House Prices Euro Area

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Abstract

Keywords:

JEL Codes:

1 Empirical part

1.1 Notes

- Check whether time to build (time to get building permit) is also important to predict house price variation or house price responses to mp shock on other levels (e.g. within country on city level). Could one do a similar exercise on city level for Germany or US? Has somebody done this? -> check spatial literature. So far we only have OECD paper for OECD member states and our own results for EA.

1.2 Set-up and findings

- Estimate cross-country panel local projections.

$$y_{n,t+h} = \alpha_n^h + \beta^h \epsilon_t^{MP} + \gamma_{inter}^h \epsilon_t^{MP} \times Inter_n + controls + u_{n,t+h}$$

- $y_{n,t}$: Log real property price index, price to rent ratio (in country n).
- ϵ_t^{MP}: High frequency MP shocks. Based on OIS at one year horizon from Altavilla et al.(2019). Applied poor man's approach. Aggregated to quarterly frequency.

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- Controls: 6 lags of LHS variable, log GDP, log HICP, EONIA, MP shock, MP shock X interaction.
- Interactions: Time to get building permit, maximum applicable capital gains tax on housing, LTV constraints, share of HH with ARMs, Homeownership rate, WH2M share.
- Sample: 2000Q1 to 2019Q4.

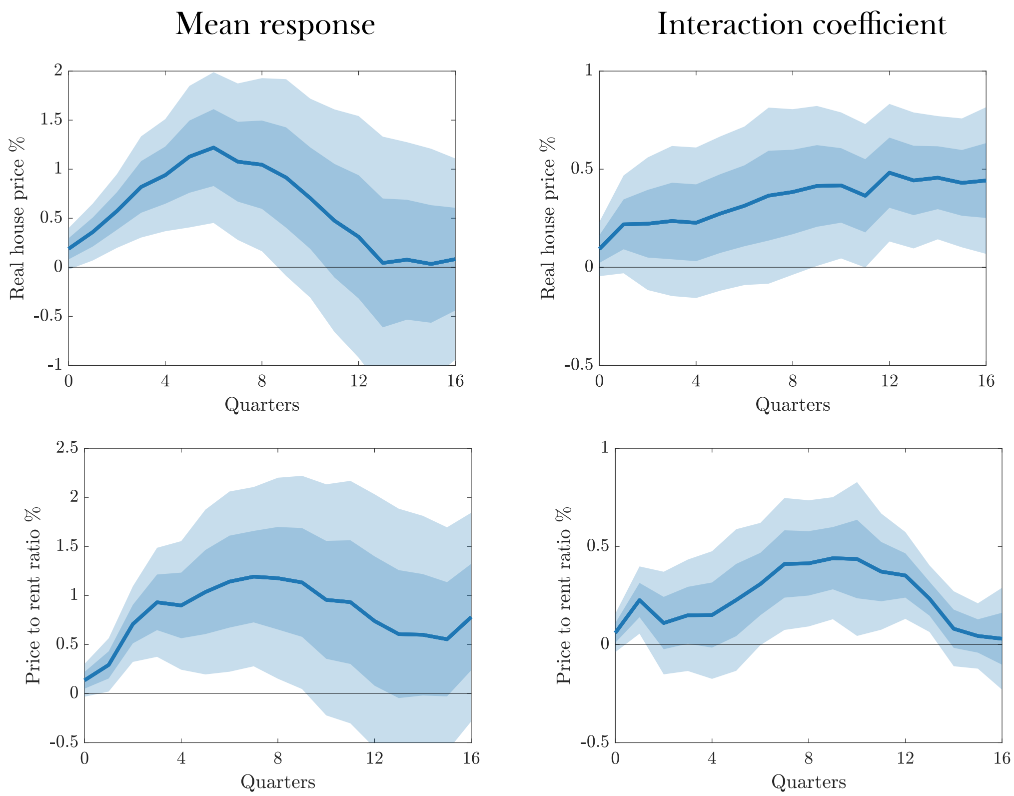

Figure 1: House price and Price to rent responses to an expansionary MP shock. Interaction term: Time to get building permit (World Bank). CI: 68% and 95%, (Driscoll & Kraay).

