J. P. Morgan Quant Mentorship Program 2022

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Case Study -A: Derivatives

Q1: Continuous Compounding

a.) What's the amount you get if you invest $10,000, on 15th Jan 2022, at an interest rate of 5% per annum, compounded semi-annually after 10 years?

Solution:

Compound interest (also known as compounding interest) is the interest on a loan or deposit that is computed using both the initial principal and the interest accumulated over time. It's also known as "interest on interest," because it accelerates the growth of a sum compared to simple interest, which is calculated just on the principal amount.

Compound interest accrues at a rate determined by the frequency of compounding, with the higher the number of compounding periods, the higher the compound interest rate.

Compound Interest can be calculated by the formula :-

$$\mathbf{A = P(1 + \frac{r}{n})^{nt}}$$

- A = Accrued amount.
- P = Principal amount
- r = Annual nominal interest rate as a decimal
- R = Annual nominal interest rate as a percent
- r = R/100
- n = number of compounding periods per unit of time
- t = time in decimal years

Now if we invest $10,000, on 15th Jan 2022, at an interest rate of 5% per annum, compounded semi-annually after 10 years then Amount will be,

Given,

P=$10,000

R=5%

r=5/100 = 0.05

t=10 years

n=2(compounded semi-annually)

$$A = P(1 + \frac{r}{n})^{nt}$$

$$A = 10{,}000(1 + \frac{.05}{2})^{20}$$

$$A = 10{,}000(1 + 0.025)^{20}$$

$$\boxed{A = \$16{,}386.16}$$

b.) What would be the final amount if the invested amount was compounded weekly?

Solution:

Given,

P=$10,000

R=5%

r=5/100 = 0.05

t=10 years

n=52(compounded weekly)

$$A = P(1 + \frac{r}{n})^{nt}$$

$$A = 10{,}000(1 + \frac{.05}{52})^{520}$$

$$A = 10{,}000(1 + \ 0.000961538)^{520}$$

$$\boxed{A = \$16{,}483.25}$$

c.) If compounded daily?

Solution:

Given,

P=$10,000

R=5%

r=5/100 = 0.05

t=10 years

n=365(compounded daily)

$$A = P(1 + \frac{r}{n})^{nt}$$

$$A = 10{,}000(1 + \frac{.05}{365})^{3650}$$

$$A = 10{,}000(1 + 0.000136986)^{3650}$$

$$\boxed{A = \$16{,}486.63}$$

d.) As you might have noticed, the amount is increasing with each reduction in the compounding time period. Can you calculate the formulation if the compounding is done continuously instead of discrete time intervals like "daily" or "weekly"? First try and calculate in terms of the following parameters, and then plug in the values from part (a) to compare with discrete intervals! Parameters - \$(P) principal, (r) annual interest rate and (T) years.

Solution: Now if the compounding is done continuously then

n=1,2,52,365……∞

$$A = P(1 + \frac{r}{n})^{nt}$$

$$b = (1 + \frac{r}{n})^{n}$$

$$A = Pb^{t}$$

For n→∞

$$A = \lim_{n\to\infty} Pb^{t}$$

$$\text{Using } \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^{n} = e^{x}$$

$$\lim_{n\to\infty} b = \lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^{n} = e^{r}$$

$$\boxed{\mathbf{A = Pe^{rt}}}$$

Now Plugging in the values from part (a)

Given,

P=\$10,000

R=5%

r=5/100 = 0.05

t=10 years

$$A = Pe^{rt}$$

$$A = 10{,}000e^{0.5}$$

$$\boxed{\mathbf{A = \$16,487.212}}$$

the amount is increasing with each reduction in the compounding time period because a lender may use more aggressive compounding on a monthly or quarterly basis, increasing the amount owed by the borrower.

Q2: Call/Put Option Pricing

a.) Money today is more valuable than money tomorrow. Why? Simply because you can invest the money today and earn a risk-free (i.e., FD-like) interest 'r', continuously compounded. So, if you were in the market looking for a call option at strike $K, expiry in T years from now, for a stock which is currently trading at $S, how much would you be willing to pay today? Remember, the strike $K is in the future, i.e. at time T, so it is less valuable right now than it will be at time T. It's helpful to think of part (d) in the first question and use that result here (If your final amount is $K and time to maturity/expiry is T years, what's the principal amount…). Assume r is the annual "risk-free" interest rate?

Solution:

Given:

Strike Amount = $K

Expiry time = T years

Interest rate = r

Current Trading Amount = $K

Since it is continuously compounded,

From

$$A = P(1 + \frac{r}{n})^{nt}$$

We get,

$$A = Pe^{rt}$$

As K is the strike amount for expiry period of T years, we can replace A with K and get the principal amount P with which we can calculate the premium amount,

$$K = Pe^{rT},$$

Therefore ,

$$\boxed{\mathbf{P = \frac{K}{e^{rT}}}}$$

I would be willing to pay **$(S-P)** today

Premium = $$(S - \frac{K}{e^{rT}})$ > 0 (Assumption: $(S - \frac{K}{e^{rT}})$ > 0 , always)

b.) Repeat the analysis from part (a), this time for a put option with the same parameters – Strike $K, Time to expiry T years, Annual interest rate r.

Solution:

Given:

Strike Amount = $K

Expiry time = T years

Interest rate = r

Current Trading Amount = $K

Since it is continuously compounded,

From

$$A = P(1 + \frac{r}{n})^{nt}$$

We get,

$$A = Pe^{rt}$$

As K is the strike amount for expiry period of T years, we can replace A with K and get the principal amount P with which we can calculate the premium amount,

$$K = Pe^{rT}$$

Therefore ,

$$\boxed{P = \frac{K}{e^{rT}}}$$

I would be willing to pay **$(P-S)** today

Premium = $$(\frac{K}{e^{rT}} - S)$ > 0 (Assumption: $(\frac{K}{e^{rT}} - S)$ > 0 , always)

Q3: BSM pricing

a.) Stock volatility (σ) is meant to account for uncertainty is stock price movement between t and T. If we are very close to expiry, it stands to reason that σ won't be able to impact the call/put option prices by much. For t–> T, find the BSM price for both call and put options. Do these agree with the formulae you derived in the previous ques. without inculcating σ?

Since, in real world markets react to events like news, which contribute to the price of the stocks going up and down, therefore we consider a small positive fraction, volatility (σ), which represents this uncertainity in stock price movement.

Solution:

Black-Scholes-Merton (BSM) is a model which includes this fraction of uncertainity for calculating stock prices into its call/put options. The BSM model is used to calculate stock option fair pricing based on six variables: volatility, type, underlying stock price, strike price, period, and risk-free rate. It is based on the hedging principle and aims to eliminate risks connected with underlying assets and stock options' volatility.

Given,

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t)$$

$$= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$$

where,

- $C(St, t)$ - BSM price of a call option at time t and stock price St (in \$)
- $P(St, t)$ - BSM price of a put option at time t and stock price St (in \$)
- St - Stock price at time t (in \$)
- σ - Stock volatility (absolute, not percent)
- K - Option strike price (in \$)

- T - Time to maturity (in years)
- N - CDF of the standard normal distribution
- r - annual risk free rate of interest (absolute, not percent)

The CDF of the standard normal distribution is denoted by the N function:

$$N(x)=P(Z\leq x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\left(\frac{-u^2}{2}\right)} du$$

Here are some properties of the N function that can be shown from its definition.

1. $\lim_{x\to\infty} N(x)=1,\ \lim_{x\to-\infty} N(x)=0;$
2. $N(0)=\frac{1}{2};$
3. $N(-x)=1-N(x)$, for all $x\in R$.

BSM Price for Call option:

$$\lim_{t\to T} C(S_t, t) = \lim_{t\to T} \left(N(d_1)S_t - N(d_2)Ke^{-r(T-t)}\right)$$

Plugging the values

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$d_2 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] - \sigma\sqrt{T-t}$$

$$\lim_{t\to T} C(S_t, t) = \lim_{t\to T} \left(N\left(\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]\right)S_t - N\left(\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] - \sigma\sqrt{T-t}\right)Ke^{-r(T-t)}\right)$$

$$\lim_{t\to T} C(S_t,t) = \left(N\left(\frac{1}{\sigma\sqrt{T-T}}\left[\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-T)\right]\right)S_t - N\left(\frac{1}{\sigma\sqrt{T-T}}\left[\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-T)\right]-\sigma\sqrt{T-T}\right)Ke^{-r(T-T)}\right)$$

$$\lim_{t\to T} C(S_t,t) = \left(N\left(\frac{1}{\sigma\sqrt{0}}\left[\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(0)\right]\right)S_t - N\left(\frac{1}{\sigma\sqrt{0}}\left[\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(0)\right]-\sigma\sqrt{0}\right)Ke^{-r(0)}\right)$$

$$\lim_{t\to T} C(S_t,t) = (N(\infty)S_t - N(\infty)K)$$

Using,

$$\lim_{x\to\infty} N(x)=1,$$

$$\boxed{\lim_{t\to T} \mathbf{C}(\mathbf{S_t},\mathbf{t}) = \mathbf{S_t} - \mathbf{K}}$$

BSM Price for Put option:

$$P(S_t,t) = Ke^{-r(T-t)} - S_t + C(S_t,t)$$

$$= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S_t$$

$$\lim_{t\to T} P(S_t,t) = \lim_{t\to T} \left(Ke^{-r(T-t)} - S_t + C(S_t,t)\right)$$

$$\lim_{t\to T} P(S_t,t) = \lim_{t\to T} Ke^{-r(T-t)} - \lim_{t\to T} S_t + \lim_{t\to T} C(S_t,t)$$

$$\lim_{t\to T} P(S_t,t) = Ke^{-r(T-T)} - S_t + S_t - K$$

$$\lim_{t\to T} P(S_t,t) = K - S_t + S_t - K$$

$$\boxed{\lim_{t\to T} \mathbf{P}(\mathbf{S_t},\mathbf{t}) = \mathbf{0}}$$

These values are not same as derived in the previous question because we have considered an ideal case wherein events don't affect the stock prices. But in Q3 we have taken into consideration the uncertainity factor, volatility($\boldsymbol{\sigma}$), which reflects in the difference in the derived formulae.

b.) For 3 given points in time (t = 0, 0.5 and 1 $years\ respectively$), plot the BSM price of a call option C, with stock price S_t as the x-axis; consider the range of S_t to be from \$1 to \$100, K = \$50, r = 12 % p.a., σ = 0.30, T = 1 year.

