<u>STOCHASTIC SIMULATION TECHNIQUES AND STRUCTURAL RELIABILITY</u>

Assignment 3

Random Processes

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Task 1

Consider a sequence of piles arranged along a line. The piles are load-tested sequentially. Because of the proximity of the piles, if one fails the load test, there is a 40% probability that the next one in the sequence will also fail the test. Conversely, if a pile passes the load test, the probability that the next will also pass the load test is 60%. What is the one-step transition probability matrix for this problem?

Solution:

Given:

- If a pile fails, the next pile fails with probability of 0.4.
- If a pile passes, the next pile passes with probability of 0.6.

Lets,

- State F = The pile fails the load test.
- State P = The pile passes the load test.

If the current pile passes then,

- Probability of the next pile also passes: P(P|P) = 0.6
- Probability of the next pile fails: P(F|P) = 1-.06 = 0.4

If the current pile fails then,

- Probability of the next pile also fails: P(F|F) = 0.4
- Probability of the next pile passes: P(P|F) = 1-0.4 = 0.6

Then the one step transition matrix P is:

$$P = \begin{bmatrix} P(P|P) & P(P|F) \\ P(F|P) & P(F|F) \end{bmatrix}$$

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}$$

Task 2

On any given day, a company undertakes either zero, one, or two site investigations. The next day the number of sites investigated can be either zero, one, or two again, but there is some dependence from day to day. This is a simple threestate, discrete-time Markov chain. Suppose that the one-step transition matrix for this problem appears as follows

(a) *What is the two-step transition matrix for this problem?*

Solution:

Given transition matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0.0 \\ 0.2 & 0.3 & 0.5 \\ 0.0 & 0.4 & 0.6 \end{bmatrix}$$

The n step transition matrix $\mathrm{P}^n = \mathrm{P}^{n-1} * \mathrm{P}$.

Ie. $\mathrm{P}^2 = \mathrm{P} * \mathrm{P}$

$$\mathrm{P}^2 = \begin{bmatrix} 0.55 & 0.3 & 0.15 \\ 0.2 & 0.35 & 0.5 \\ 0.08 & 0.36 & 0.56 \end{bmatrix}$$

(b) *Is state 0 transient or recurrent?*

Solution:

State 0 is transient because the probability of returning to state 0 from state 0 in one step is P_{00}=0.7, which is less than 1. This indicates there is a non-zero probability that the chain might never return to state 0 after leaving it.

Task 3

In an electronic load-measuring system, under certain adverse conditions, the probability of an error on each sampling cycle depends on whether/or not it was preceded by an error. We will define 1 as the error state and 2 as the non-error state. Suppose the probability of an error if preceded by an error is 0.70, the probability of an error if preceded by a non-error is 0.25, and thus the probability of a non-error if preceded by an error is 0.3, and the probability of a non-error if preceded by a non-error is 0.75. 1

(a) *What is the two-step, ..., seven-step transition matrices?*

Solution:

Given transition matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.25 & 0.75 \end{bmatrix}$$

$$\mathrm{P}^{n} = \mathrm{P}^{n-1} * \mathrm{P}$$

Two-Step transition Matrix,

$$\mathrm{P}^{2} = \begin{bmatrix} 0.5650 & 0.4350 \\ 0.3625 & 0.6375 \end{bmatrix}$$

Three-Step transition Matrix,

$$\mathrm{P}^{3} = \begin{bmatrix} 0.5043 & 0.4957 \\ 0.4131 & 0.5869 \end{bmatrix}$$

Four-Step transition Matrix,

$$\mathrm{P}^{4} = \begin{bmatrix} 0.4769 & 0.5231 \\ 0.4359 & 0.5641 \end{bmatrix}$$

Five-Step transition Matrix,

$$P^5 = \begin{bmatrix} 0.4646 & 0.5354 \\ 0.4462 & 0.5538 \end{bmatrix}$$

Six-Step transition Matrix,

$$P^6 = \begin{bmatrix} 0.4591 & 0.5409 \\ 0.4508 & 0.5492 \end{bmatrix}$$

Seven-Step transition Matrix,

$$P^7 = \begin{bmatrix} 0.4566 & 0.5434 \\ 0.4528 & 0.5472 \end{bmatrix}$$

(b) *Does the process converge to a steady state?*

Solution:

By continuing for more steps,

Eight-Step transition Matrix

$$P^8 = \begin{bmatrix} 0.4555 & 0.5445 \\ 0.4538 & 0.5462 \end{bmatrix}$$

Nine-Step transition Matrix

$$P^9 = \begin{bmatrix} 0.4550 & 0.5450 \\ 0.4542 & 0.5458 \end{bmatrix}$$

Ten-Step transition Matrix

$$P^{10} = \begin{bmatrix} 0.4547 & 0.5453 \\ 0.4544 & 0.5456 \end{bmatrix}$$

Eleven-Step transition Matrix

$$P^{11} = \begin{bmatrix} 0.4546 & 0.5454 \\ 0.4545 & 0.5455 \end{bmatrix}$$

Twelve-Step transition Matrix

$$P^{12} = \begin{bmatrix} 0.4546 & 0.5454 \\ 0.4545 & 0.5455 \end{bmatrix}$$

Thirteen-Step transition Matrix,

$$P^{13} = \begin{bmatrix} 0.4546 & 0.5454 \\ 0.4545 & 0.5455 \end{bmatrix}$$

From the last three values of P we can understand that the P has a steady state.

Task 4

Considering the transition diagram in Figure 1 for a three-state discrete-time Markov chain answer the following questions:

(a) *Is state 2 transient or recurrent*

Solution:

From the diagram the transition matrix,

$$P = \begin{bmatrix} 0.5 & 0.1 & 0.4 \\ 0.0 & 1.0 & 0.0 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

Since the probability of remaining in state 2 is less than 1 (0.3) and state 2 does not guarantee a return once left. This indicates that state 2 is not recurrent and is therefore classified as transient.

(b) *Compute the probabilities that, starting in state 0, state 2 is reached for the first time in one, two, or three time steps*

Solution:

The probabilities of reaching state 2 from state 0

In one step,

$$P02 = P(0 \mid 2) = 0.4$$

In two steps:

$$P02 = P(0 \mid 0) * P(0 \mid 2)$$

$$= 0.5 * 0.4 = 0.2$$

In three steps:

$$P02 = P(0 \mid 0) * P(0 \mid 0) * P(0 \mid 2)$$

$$= 0.5 * 0.5 * 0.4 = 0.1$$

c) *Estimate (or make a reasonable guess at) the probability that state 2 is reached from state 0.*

<u>Solution:</u>

By recursive and immediate probabilities:

$$F02 = P02 + (P00 * F02)$$

$$F02 = 0.4 + (0.5 * F02)$$

$$F02 = 0.4 + (0.5 * F02)$$

$$F02 - (0.5 * F02) = 0.4$$

$$F02(1 - 0.5) = 0.4$$

$$F02 = 0.4 / (0.5)$$

$$F02 = 0.8$$

The probability that state 2 from state 0 is 0.8

Task 5

A sequence of soil samples are taken along the line of railway. The samples are tested and classified into three states:

1. *Good*
2. *Fair (needs some remediation)*
3. *Poor (needs to be replaced)*

After taking samples over a considerable distance, the geotechnical engineer in charge notices that the soil classifications are well modeled by a three-state stationary Markov chain with the transition probabilities

(a) *What are the steady-state probabilities?*

Solution:

From the diagram the transition matrix,

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.3 & 0.7 \end{bmatrix}$$

Steady-state probabilities,

$$\pi * P = \pi$$

$$\pi_1 * P_{11} + \pi_2 * P_{21} + \pi_3 * P_{31} = \pi_1$$

$$\pi_1 * P_{12} + \pi_2 * P_{22} + \pi_3 * P_{32} = \pi_2$$

$$\pi_1 * P_{13} + \pi_2 * P_{23} + \pi_3 * P_{33} = \pi_3$$

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Substituting,

$$0.6 * \pi_1 + 0.3 * \pi_2 + 0.0 * \pi_3 = \pi_1$$

$$0.2 * \pi_1 + 0.4 * \pi_2 + 0.3 * \pi_3 = \pi_2$$

$$0.2 * \pi_1 + 0.3 * \pi_2 + 0.7 * \pi_3 = \pi_3$$

This gives,

$$\pi_1 = \frac{3}{4} * \pi_2$$

$$\pi_3 = \frac{3}{2} * \pi_2$$

Solving,

$$\pi_1 = \frac{3}{13}$$

$$\pi_2 = \frac{4}{13}$$

$$\pi_3 = \frac{6}{13}$$

(b) *On average, how many samples must be taken until the next sample to be classified as poor is encountered?*

<u>Solution:</u>

Using geometric distribution, Average number of steps = $\frac{1}{\pi_3}$ = $\frac{13}{6}$ = 2.17 samples