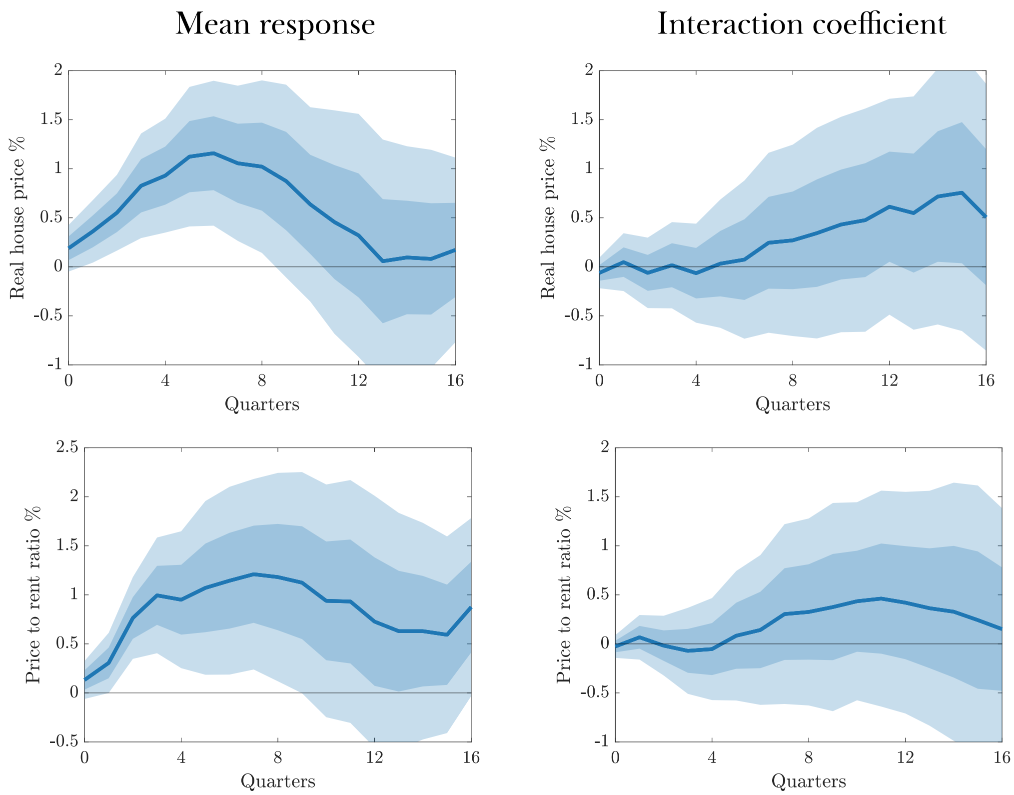

Figure 2: House price and Price to rent responses to an expansionary MP shock. Interaction term: Maximum applicable tax rate - capital gains tax (ECB 2009). CI: 68% and 95%, (Driscoll & Kraay).