Solution: Given:

K = \$50

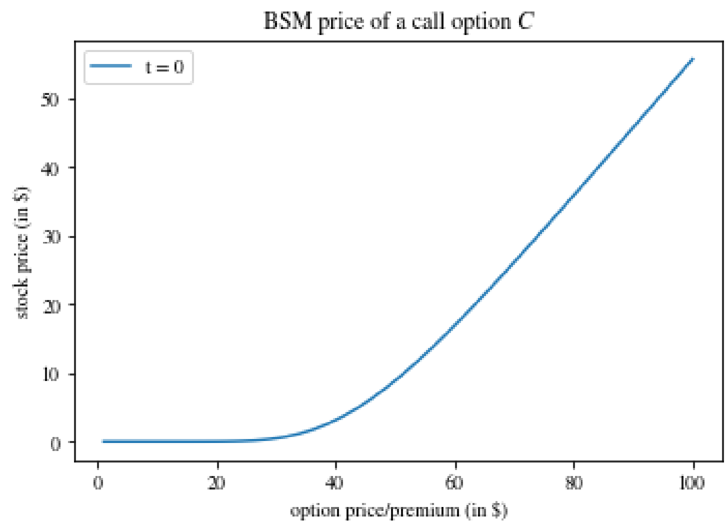
r = 12%pa

σ = 0.30

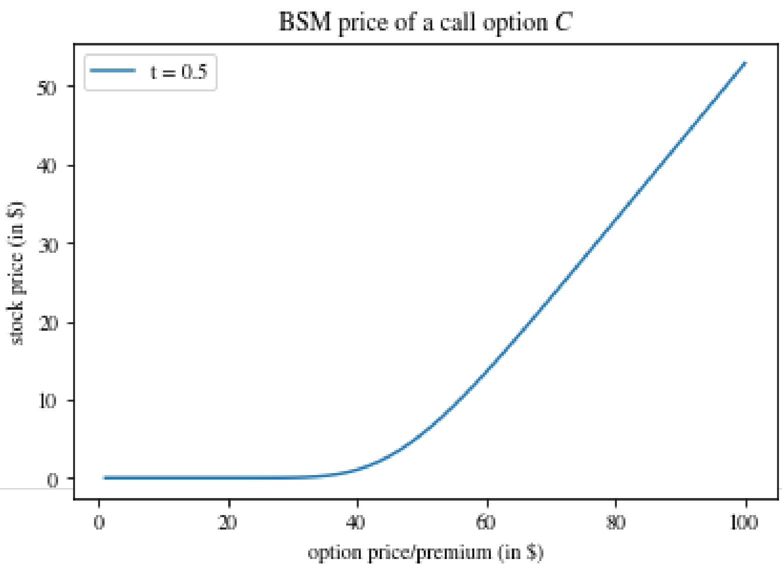
T = 1 year.

Plots of BSM Price of a Call Option(C) with Stock Price(S_t ranges from \$1 to \$100) for :

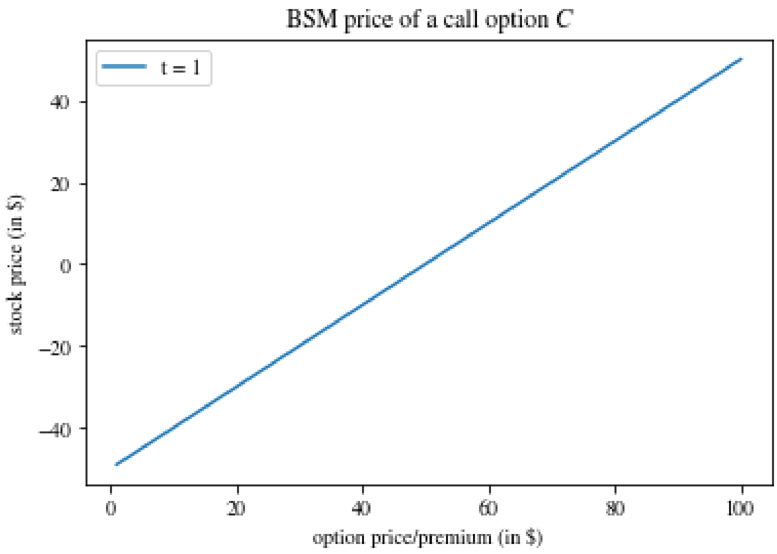
1. t=0 Year



2. t=0.5 Year

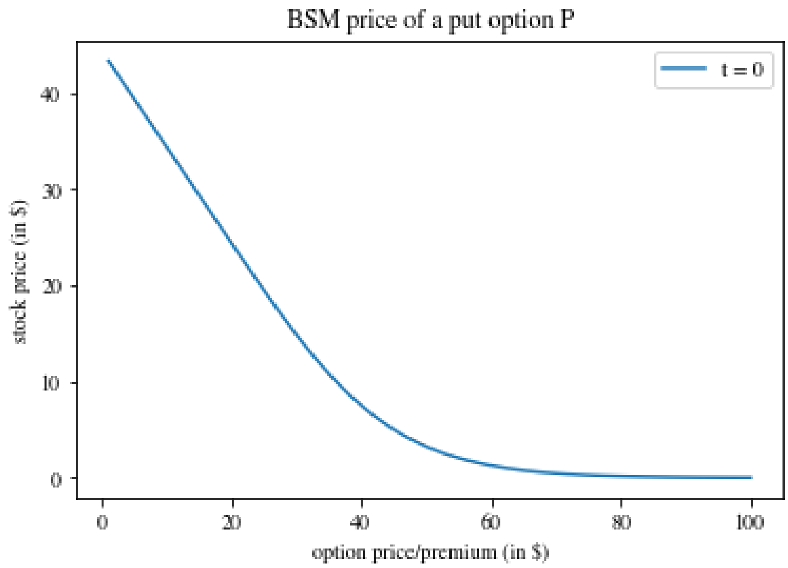


3. t=1 Year

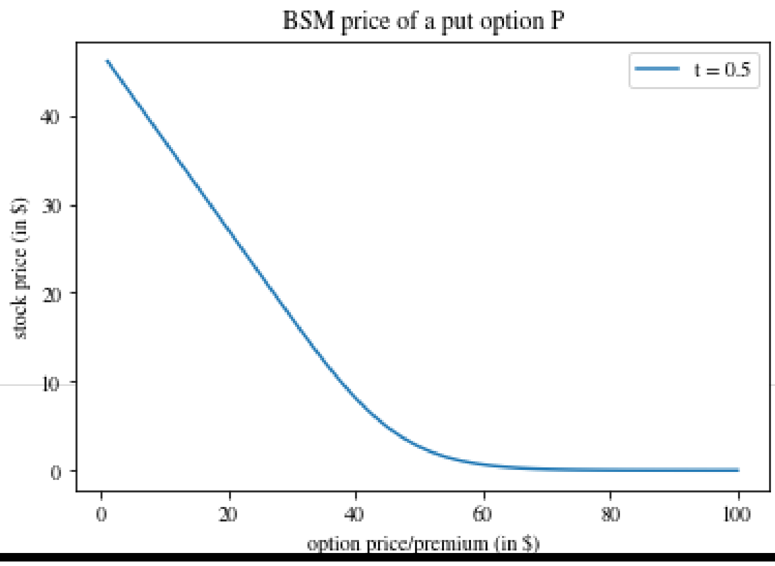


Plots of BSM Price of a Put Option(P) with Stock Price(S_t ranges from $1 to $100) for :

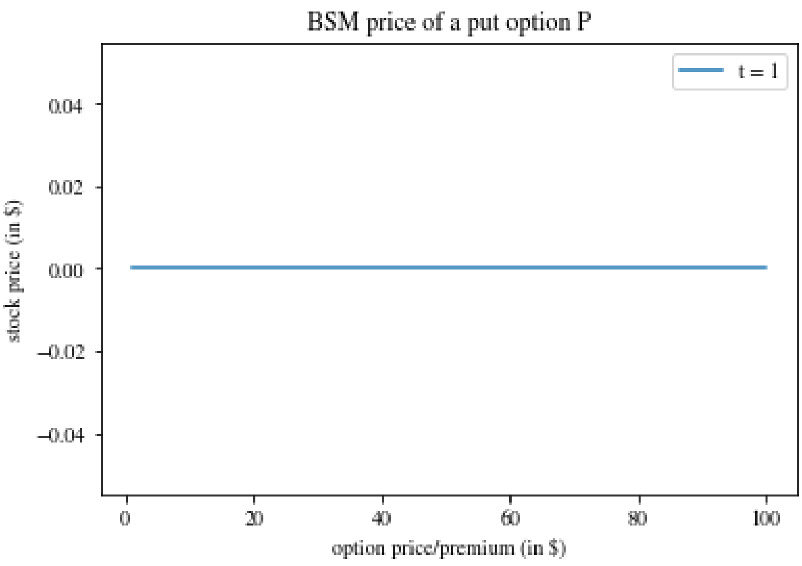
1. t=0 Year



2. t=0.5 Year



3. t=1 Year



Q4. Delta calculation

a. With the BSM pricing formulae given in the previous question, analytically calculate the delta of call and put options in terms of t, St , T, σ and K. Calculate these deltas in the six scenarios mentioned in the previous question for St = $125.

