6}{5}$

$A_0 = 12!\binom{6}{6}$

$$\cancel{\frac{11! + 10! + 9! + 8! + 7! + 6!}{12!}}$$

$$\frac{6!}{12!} + 6\frac{7!}{12!} + \frac{6!}{2!4!}\frac{8!}{12!} + \frac{6!}{3!3!}\frac{9!}{12!} + \frac{6!}{2!4!}\frac{10!}{12!} + \frac{6!}{1!5!}\frac{11!}{12!}$$

$$\frac{1}{12!}(301826160) = 0.63$$

21:34
21:54
$\boxed{0:20}$

Problem 8. [12 points] Each time a baseball player bats, he hits the ball with some probability. The table below gives the hit probability and number of chances to bat next season for five players. $\frac{10}{12}$

player	prob. of hit	# chances to bat
Player A	1/3	300
Player B	1/4	200
Player C	1/4	400
Player D	1/5	250
Player E	2/5	500

✓**(a)** Let X be the total number times these five players hit the ball next season. What is $\text{Ex}(X)$?

$$E[X] = E[A+B+C+D+E] = E[A] + E[B] + \cdots + E[E]$$

$$= \frac{1}{3} \cdot 300 + \frac{1}{4} \cdot 200 + \frac{1}{4} \cdot 400 + \frac{1}{5} \cdot 250 + \frac{2}{5} \cdot 500$$

$$= 100 + 50 + 100 + 50 + 200$$

$$\underline{= 500}$$

✓**(b)** Give a nontrivial upper bound on $\Pr(X \geq 1500)$ and justify your answer. *Do not* assume that hits happen mutually independently.

Markov: $\underline{P(X \geq 1500) \leq \frac{E[X]}{1500} = \frac{500}{1500} = \frac{1}{3}}$

I don't recognize the form of Chernoff used in the solutions. My derived form is not the same and is off by a factor of $\sim e^{-6}$.

× **(c)** Using a Chernoff inequality, give a nontrivial upper bound on $\Pr(X \leq 400)$. For this part, you *may* assume that all hits happen mutually independently.

Chernoff: $P(X \geq cE[X]) \leq e^{-zE[X]}$ $\qquad z = c\ln c - c + 1$

~~$P(X < 400) = 1 - P(X \geq 400)$~~

~~$1 - P(X \geq cE[X]) \geq 1 - e^{-zE[X]}$~~

~~$P(X \leq cE[X]) \geq 1 - e^{-zE[X]}$~~

$X \leq 400 \Rightarrow |u - X| \geq u - 400 \quad ; \quad E[u - X] = |E[u] - E[X]| = |u - E[X]|$

Let $u = 1650$ (max. possible # of hits)

$P(u - X \geq cE[u - X]) \leq e^{-zE[u-X]}$

$P(u - X \geq cu - cE[X]) \leq e^{-z(u - E[X])}$

$P(-X \geq (c-1)u - cE[X]) \leq e^{-zu} \cdot e^{zE[X]}$

$P(X \leq cE[X] - (c-1)u) \leq e^{-zu} e^{zE[X]}$

$cE[X] - (c-1)u = 400$

$500c - (c-1) \cdot 1650 = 400 \Rightarrow -1150c + 1650 = 400 \Rightarrow c = \frac{1250}{1150} = \frac{25}{23}$

$z = c\ln c - c + 1 = 0.003676$

$zu - zE[X] = 4.227$

$e^{-4.227} = 0.0146$ The given answer here is e^(-10)

$P(X \leq 400) \leq 0.0146$