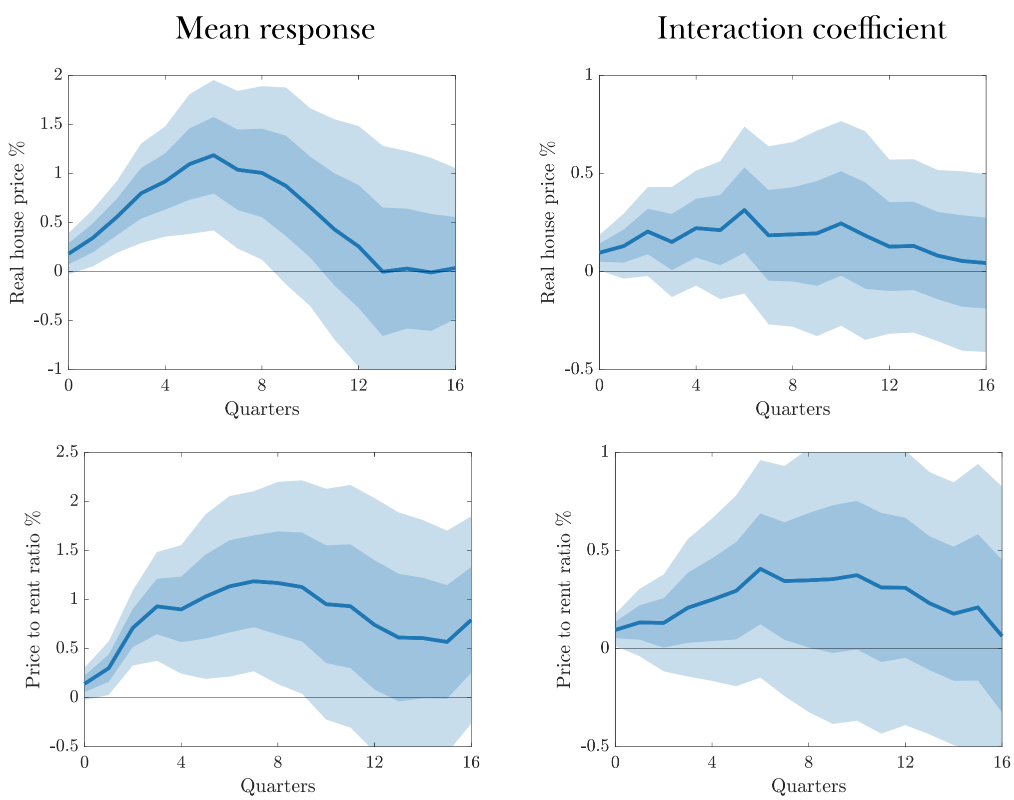

Figure 3: House price and Price to rent responses to an expansionary MP shock. Interaction term: LTV constraints (Catte et al. 2004). CI: 68% and 95%, (Driscoll & Kraay).

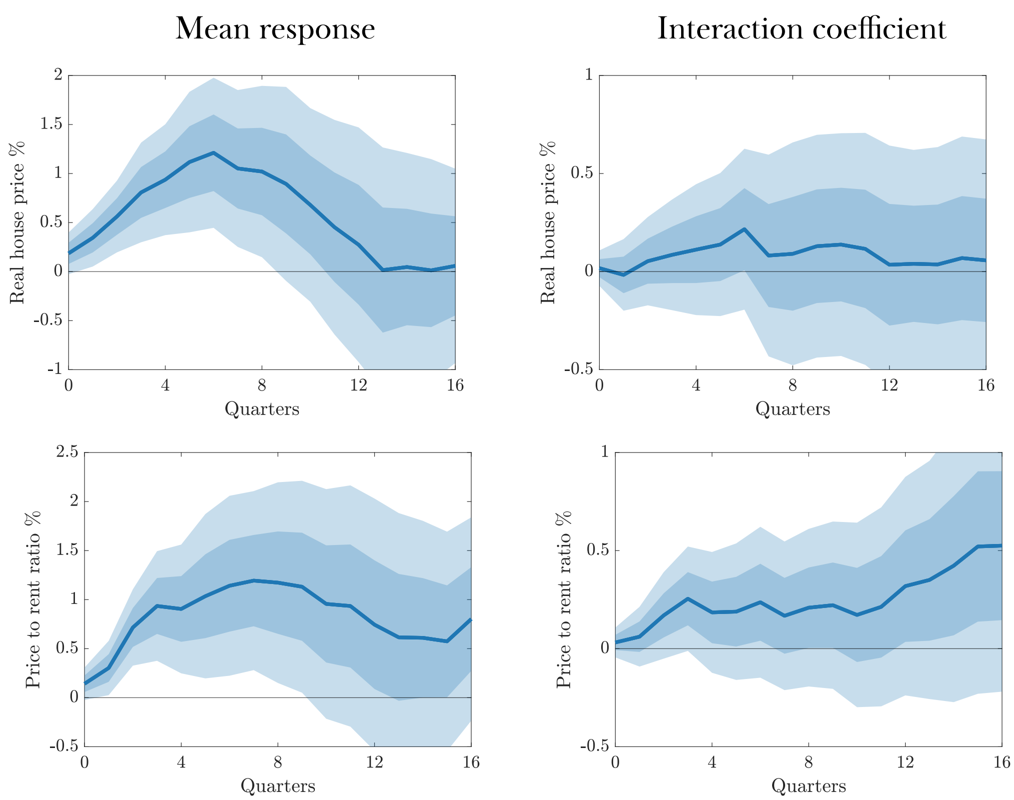

Figure 4: House price and Price to rent responses to an expansionary MP shock. Interaction term: Share of HH with ARM (HFCS). CI: 68% and 95%, (Driscoll & Kraay).