Solution:

Given:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$C(S_t, t) = N(d_1)S_t - N(d_2)Ke^{-r(T-t)}$$

Partial Differentiating $C(S_t, t)$ $w.r.t$ S_t

$$\frac{d\,(C)}{d(S_t)} = \frac{d\,(N(d_1))}{d(d1)}\frac{d\,(d_1)}{d(S_t)}S_t + N(d_1) - \frac{d\,(N(d_2))}{d(d_2)}\frac{d\,(d_2)}{d(S_t)}Ke^{-r(T-t)}$$

For Standard Normal Distribution, PDF(Probability Distribution Function)(f(x)) is the Derivative of CDF(Cumulative Distributive Function)(N(x))

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$f(\mathrm{d}_1) = \frac{d\,(\mathrm{N}(\mathrm{d}_1))}{d(\mathrm{d}1)}$$

$$f(\mathrm{d}_1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathrm{d}_1{}^2}{2\sigma^2}}$$

$$\mathrm{d}_1 = \frac{1}{\sigma\sqrt{\mathrm{T}-\mathrm{t}}}\left[\ln\left(\frac{\mathrm{S}_\mathrm{t}}{\mathrm{K}}\right) + \left(\mathrm{r} + \frac{\sigma^2}{2}\right)(\mathrm{T}-\mathrm{t})\right]$$

$$\frac{\mathrm{d}(\mathrm{d}_1)}{d(\mathrm{S}_\mathrm{t})} = \frac{1}{\sigma\sqrt{\mathrm{T}-\mathrm{t}}}\left(\frac{K}{\mathrm{S}_\mathrm{t}}\right)$$

$$f(\mathrm{d}_2) = \frac{d\,(\mathrm{N}(\mathrm{d}_2))}{d(\mathrm{d}_2)}$$

$$f(\mathrm{d}_2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathrm{d}_2{}^2}{2\sigma^2}}$$

$$\mathrm{d}_2 = \mathrm{d}_1 - \sigma\sqrt{\mathrm{T}-\mathrm{t}}$$

$$\frac{d\,(\mathrm{d}_2)}{d(\mathrm{S}_\mathrm{t})} = \frac{d\,(\mathrm{d}_1)}{d(\mathrm{S}_\mathrm{t})}$$

$$\frac{d\,(\mathrm{d}_2)}{d(\mathrm{S}_\mathrm{t})} = \frac{1}{\sigma\sqrt{\mathrm{T}-\mathrm{t}}}\left(\frac{K}{\mathrm{S}_\mathrm{t}}\right)$$

Plugging the values in the equation

$$\frac{d\,(\mathrm{C})}{d(\mathrm{S}_\mathrm{t})} = \frac{d\,(\mathrm{N}(\mathrm{d}_1))}{d(\mathrm{d}1)}\frac{d\,(\mathrm{d}_1)}{d(\mathrm{S}_\mathrm{t})}\mathrm{S}_\mathrm{t} + N(\mathrm{d}_1) - \frac{d\,(\mathrm{N}(\mathrm{d}_2))}{d(\mathrm{d}_2)}Ke^{-r(T-t)}$$

$$\frac{d\,(\mathrm{C})}{d(\mathrm{S_t})} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathrm{d_1}^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{\mathrm{T-t}}}\left(\frac{K}{\mathrm{S_t}}\right)\mathrm{S_t} + N(\mathrm{d_1}) - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathrm{d_2}^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{\mathrm{T-t}}}\left(\frac{K}{\mathrm{S_t}}\right)Ke^{-r(T-t)}$$

$$\frac{d\,(\mathrm{C})}{d(\mathrm{S_t})} = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\mathrm{K}e^{-\frac{\mathrm{d_1}^2}{2\sigma^2}}}{\sigma\sqrt{\mathrm{T-t}}} + N(\mathrm{d_1}) - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathrm{d_2}^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{\mathrm{T-t}}}\left(\frac{K^2e^{-r(T-t)}}{\mathrm{S_t}}\right)$$

$$\frac{d\,(\mathrm{C})}{d(\mathrm{S_t})} = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\mathrm{K}e^{-\frac{\mathrm{d_1}^2}{2\sigma^2}}}{\sigma\sqrt{\mathrm{T-t}}} + N(\mathrm{d_1}) - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathrm{d_2}^2}{2\sigma^2}} \frac{1}{\sigma\sqrt{\mathrm{T-t}}}\left(\frac{K^2e^{-r(T-t)}}{\mathrm{S_t}}\right)$$

It is important to note that the first and last terms in the returned expression for diff_C are negligible. This concludes that**:**

$$\boxed{\frac{\boldsymbol{d}\,(\mathbf{C})}{\boldsymbol{d}(\mathbf{S_t})} \approx \boldsymbol{N}(\mathbf{d_1})}$$

$$\mathrm{P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t)}$$

Partial Differentiating $\mathrm{P(S_t, t)}$ $w.r.t\ \ S_t$

$$\frac{d\,(\mathrm{P})}{d(\mathrm{S_t})} = \frac{d\,(\mathrm{Ke^{-r(T-t)}})}{d(\mathrm{S_t})} - \frac{\mathrm{d(S_t)}}{d(\mathrm{S_t})} + \frac{d\,(\mathrm{C(S_t, t)})}{d(\mathrm{S_t})}$$

$$\frac{d\,(\mathrm{P})}{d(\mathrm{S_t})} = -1 + N(\mathrm{d_1})$$

It is important to note that**:**

$$\frac{\boldsymbol{d}\,(\mathbf{P})}{\boldsymbol{d}(\mathbf{S_t})} = -\boldsymbol{N}(-\mathbf{d_1})$$

Case	Option	Time	Delta(Analytical)	Delta(Numerical)	Error %
1	Call	0	0.9998435037893203	0.9998432248412712	2.7899178808574987e-05
2	Call	0.5	0.9999996999537609	0.9999992424638364	4.574902710645463e-05
3	Call	1	1	1	-
4	Put	0	-0.00015649621067970187	-0.00015677515871973316	0.17792872436503807
5	Put	0.5	-3.0004623907675665e-07	-7.575361513627405e-07	60.39182571854822
6	Put	1	0	0	-

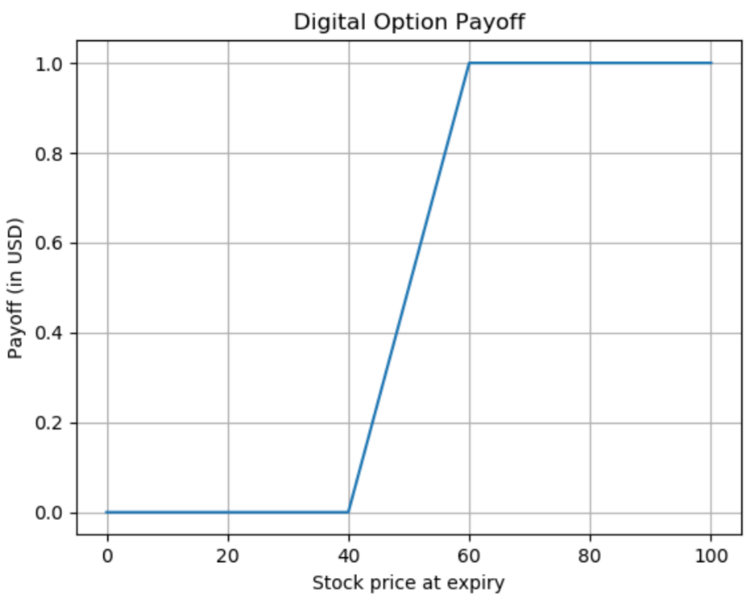
It is clearly seen that error is negligible in most of the cases which means that the analytical and numerical deltas were very close. Hence, the below equations can be used to calculate delta of Call and Put options:

$$\frac{d\,(\mathbf{C})}{d(\mathbf{S_t})} \approx N(\mathbf{d_1}) \quad \& \quad \frac{d\,(\mathbf{P})}{d(\mathbf{S_t})} = -N(-\mathbf{d_1})$$

Q5. a. In the previous question, you learnt how to calculate delta for call/put options. From the theory above you can now build a riskless portfolio if you are given a bunch of calls and puts pretty easily. Since creating riskless portfolios for calls and puts is so easy (calculating deltas from the formulae you derived in the last question and buying/selling the corresponding number of underlying stocks), its usual practice to break down complex derivative payoffs into a linear combination of calls and puts, and hedge the weighted sum of their deltas. One such derivative with the below payoff is called a digital option. Show that this payoff can be broken down into an equivalent linear combination of call and put options. Once you do, show how delta hedging can be done as a linear combination of the delta 'contributions' from said call and put options. What will be the final riskless portfolio in this case if you start with n units of the below digital option.