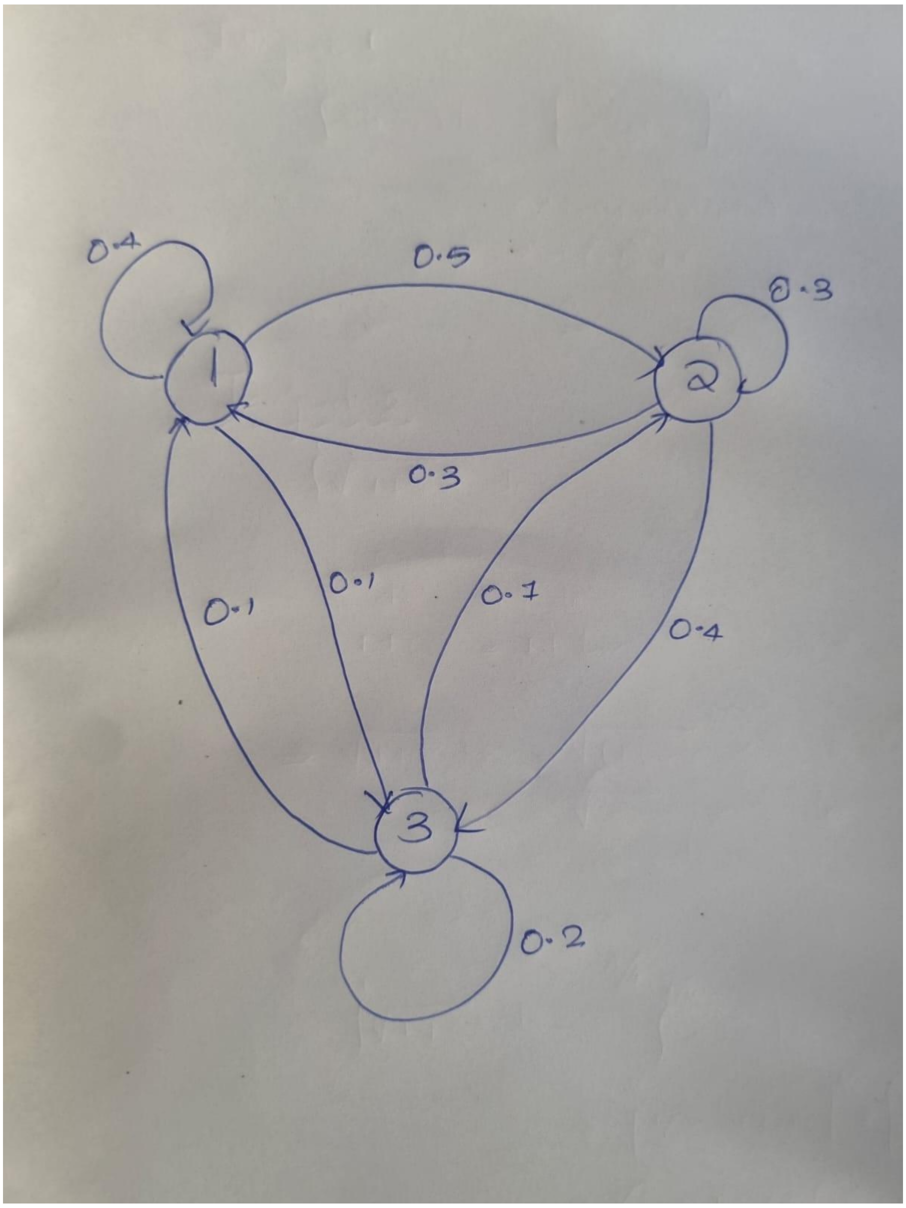
Task 6

The water table at a particular site may be idealized into three states: low, moderate, and high. Because of the probabilistic nature of rainfall patterns, irrigation pumping, and evaporation, the water table level may shift from one state to another between seasons as a Markov chain. The one-step transition matrix is given by:

where low, moderate, and high-water table levels are denoted by states 1, 2, and 3, respectively.

(a) *Draw the transition diagram for this problem.*

Solution:



Fig(1):Transition diagram for one step transition

(b) *Suppose that for season 1 you predict that there is an 80% probability the water table will be high at the beginning of season 1 on the basis of extended weather reports. Also, if it is not high, the water table will be three times as likely to be moderate as it is to be low. On the basis of this prediction, what is the probability that the water table will be high at the beginning of season 2?*

Solution:

P(High), π_1= 0.8

P(Moderate) : P(Low) = 3:1

Then,

P(Moderate), π_2= 3 * P(Low)

P(Low), π_3= x

We Know,

$$\pi_1 + \pi_2 + \pi_3 \ = \ 1$$

Solving,

$$\pi_2 \ = 0.15$$

$$\pi_3 \ = 0.05$$

Initial probability,

$$\pi(0) = \{0.05, \quad 0.15, \quad 0.8\}$$

Then,

$$\pi(1) = \ \pi(0) * P$$

$$\pi(1) = \{\pi_1 \quad \pi_2 \, , \quad \pi_3 \}$$

$$= [0.05, \quad 0.15, \quad 0.8] * \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.7 & 0.2 \end{bmatrix}$$

$$\pi(1) = \ \{0.145, \quad 0.63, \quad 0.225\}$$

(c) *What is the steady-state probability that the water table will be high in any one season?*