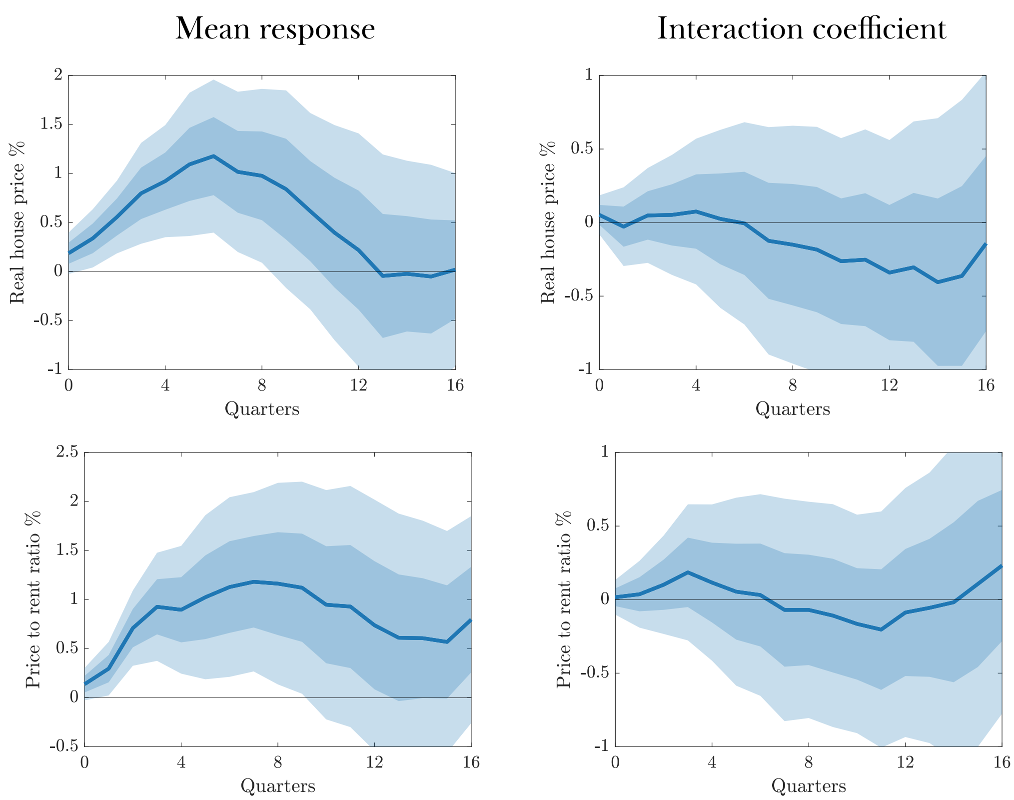

Figure 5: House price and Price to rent responses to an expansionary MP shock. Interaction term: Homeownership rate (HFCS). CI: 68% and 95%, (Driscoll & Kraay).