Note: the below payoff has been plotted against stock price at expiry, i.e. S_T, rather than S_t. Take r = 12 % p.a., σ = 0.3 and T = 1 year, and create riskless portfolios for 3 scenarios at t = 0:

(i) St = **$30,**
(ii) St = **$50, and**
(iii) St = **$70.**



Solution:
Delta hedging is an options strategy that seeks to be directionally neutral by establishing offsetting long and short positions in the same underlying. By reducing directional risk, delta hedging can isolate volatility changes for an options trader. One of the drawbacks of delta hedging is the necessity of constantly watching and adjusting positions involved. It can also incur trading costs as delta hedges are added and removed as the underlying price changes.

Delta for Long Call is taken as positive.

Delta for Short Call is taken as negative.

Delta for Long Put is taken as negative.

Delta for Short Put is taken as positive.

Given

annual risk free rate of interest, r = 12% p.a.

Stock volatility, σ= 0.3

Time to maturity (in years), T = 1 year

The given graph resembles with **Bull Call**

Using the given payoff and the given data, we can say that

This payoff can be broken down into an equivalent linear combination of call and put options as follows:

i. Long call option at Strike $40

+

ii. Short call option at Strike $60.

I. S_t = $30
II. S_t = $50
III. S_t = $70

Delta for Long Call is taken as positive.

Delta for Short Call is taken as negative.

Long call option at Strike $40 = Delta_call($S_t$,40)

Short call option at Strike $60 = -Delta_call($S_t$,60)

Delta = Delta_call(S_t , 40) -Delta_call(S_t , 60)

Now for Delta_I = Delta_call(S_t _I, 40) -Delta_call(S_t _I, 60)

Delta_I = 0.3019576

Riskless Portfolio for S_t = 30

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_I*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_I*n) of stock.**

Now For Delta_II **,**

Delta_II = Delta_call(S_t _II, 40) -Delta_call(S_t _II, 60)

Delta_II = 0.425023

Riskless Portfolio for St = 50

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_II*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_II*n) of stock.**

In this case, we will short more stocks than case I.

Now For Delta_III **,**

Delta_III = Delta_call(S_t _III, 40) -Delta_call(S_t _III, 60)

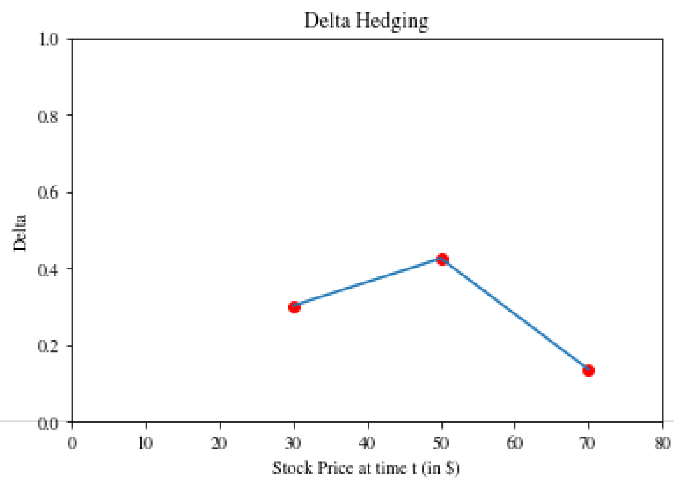
Delta_III = 0.1359369

Riskless Portfolio for St = 70

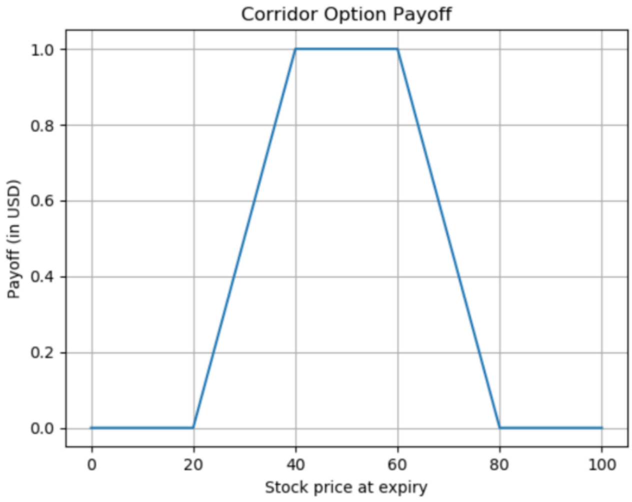
As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_III*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_III*n) of stock.**

In this case, we will buy even more stocks than case II.



b.) The strategy used in part (a) can theoretically be applied to any derivative with a piecewise linear payoff vs S_t. In th.at spirit, consider the following derivative given in the below figure.



Take r = 12 % p.a., σ = 0.3 and T = 1 year, and create riskless portfolios for 5 scenarios at t = 0 (assuming that we start with n units of this derivative):

(i) St = $10,

(ii) St = $30,

(iii) St = $50,

(iv) St = $70, and

(v) St = $90

Solution:

Given

annual risk free rate of interest, r = 12% p.a.

Stock volatility, σ= 0.3

Time to maturity (in years), T = 1 year

t=0

Using the given payoff and the given data, we can say that

This payoff can be broken down into an equivalent linear combination of call and put options as

i. Long call option at Strike $20

+

ii. Short call option at Strike \$40

+

iii. Long put option at Strike \$60

+

iv. Short put option at Strike \$80

Delta for Long Call is taken as positive.

Delta for Short Call is taken as negative.

Delta for Long Put is taken as negative.

Delta for Short Put is taken as positive.

Long call option at Strike \$20 = Delta_call(S_t, 20)

Short call option at Strike \$40 = -Delta_call(S_t, 40)

Long put option at Strike \$60 = -Delta_call(S_t, 60)

Short put option at Strike \$80 = Delta_call(S_t, 80)

Delta = Delta_call(S_t , 20) -Delta_call(S_t , 40) + Delta_put(S_t , 60) - Delta_put(S_t , 80)

Now for Delta_i = Delta_call(S_t _i, 20) -Delta_call(S_t _i, 40) +Delta_put(S_t _i, 60) -Delta_put(S_t _i, 80)

Delta_i = 0.04140507178631403

Riskless Portfolio for S_t = 10

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_i*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_i*n) of stock.**

Now for Delta_ii = Delta_call(S_t _ii, 20) -Delta_call(S_t _ii, 40) + Delta_put(S_t _ii, 60) -Delta_put(S_t _ii, 80)

Delta_ii = 0.6658114011307994

Riskless Portfolio for S_t = 30

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_ii*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_ii*n) of stock.**

Now for Delta_iii = Delta_call(S_t _iii, 20) -Delta_call(S_t _iii, 40) + Delta_put(S_t _iii, 60) - Delta_put(S_t _iii, 80)

Delta_iii = 0.42003862136533243

Riskless Portfolio for S_t = 50

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_iii*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_iii*n) of stock.**

Now for Delta_iv = Delta_call(S_t _iv, 20) -Delta_call(S_t _iv, 40) + Delta_put(S_t _iv, 60) - Delta_put(S_t _iv, 80)

Delta_iv = 0.32235202614604397

Riskless Portfolio for S_t = 70

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_iv*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_iv*n) of stock.**

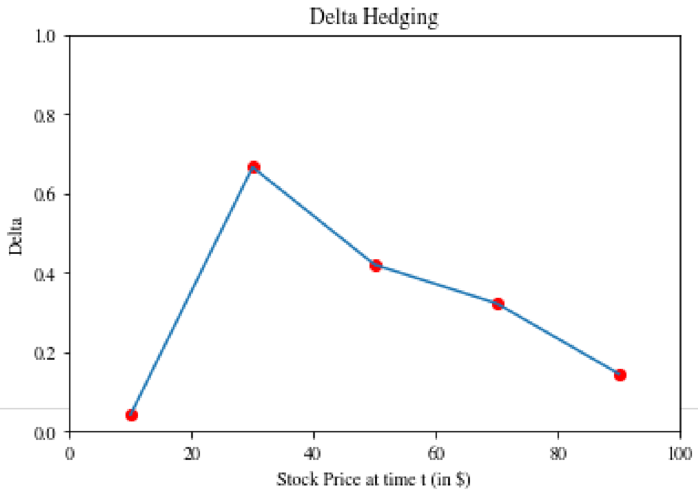
Now for Delta_v = Delta_call(S_t _v, 20) -Delta_call(S_t _v, 40) + Delta_put(S_t _v, 60) -Delta_put(S_t _v, 80)

Delta_v = 0.1449373252902968

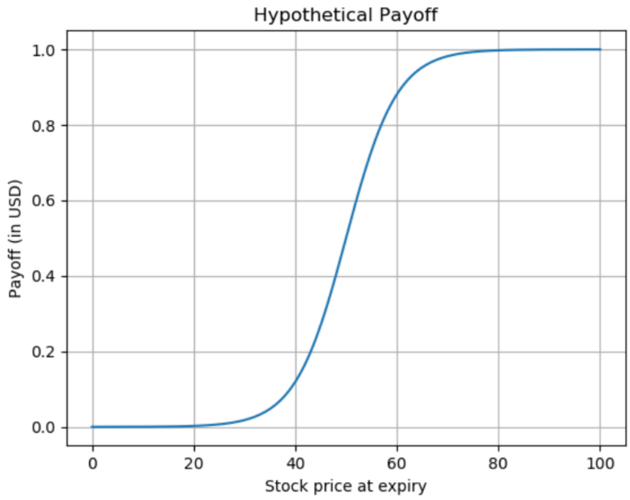
Riskless Portfolio for S_t= 90

As we can see, delta for the above combination of options is positive.

