PROVE.

$$1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n} \quad , n \geq 2$$

1) REARRANGE:

$$\frac{1}{n} + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} < 1$$

2) WE SHOW THAT AS n GROWS, DENOMINATOR WILL BE ALWAYS BIGGER THAN NUMERATOR, MEANING THE EXPRESSION < 1.

$n = 2 \quad \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1$

$n = 3 \quad \frac{1}{3} + \frac{1}{4} + \frac{1}{9}$

$n = 4 \quad \frac{1}{4} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$

$n = 5 \quad \frac{1}{5} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$

3) AS WE CAN SEE THE FIRST ELEMENT OF THE EXPRESSION GETS SMALLER BY:

$\frac{1}{n-1} - \frac{1}{n}$, AND THIS IS ALWAYS BIGGER THAN THE ADDED NEW ELEMENT IN LAST POSITION: $\frac{1}{n^2}$

$$\frac{1}{n-1} - \frac{1}{n} > \frac{1}{n^2} \Rightarrow \frac{n-(n-1)}{n \cdot (n-1)} > \frac{1}{n^2} \Rightarrow$$

$\Rightarrow \frac{1}{n^2 - n} > \frac{1}{n^2}$, $n^2 - n$ is always smaller than n^2, so the expression is always true.

Q.E.D.

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