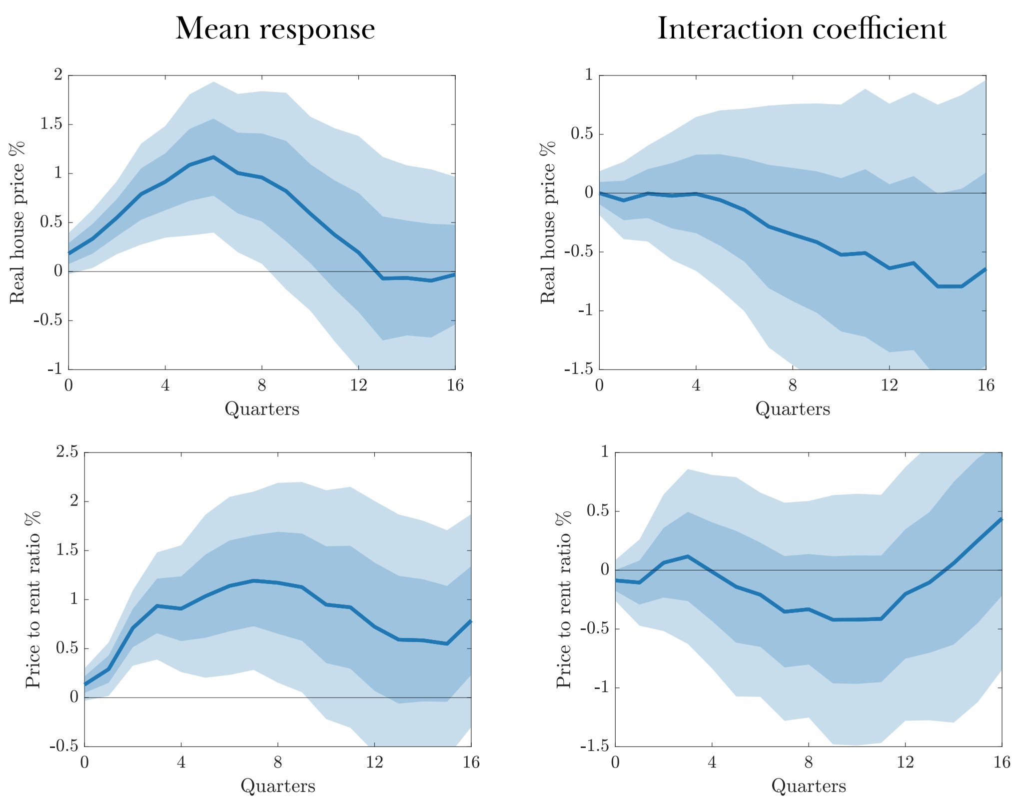

Figure 6: House price and Price to rent responses to an expansionary MP shock. Interaction term: WH2M (HFCS). CI: 68% and 95%, (Driscoll & Kraay).

- Estimate cross-country panel local projections.

$$y_{n,t+h} = \alpha_n^h + \beta^h \epsilon_t^{MP} + \gamma_{inter}^h \epsilon_t^{MP} \times Inter_n + controls + u_{n,t+h}$$

- $y_{n,t}$: Log of gross fixed capital formations for dwellings.
- ϵ_t^{MP}: High frequency MP shocks. Based on OIS at one year horizon from Altavilla et al.(2019). Applied poor man's approach. Aggregated to quarterly frequency.
- Controls: 6 lags of LHS variable, log of real property price index, log HICP, EONIA, MP shock, MP shock X interaction.
- Interactions: Time to get building permit.
- Sample: 2000Q1 to 2019Q4.

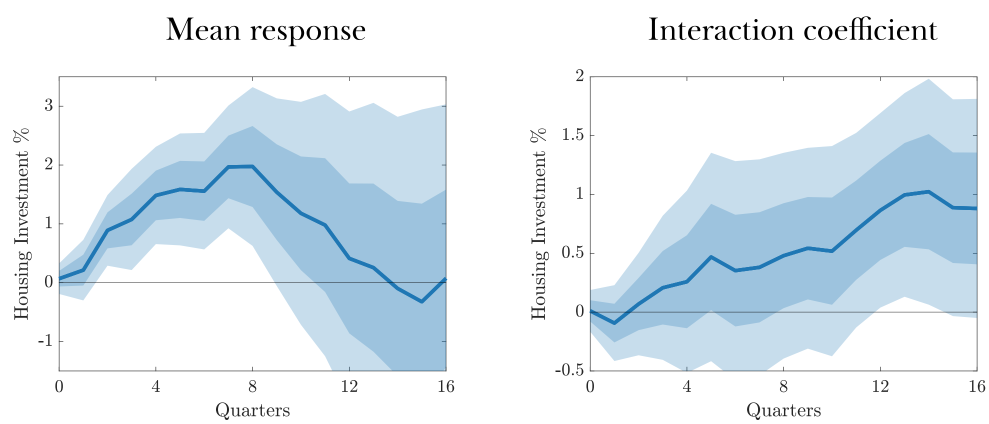

Figure 7: Housing investment response to an expansionary MP shock. Interaction term: Time to get building permit (World Bank). CI: 68% and 95%, (Driscoll & Kraay).

2 Model details (version 5)

In the following we describe our two-country currency-union model with home bias. Foreign variables are denoted by an asterisk (*). So for instance $c^*_{H,t}$ denotes foreign consumption of the domestic good (H for home), while $c^*_{F,t}$ would denote foreign consumption of the foreign good (F). For reasons of expositional brevity, only the home economy is described where the corresponding details for the foreign economy are apparent from the context.

ξ_t is a vector that comprises the shocks in the model, all of which are symmetric between H and F.