Hence, We can **short (Delta_v*n) units** of the underlying stock directly, your total exposure to the market (called a portfolio) will have **n units of the digital option and (Delta_v*n) of stock.**



c.) An interesting extension of this concept is to consider a case where payoff vs S_t is not piecewise linear, like the payoff given below.



Describe a general purpose strategy/technique/hypothesis to create riskless portfolios for such a payoff for any given S_0. To test your strategy/technique/hypothesis, create riskless portfolios for the 5 scenarios and parameters given in part (b).

Solution:

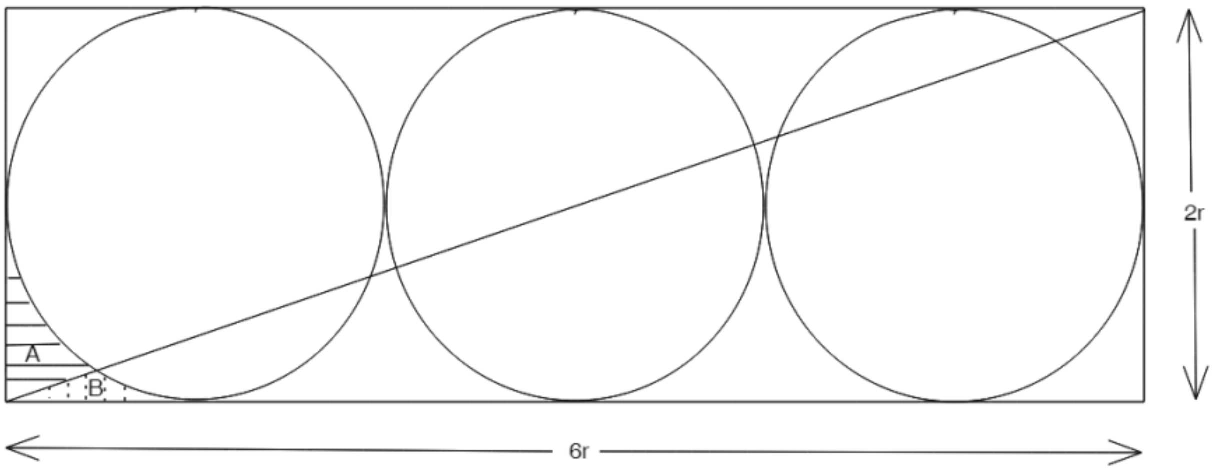
This hypothetical payoff resembles the payoff of part(a),so it can be a combination of Long and Short Calls with continuous strike prices

Case Study -B

Q1: Circles?

Filename: URJA_GUPTA_SECTION_B_Q1_MAIN.MLX

a.) Consider the below figure. There are three circles of equal radius 'r' arranged side by side inside a rectangle of length '6r' and breadth '2r'. A diagonal is drawn in the rectangle. Area 'A' is the bounded region to the left of the first circle, above the intersecting diagonal shaded with solid horizontal lines. Area 'B' is the area between the first circle and the diagonal shaded with dotted lines.



Let AreaRatio be defined as:

AreaRatio = area of 'A' / (area of 'A' + area of 'B')

Q1.1. What is the AreaRatio in the set up described above?

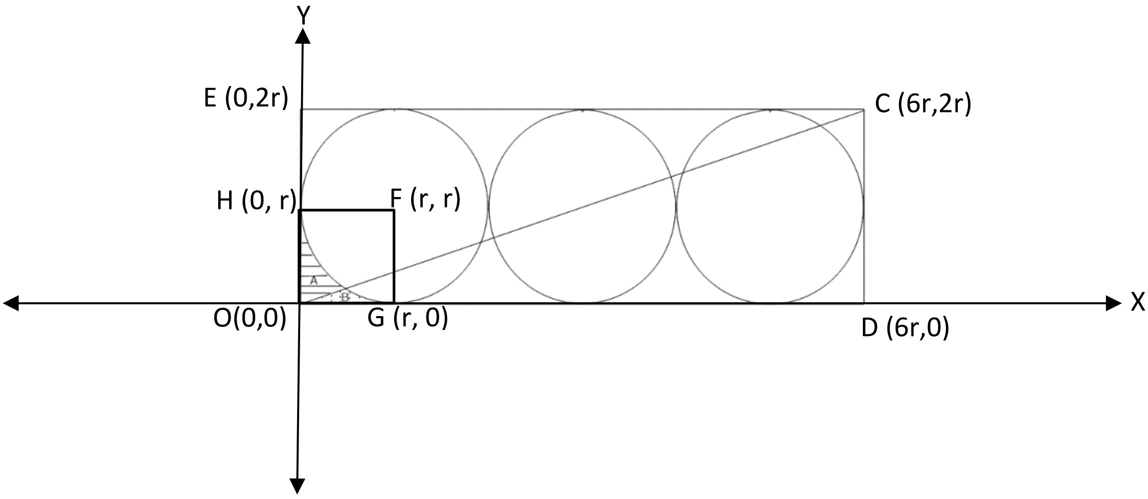
Solution: Assuming Corner point of Rectangle as Origin(0,0) and marking the coordinates of different points.

Given :

Radius of circle = r

Length = 6r

Breadth = 2r



Writing the equation of line OC :-

$$(x_1, y_1) = (0,0)$$

$$(x_2, y_2) = (6r, 2r)$$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) * \ x - x_1$$

$$y - 0 = \left(\frac{2r - 0}{6r - 0}\right) * \ x - 0$$

$$y = \frac{x}{3}$$

Writing the equation of Circle with Center F (r, r) and radius r :-

$$(x - r)^2 + \ (y - r)^2 = r^2$$

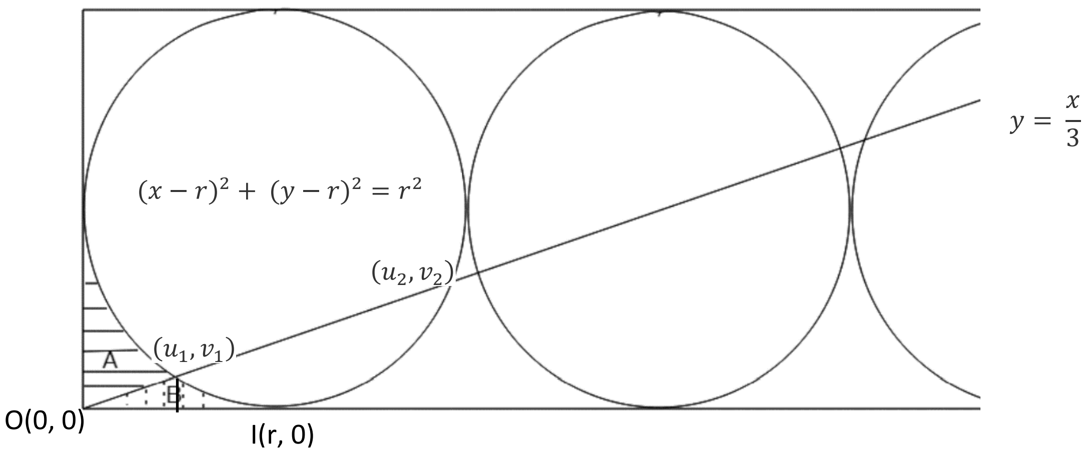
Finding the point of intersection between line and circle by substituting the values:-

$Let\ (u_1, v_1)\ and\ (u_2, v_2)\ be\ the\ coordinates\ of\ point\ of\ intersection$ then

By Solving the equation through MATLAB:-

$$(u_1, v_1) = \left(\frac{6\,r}{5} + \frac{3\,\sqrt{6}\,r}{10}, \frac{2\,r}{5} + \frac{\sqrt{6}\,r}{10} \right)$$

$$(u_2, v_2) = \left(\frac{6\,r}{5} - \frac{3\,\sqrt{6}\,r}{10}, \frac{2\,r}{5} - \frac{\sqrt{6}\,r}{10} \right)$$