Solution:

For steady state,

$$\pi * P = \pi$$

$$\pi_1 * P_{11} + \pi_2 * P_{21} + \pi_3 * P_{31} = \pi_1$$

$$\pi_1 * P_{12} + \pi_2 * P_{22} + \pi_3 * P_{32} = \pi_2$$

$$\pi_1 * P_{13} + \pi_2 * P_{23} + \pi_3 * P_{33} = \pi_3$$

And,

$$\pi_1 + \pi_2 + \pi_3 = 1$$

Substituting,

$$0.4 * \pi_1 + 0.3 * \pi_2 + 0.1 * \pi_3 = \pi_1$$

$$0.5 * \pi_1 + 0.3 * \pi_2 + 0.7 * \pi_3 = \pi_2$$

$$0.1 * \pi_1 + 0.4 * \pi_2 + 0.2 * \pi_3 = \pi_3$$

Solving,

$$\{\pi_1, \quad \pi_2, \quad \pi_3\} = \{0.25, \quad 0.45, \quad 0.30\}$$

The steady-state probability that the water table will be high in any one season is 30%.

Task 7

A bus arrives at its stops either early, on time, or late. If the bus is late at a stop, its probabilities of being early, on time, and late at the next stop are 1/6, 3 2/6, and 3/6, respectively. If the bus is on time at a stop, it is equally likely to be early, on time, or late at the next stop. If it is early at the stop, it is twice as likely to be on time at the next stop as either early or late, which are equally likely.

(a) *Why can this sequence of bus stops be modeled using a Markov chain?*

Solution:

Future state depends only on current state not the past, which is the Markov property. Thus the probabilities of the bus being early, on time, or late at the next stop depend only on its current state (early, on time, or late) and not on any previous states.

(b) *Find the one-step transition matrix P.*

Solution:

The one-step transition matrix,

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

(c) *If the bus is early at the first stop, what is the probability that it is still early at the third stop? What is this probability at the fourth stop?*

Solution:

Two-Step transition Matrix

$$P^2 = \begin{bmatrix} 0.2708 & 0.3750 & 0.3542 \\ 0.2500 & 0.3889 & 0.3611 \\ 0.2361 & 0.3611 & 0.4028 \end{bmatrix}$$

The probability that it is still early at the third stop, P0 = 0.2708

Three-Step transition Matrix

$$P^3 = \begin{bmatrix} 0.2517 & 0.3785 & 0.3698 \\ 0.2523 & 0.3750 & 0.3727 \\ 0.2465 & 0.3727 & 0.3808 \end{bmatrix}$$

The probability that it is still early at the third stop, P0 = 0.2517

***(d)** If the controller estimates the bus to have probabilities of 0.1, 0.7, and 0.2 of being early, on time, or late at the first stop, what now is the probability that the bus is early at the third stop?*

Solution:

$$\{\pi_1, \quad \pi_2, \quad \pi_3\} = [0.1, \quad 0.7, \quad 0.2]$$

So,

$$\pi(n) = \pi(0) * P$$

$$\pi(3) = \pi(0) * P^2$$

$$\{\pi_1, \quad \pi_2, \quad \pi_3\} = [0.1, \quad 0.7, \quad 0.2] * \begin{bmatrix} 0.2708 & 0.3750 & 0.3542 \\ 0.2500 & 0.388 & 0.3611 \\ 0.2361 & 0.3611 & 0.4028 \end{bmatrix}$$

$$\pi(1) = \{0.2493, \quad 0.3819, \quad 0.3687\}$$

The probability of being early at the third stop is 0.2493.

***(e)** After many stops, at what fraction of stops is the bus early on average?*

Solution:

Steady-state for long run average,

$$\pi(1) = \pi(0) * P$$

$$\{\pi_1, \quad \pi_2, \quad \pi_3\} = [\pi_1, \quad \pi_2, \quad \pi_3] * \begin{bmatrix} 0.2500 & 0.5000 & 0.2500 \\ 0.333 & 0.333 & 0.3333 \\ 0.1667 & 0.3333 & 0.5000 \end{bmatrix}$$

$$0.25 * \pi_1 + 0.3333 * \pi_2 + 0.1667 * \pi_3 = \pi_1$$

$$0.50 * \pi_1 + 0.3333 * \pi_2 + 0.3333 * \pi_3 = \pi_2$$

$$0.25 * \pi_1 + 0.3333 * \pi_2 + 0.50 * \pi_3 = \pi_3$$

And,

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\{\pi_1, \quad \pi_2, \quad \pi_3\} = \{0.25, \quad 0.375, \quad 0.375\}$$

So, bus is early at 25% of stops on average.