Now The area under the curve

$$Area\ of\ 'B_1': y = \frac{x}{3}\ from\ (0,0) \to (u_1, v_1)$$

$$Area\ of\ 'B_2': y = r - \sqrt{r^2 - (x-r)^2}\ from\ (u_1, v_1) \to (r, 0)$$

$$Area\ of\ 'B' = Area\ of\ 'B_1' + Area\ of\ 'B_2'$$

Integrating within the limits

$$\int_0^r B = \int_0^r (B_1 + B_2)$$

$$\int_0^r B = \int_0^{u_1} B_1 + \int_{u_1}^r B_2$$

$$\int_0^r B = \int_0^{u_1} \frac{x}{3} + \int_{u_1}^r r - \sqrt{r^2 - (x-r)^2}$$

Plugging the values $u_1 = \frac{6\,r}{5} - \frac{3\,\sqrt{6}\,r}{10}$ and Integrating through MATLAB

$$\int_0^r B = -\frac{3\,r^2\,(4\,\sqrt{6} - 11)}{100} + \frac{r\,\sigma_1}{10} - r\,\left(\frac{r}{5} - \frac{3\,\sqrt{6}\,r}{10}\right) - \frac{r^2 \operatorname{asin}\left(\frac{3\,\sqrt{6}}{10} - \frac{1}{5}\right)}{2} - \frac{3\,\sqrt{6}\,r\,\sigma_1}{20}$$

$$\text{Where, } \sigma_1 = \sqrt{\frac{3\,\sqrt{6}\,r^2}{25} + \frac{21\,r^2}{50}}$$

$$Area\ of\ 'B' = \frac{r\,\sigma_1}{10} - \frac{3\,r^2\,(4\,\sqrt{6} - 11)}{100} - r\,\left(\frac{r}{5} - \frac{3\,\sqrt{6}\,r}{10}\right) - \frac{r^2 \operatorname{asin}\left(\frac{3\,\sqrt{6}}{10} - \frac{1}{5}\right)}{2} - \frac{3\,\sqrt{6}\,r\,\sigma_1}{20}$$

$$where$$

$$\sigma_1 = \sqrt{\frac{3\ \sqrt{6}\ r^2}{25} + \frac{21\ r^2}{50}}$$

Now for

$$Area\ of\ 'A' = Area\ of\ Square\ 'OGFH' - Area\ of\ Quadarant\ 'FGH' - \ Area\ of\ 'B'$$

$$Area\ of\ Square\ 'OGFH' = r^2$$

$$Area\ of\ Quadarant\ 'FGH' = \frac{\pi r^2}{4}$$

$$Area\ of\ 'B' = \frac{r\ \sigma_1}{10} - \frac{3\ r^2\ (4\ \sqrt{6} - 11)}{100} - r\ \left(\frac{r}{5} - \frac{3\ \sqrt{6}\ r}{10}\right) - \frac{r^2 \operatorname{asin}\left(\frac{3\ \sqrt{6}}{10} - \frac{1}{5}\right)}{2} - \frac{3\ \sqrt{6}\ r\ \sigma_1}{20}$$

$$where$$

$$\sigma_1 = \sqrt{\frac{3\ \sqrt{6}\ r^2}{25} + \frac{21\ r^2}{50}}$$

$$Area\ of\ 'A' = r^2 - \frac{\pi r^2}{4} - \frac{r\ \sigma_1}{10} - \frac{3\ r^2\ (4\ \sqrt{6} - 11)}{100} - r\ \left(\frac{r}{5} - \frac{3\ \sqrt{6}\ r}{10}\right) - \frac{r^2 \operatorname{asin}\left(\frac{3\ \sqrt{6}}{10} - \frac{1}{5}\right)}{2} - \frac{3\ \sqrt{6}\ r\ \sigma_1}{20}$$

$$where$$

$$\sigma_1 = \sqrt{\frac{3\ \sqrt{6}\ r^2}{25} + \frac{21\ r^2}{50}}$$

Now

$$AreaRatio = \frac{Area\ of\ 'A'}{Area\ of\ 'A' + \ Area\ of\ 'B'}$$

Plugging the Values from above

$We\ get,$

$$AreaRatio = 0.70743$$

$= 70.743\%$

MATLAB Code:-

```
syms x y r n p
r=1
n=3
eqns = [(x-r)^2 + (y-r)^2 == r*r, y - (1/(n))*x == 0 ]; %Putting the equation of circle and line
vars = [x, y];                          %finding the point of intersections
[solx, soly] = solve(eqns,vars)                %Intersection points
x1=solx(1)                              %u1=x1
y1=soly(1)                              % v1=y1
fun1 = @(x)(1/(n))*x;
q1 = int(fun1,x,0,x1)                      %Integrating the Area B1
fun2 = @(x)(r-sqrt((r^2)-((x-r)^2)));
q2 = int(fun2,x,x1,r)                      %Integrating the Area B2
B=double(q1+q2)                            %Area of B
A=double((((r^2)-(pi*(r^2))/4)-B))               %Area of A
Arearatio=double((A)/(A+B))                    %AreaRatio
sprintf('%0.5f', Arearatio)
```

Q1.2. What is the range of AreaRatio?

Solution: Now If there are 'n' circles arranged inside a rectangle of dimensions '2nr' by '2r' in a similar fashion,

Then,

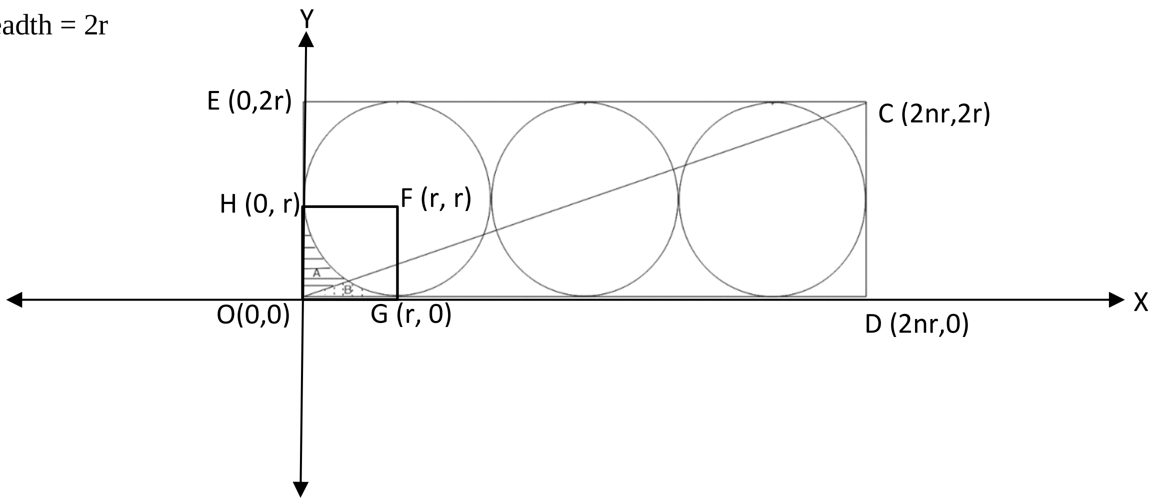
Assuming Corner point of Rectangle as Origin(0,0) and marking the coordinates of different points.

Given :

Radius of circle = r

Length = 2nr

Breadth = 2r



Writing the equation of line OC :-

$$(x_1, y_1) = (0,0)$$

$$(x_2, y_2) = (6r, 2r)$$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) * \ x - x_1$$

$$y - 0 = \left(\frac{2r - 0}{2nr - 0}\right) * \ x - 0$$

$$y = \frac{x}{n}$$

Writing the equation of Circle with Center F (r, r) and radius r :-

$$(x - r)^2 + \ (y - r)^2 = r^2$$

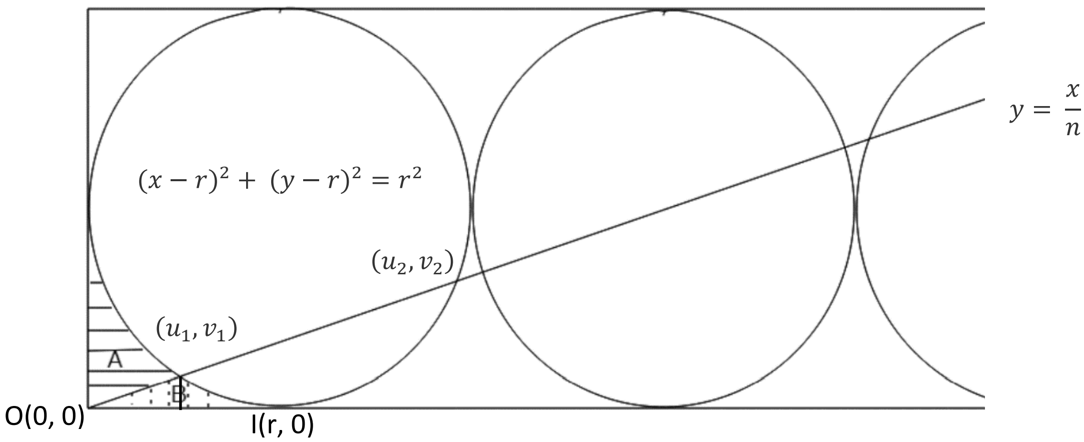
Finding the point of intersection between line and circle by substituting the values:-

$Let\ (u_1, v_1)\ and\ (u_2, v_2)\ be\ the\ coordinates\ of\ point\ of\ intersection$ then

By Solving the equation through MATLAB:-

$$(u_1, v_1) = \left(\frac{n\ (r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r)}{n^2 + 1}, \frac{r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r}{n^2 + 1}\right)$$

$$(u_2, v_2) = \left(\frac{n\ (r + n\,r - \sqrt{2}\ \ \sqrt{n}\ r)}{n^2 + 1}, \frac{r + n\,r - \sqrt{2}\ \ \sqrt{n}\ r}{n^2 + 1}\right)$$



Now The area under the curve

$$Area\ of\ 'B_1':y = \frac{x}{n}\ from\ (0,0) \rightarrow (u_1, v_1)$$

$$Area\ of\ 'B_2':y = r - \sqrt{r^2 - (x-r)^2}\ from\ (u_1, v_1) \rightarrow (r, 0)$$

$$Area\ of\ 'B' = Area\ of\ 'B_1' + Area\ of\ 'B_2'$$

Integrating within the limits

$$\int_0^r B = \int_0^r (B_1 + B_2)$$

$$\int_0^r B = \int_0^{u_1} B_1 + \int_{u_1}^r B_2$$

$$\int_0^r B = \int_0^{u_1} \frac{x}{n} + \int_{u_1}^r r - \sqrt{r^2 - (x-r)^2}$$

Plugging the values $u_1 = \frac{n\ (r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r)}{n^2+1}$ and Integrating through MATLAB

$$\int_0^r B = \frac{n\ (r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r)}{n^2+1} + r\,\sigma_1 - \frac{\sqrt{r^2 - \sigma_1{}^2}\ \sigma_1}{2} - \frac{r^2 \operatorname{asin}\left(\frac{\sigma_1}{\sqrt{r^2}}\right)}{2}$$

Where, $\sigma_1 = r - \frac{n\ (r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r)}{n^2+1}$

$$Area\ of\ 'B' = r\,\sigma_1 - \frac{\sqrt{r^2 - \sigma_1{}^2}\ \sigma_1}{2} - \frac{r^2 \operatorname{asin}\left(\frac{\sigma_1}{\sqrt{r^2}}\right)}{2} + \frac{n\ (r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r)^2}{2\ (n^2+1)^2}$$

$$where$$

$$\sigma_1 = r - \frac{n\ (r + n\,r + \sqrt{2}\ \ \sqrt{n}\ r)}{n^2+1}$$

Now for

$$Area\ of\ 'A' = Area\ of\ Square\ 'OGFH' - Area\ of\ Quadarant\ 'FGH' - Area\ of\ 'B'$$

$$Area\ of\ Square\ 'OGFH' = r^2$$

$$Area\ of\ Quadarant\ 'FGH' = \frac{\pi r^2}{4}$$

$$Area\ of\ 'B' = r\,\sigma_1 - \frac{\sqrt{r^2 - {\sigma_1}^2}\ \sigma_1}{2} - \frac{r^2 \operatorname{asin}\left(\frac{\sigma_1}{\sqrt{r^2}}\right)}{2} + \frac{n\ \left(r + n\,r + \sqrt{2}\ \sqrt{n}\ r\right)^2}{2\ \left(n^2+1\right)^2}$$

$$where$$

$$\sigma_1 = r - \frac{n\ \left(r + n\,r + \sqrt{2}\ \sqrt{n}\ r\right)}{n^2+1}$$

$$Area\ of\ 'A' = r^2 - \frac{\pi r^2}{4} - r\,\sigma_1 - \frac{\sqrt{r^2 - {\sigma_1}^2}\ \sigma_1}{2} - \frac{r^2 \operatorname{asin}\left(\frac{\sigma_1}{\sqrt{r^2}}\right)}{2} + \frac{n\ \left(r + n\,r + \sqrt{2}\ \sqrt{n}\ r\right)^2}{2\ \left(n^2+1\right)^2}$$

$$where$$

$$\sigma_1 = r - \frac{n\ \left(r + n\,r + \sqrt{2}\ \sqrt{n}\ r\right)}{n^2+1}$$

Now

$$AreaRatio = \frac{Area\ of\ 'A'}{Area\ of\ 'A' +\ Area\ of\ 'B'}$$

Plugging the Values from above

We get,

$$AreaRatio = \frac{1125899906842624\,n\ \left(n + \sqrt{2}\ \sqrt{n} + 1\right)}{241620187839069\ \left(n^2+1\right)} - \frac{562949953421312\,\sigma_1\ \sqrt{1 - {\sigma_1}^2}}{241620187839069} - \frac{562949953421312\,n\ \left(n + \sqrt{2}\ \sqrt{n} + 1\right)^2}{241620187839069\ \left(n^2+1\right)^2} - \frac{562949953421312 \operatorname{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\ \left(n + \sqrt{2}\ \sqrt{n} + 1\right)}{n^2+1} - 1$$

Now For range we put values n=1,2.3……∞

Now for n=1

$$AreaRatio = 0.50000$$

For n= ∞

$$AreaRatio \rightarrow 1$$

$$So, Range\ of\ AreaRatio\ is\ \ [0.5000, 1)$$

for n= n=1,2.3……∞

MATLAB Code:-

```
syms x y r n p
r=1                                    %Treating r as unit Constant as it will get cancel
n=1                                    %putting n =1 (minimum range)and n= inf(maximum range) for range
eqns = [(x-r)^2 + (y-r)^2 == r*r, y - (1/(n))*x == 0 ]; %Putting the equation of circle and line
vars = [x, y];                         %finding the point of intersections


[solx, soly] = solve(eqns,vars)        %Intersection points
x1=solx(1)                             %u1=x1
y1=soly(1)                             % v1=y1
fun1 = @(x)(1/(n))*x;
q1 = int(fun1,x,0,x1)                  %Integrating the Area B1
fun2 = @(x)(r-sqrt((r^2)-((x-r)^2)));
q2 = int(fun2,x,x1,r)                  %Integrating the Area B2
B=(q1+q2)                              %Area of B
A=((((r^2)-(pi*(r^2))/4)-B))           %Area of A
Arearatio=double((A)/(A+B))            %AreaRatio
```

Q1.3. What is the minimum value of ‘n’ for which the AreaRatio is greater than or equal to:

a. 50%

b. 70%

c. 90%

d. 99.9%

e. 99.99%

f. 100%

Solution:

Let Given AreaRatio/100 be 'P'

Now from Q(1.2) we have concluded that

$$AreaRatio = \frac{Area\ of\ 'A'}{Area\ of\ 'A' +\ Area\ of\ 'B'}$$

$AreaRatio$
$=$
$$\frac{1125899906842624\,n\ (n+\sqrt{2}\ \sqrt{n}+1)}{241620187839069\ (n^2+1)} - \frac{562949953421312\,\sigma_1\ \sqrt{1-\sigma_1{}^2}}{241620187839069} - \frac{562949953421312\,n\ (n+\sqrt{2}\ \sqrt{n}+1)^2}{241620187839069\ (n^2+1)^2} - \frac{562949953421312\,\operatorname{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\ (n+\sqrt{2}\ \sqrt{n}+1)}{n^2+1} - 1$$

We will be equating the AreaRatio with the P to find the value of n, we will be using the ceil function in MATLAB to calculate integral values of n

So,

$$AreaRatio = P$$

a.)**Then for P=0.5**

$AreaRatio$
$=$
$$\frac{1125899906842624\,n\ (n+\sqrt{2}\ \sqrt{n}+1)}{241620187839069\ (n^2+1)} - \frac{562949953421312\,\sigma_1\ \sqrt{1-\sigma_1{}^2}}{241620187839069} - \frac{562949953421312\,n\ (n+\sqrt{2}\ \sqrt{n}+1)^2}{241620187839069\ (n^2+1)^2} - \frac{562949953421312\,\operatorname{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\ (n+\sqrt{2}\ \sqrt{n}+1)}{n^2+1} - 1$$

0.5
$=$
$$\frac{1125899906842624\,n\ (n+\sqrt{2}\ \sqrt{n}+1)}{241620187839069\ (n^2+1)} - \frac{562949953421312\,\sigma_1\ \sqrt{1-\sigma_1{}^2}}{241620187839069} - \frac{562949953421312\,n\ (n+\sqrt{2}\ \sqrt{n}+1)^2}{241620187839069\ (n^2+1)^2} - \frac{562949953421312\,\operatorname{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\ (n+\sqrt{2}\ \sqrt{n}+1)}{n^2+1} - 1$$

After Solving through MATLAB We get,

n=1

b.) for P=0.7

$AreaRatio$
$=$
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

0.7
$=$
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

After Solving through MATLAB We get,

n=3

c.) for P=0.9

$AreaRatio$
$=$
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

0.9
$=$
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

After Solving through MATLAB We get,

$$\boxed{n=15}$$

d.)for P=0.999

$AreaRatio$
$=$

$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)} - \frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069} - \frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2} - \frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1} - 1$$

0.999
$=$

$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)} - \frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069} - \frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2} - \frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1} - 1$$

After Solving through MATLAB We get,

$$\boxed{n=2240}$$

e.)for P=0.9999

$AreaRatio$
$=$

$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)} - \frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1{}^2}}{241620187839069} - \frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2} - \frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069} - \frac{884279719003555}{241620187839069}$$

where

$$\sigma_1 = \frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1} - 1$$

0.9999
=
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

After Solving through MATLAB We get,

$$\boxed{\mathbf{n=23012}}$$

f.)for P=1

$AreaRatio$
=
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

1
=
$$\frac{1125899906842624\,n\,(n+\sqrt{2}\,\sqrt{n}+1)}{241620187839069\,(n^2+1)}-\frac{562949953421312\,\sigma_1\,\sqrt{1-\sigma_1^2}}{241620187839069}-\frac{562949953421312\,n\,(n+\sqrt{2}\,\sqrt{n}+1)^2}{241620187839069\,(n^2+1)^2}-\frac{562949953421312\,\mathrm{asin}(\sigma_1)}{241620187839069}-\frac{884279719003555}{241620187839069}$$

where

$$\sigma_1=\frac{n\,(n+\sqrt{2}\,\sqrt{n}+1)}{n^2+1}-1$$

After Solving through MATLAB We get,

$$\boxed{\mathbf{n=\infty}}$$

MATLAB Code:-

```
syms x y r n p
  r=1                                                      %Treating r as unit Constant as it will get cancel
  ans=zeros()
  eqns = [(x-r)^2 + (y-r)^2 == r*r, y - (1/(n))*x == 0 ]; %Putting the equation of circle and line
  vars = [x, y];                                           %finding the point of intersections
  [solx, soly] = solve(eqns,vars)                          %Intersection points
  x1=solx(2)                                               %u1=x1
  y1=soly(2)                                               % v1=y1
  fun1 = @(x)(1/(n))*x;
  q1 = int(fun1,x,0,x1)                                    %Integrating the Area B1
  fun2 = @(x)(r-sqrt((r^2)-((x-r)^2)));
  q2 = int(fun2,x,x1,r)                                    %Integrating the Area B2
  B=(q1+q2)                                          %Area of B
  A=((((r^2)-(pi*(r^2))/4)-B))                       %Area of A
  Arearatio=((A)/(A+B))                              %AreaRatio
  p = [0.5; 0.7; 0.9; 0.999; 0.9999; 1]
  n_hat = zeros(length(p),1);
  for i=1:length(p)
      if p(i,1)==1
          n_hat(i,1)=inf;
          continue
      end
      n_hat(i,1) = ceil(solve(Arearatio-p(i,1)==0));%Calculating n for different p
  end
  n_hat
```

Filename:
URJA_GUPTA_SECTION_B_Q2_MAIN.PY
URJA_GUPTA_SECTION_B_Q2_MAIN1.IPYNB
URJA_GUPTA_SECTION_B_Q2_MAIN2.HTML

Q2: Keep Calm and Carry on …

Q2.1) Given the language model parameters above, can you guess the most likely language transition string and corresponding probability for:

a. Cojelo Con Take It Easy

b. Con Take It Easy

c. Easy Con

d. Cojelo Easy Take It

Solution:

Given:-

Language model parameters:

vocabulary = ("cojelo", "con", "take", “it”, “easy”)

languages = ("E", "S")

Sentence start probability :

Ps[‘E’] = 0.6

Ps[‘S’] = 0.4

Transition probability:

Pt[‘S’|’E’] = 0.7

Pt[‘E’|’E’] = 0.3

Pt[‘S’|’S’] = 0.4

Pt[‘E’|’S’] = 0.6

Emission probability :

Pe[‘cojelo’|’E’] = 0.1

Pe[‘con’|’E’] = 0.2

Pe[‘take’|’E’] = 0.3

Pe[‘it’|’E’] = 0.2

Pe[‘easy’|’E’] = 0.2

Pe[‘cojelo’|’S’] = 0.3

Pe[‘con’|’S’] = 0.3

Pe[‘take’|’S’] = 0.15

Pe[‘it’|’S’] = 0.15

Pe[‘easy’|’S’] = 0.1

The basic idea behind the Code is that we iterate through all possible transition states using bitmasks(since there are only two languages we can assume E as 0 and S as 1) . We iterate from 0 to 2^n -1 converting each number to a binary number of length n where n is the number of words in our sentence and calculate the probabilities for each state. At each iteration we update our score if the probability of the current state is greater than the score so far. We also keep track of the transition states where the score was updated lastly. Hence after we reach the end of the loop we get the state with the maximum probability and also it’s probability.
Time complexity of the algorithm is O(n*2^n)

Python Code;-

```
# A function to convert a decimal number n to a binary number with bits = length
def binary(n,length):
    m = n
    bin = []
    while m > 0:
        rem = m%2
```

```
        bin.append(rem)
        m//=2
    while len(bin) < length:
        bin.append(0)
    bin = bin[::-1]
    return bin
inp = list(input().split())
length = len(inp)

# convert each word to lowercase
for i in range(length):
    inp[i] = inp[i].lower()
pow2 = int(pow(2,length))

#dictionaries to keep track of the probabilities
eprobs = {"cojelo":0.1, "con":0.2, "take":0.3, "it": 0.2, "easy":0.2}
sprobs = {"cojelo":0.3, "con":0.3, "take":0.15, "it": 0.15, "easy":0.1}

#Initialize the score to 0 and the state to an empty list
score = 0
state = []

for i in range(pow2):
    #convert to binary with 0 -> E and 1 -> S
    bin = binary(i,length)
    prob = 1
    # calculating the probability that the transition states are binary of i
    # with 0 representing English and 1 representing Spanish.
    for j in range(length):
        if j == 0:
            if bin[j] == 0:
                prob*=0.6
                prob*=eprobs[inp[j]]
            else:
                prob*=0.4
                prob*=sprobs[inp[j]]
        else:
            if bin[j] == 0 and bin[j-1] == 0:
                prob+=0.3*eprobs[inp[j]]
            elif bin[j] == 0 and bin[j-1] == 1:
                prob+=0.6*eprobs[inp[j]]
            elif bin[j] == 1 and bin[j-1] == 0:
                prob+=0.7*sprobs[inp[j]]
            else:
                prob+=0.4*sprobs[inp[j]]

    # updating the maximum probability
    if score < prob:
        score = prob
        state = bin

# rounding off to 5 decimal places
score = (format(score,".5f"))

#print the transition states
```

```
for v in state:
    if v == 0:
        print("E", end=" ")
    else:
        print("S", end=" ")
print(":",score)
```

a.) Now in the Code we will put inp = “Cojelo Con Take It Easy”

Most Likely Transition String = E S E S E

Corresponding Probability = 0.67500

= E S E S E : 0.67500

b.) Now in the Code we will put inp = “Con Take It Easy”

Most Likely Transition String = S E S E

Corresponding Probability = 0.52500

= S E S E : 0.52500

c.) Now in the Code we will put inp= “Easy Con”

Most Likely Transition String = E S

Corresponding Probability = 0.33000

= E S : 0.33000

d.) Now in the Code we will put inp in the code= “Cojelo Easy Take It”

Most Likely Transition String = S E S E

Corresponding Probability = 0.46500

= S E S E : 0.46500

Q2.2. Write a program that takes a string (eg: “Cojelo Con”) as input and prints the most probable language string and the probability of the language string (correct to 5 decimal points) separated by (“ : “).

Solution:

Python Code;-

```
# A function to convert a decimal number n to a binary number with bits = length
def binary(n,length):
    m = n
    bin = []
    while m > 0:
        rem = m%2
        bin.append(rem)
        m//=2
    while len(bin) < length:
        bin.append(0)
    bin = bin[::-1]
```

```
  return bin

inp = list(input().split())
length = len(inp)

# convert each word to lowercase
for i in range(length):
   inp[i] = inp[i].lower()
pow2 = int(pow(2,length))

#dictionaries to keep track of the probabilities
eprobs = {"cojelo":0.1, "con":0.2, "take":0.3, "it": 0.2, "easy":0.2}
sprobs = {"cojelo":0.3, "con":0.3, "take":0.15, "it": 0.15, "easy":0.1}
#Initialize the score to 0 and the state to an empty list
score = 0
state = []

for i in range(pow2):
   #convert to binary with 0 -> E and 1 -> S
   bin = binary(i,length)
   prob = 1

   # calculating the probability that the transition states are binary of i
   # with 0 representing English and 1 representing Spanish.
   for j in range(length):
      if j == 0:
         if bin[j] == 0:
            prob*=0.6
            prob*=eprobs[inp[j]]
         else:
            prob*=0.4
            prob*=sprobs[inp[j]]
      else:
         if bin[j] == 0 and bin[j-1] == 0:
            prob+=0.3*eprobs[inp[j]]
         elif bin[j] == 0 and bin[j-1] == 1:
            prob+=0.6*eprobs[inp[j]]
         elif bin[j] == 1 and bin[j-1] == 0:
            prob+=0.7*sprobs[inp[j]]
         else:
            prob+=0.4*sprobs[inp[j]]
   # updating the maximum probability
   if score < prob:
      score = prob
      state = bin

# rounding off to 5 decimal places
score = (format(score,".5f"))
#print the transition states
for v in state:
   if v == 0:
      print("E", end=" ")
   else:
      print("S", end=" ")
print(":",score)
```