2.1 Households

All households in the economy trade a fully contingent set of Arrow-securities *within* the respective country. There is no consumption-risk-sharing *between* countries. This leads to the emergence of a representative domestic and a representative foreign households and has standard implications for the division of firm profits, and the pricing kernel applied by firms. The domestic household derives utility from consuming domestic and foreign varieties, leisure, and housing:

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t u(c_t, h_t, n_t), \quad u(c_t, h_t, n_t) = \frac{\xi_{c,t} c_t^{1-\sigma}}{1-\sigma} + \frac{\xi_{h,t} h_t^{1-\nu}}{1-\nu} - \chi \frac{n_t^{1+\varphi}}{1+\varphi}$$

$$c_t = \left[\lambda^{\varsigma} c_{H,t}^{1-\varsigma} + (1-\lambda)^{\varsigma} c_{F,t}^{1-\varsigma}\right]^{\frac{1}{1-\varsigma}}$$

$$c_{H,t} = \gamma \left[\int_0^1 c_{H,t}(j)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}}, \quad c_{F,t} = (1-\gamma) \left[\int_0^1 c_{F,t}(j^*)^{\frac{\epsilon-1}{\epsilon}} dj^*\right]^{\frac{\epsilon}{\epsilon-1}}$$

where γ is the measure of households in the home economy and λ is a preference-weighting parameter on the consumption of domestic varieties. Following Benigno (2004), γ is simultaneously the economic size of the home region, i.e. the mass of variety-producing firms. Conversely, $1-\gamma$ is the mass of consumers (and the mass of variety-producers) in region F. Analogously to country H, consumers in F aggregate the home- and foreign-good, albeit with different weights: $c_t^* = [(1-\lambda^*)^{\varsigma} c_{H,t}^{1-\varsigma} + (\lambda^*)^{\varsigma} c_{F,t}^{1-\varsigma}]^{\frac{1}{1-\varsigma}}$. Home bias arises if $\lambda, 1-\lambda^* \neq \gamma$. Throughout the paper we maintain the assumptions that any home bias is symmetric: $\gamma(1-\lambda) = (1-\gamma)(1-\lambda^*)$, and not in the 'wrong' direction, $\lambda \geq \gamma$.

The household solves standard utility maximization programs to find the optimal variety consumption pattern. We denote with $P_{H,t}(j)$ the price of one unit of domestic variety j in units of the union-wide numéraire (the union-wide currency) and with $P_{F,t}(j^*)$ the price of foreign variety j^*. For instance, the optimal consumption of domestic varieties, given some maximal expenditure E (i.e. the residual of the budget constraint of the envelope program), solves:

$$\max_{(c_{H,t}(j))_{j\in[0,1]}} \gamma \left[\int_0^1 c_{H,t}(j)^{\frac{\epsilon-1}{\epsilon}} dj\right]^{\frac{\epsilon}{\epsilon-1}} \quad \text{s.t.} \quad \gamma \int_0^1 P_{H,t}(j) c_{H,t}(j) dj = E$$

$$\text{FOC:} \quad (c_{H,t}^{\text{opt}})^{1/\epsilon} \gamma^{\frac{\epsilon-1}{\epsilon}} c_{H,t}(j)^{-1/\epsilon} = \lambda P_{H,t}(j), \ \forall j$$

$$c_{H,t}(j) = (\lambda P_{H,t}(j))^{-\epsilon} c_{H,t}^{\text{opt}} \cdot \gamma^{\epsilon-1}$$

plugging this into the definition of $c_{H,t}^{\text{opt}}$ and solving for λ, and plugging back in:

$$\implies \forall j,\ c_{H,t}^{\text{opt}}(j) = \left[\frac{P_{H,t}(j)}{P_{H,t}}\right]^{-\epsilon} \cdot c_{H,t}^{\text{opt}} \cdot \gamma^{-1}, \quad P_{H,t} := \left[\int_0^1 P_{H,t}(j)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}},$$

$$c_{H,t}^{\text{opt}} = \frac{E}{P_{H,t}}$$

where the last equality follows from plugging the first equality into the budget constraint and solving for $c_{H,t}^{\text{opt}}$. The last equality is then, again standard procedure, used to substitute out all terms $\gamma \int_0^1 P_{H,t}(j) c_{H,t}(j) dj \equiv c_{H,t} P_{H,t}$ in the envelope program below. Exactly analogous logic delivers the following equalities, price indices, and demand schedules:

$$\gamma \int_0^1 P_{H,t}(j) c_{H,t}(j) dj = c_{H,t} P_{H,t}, \quad P_{H,t} = \left[\int_0^1 P_{H,t}(j)^{1-\epsilon} dj\right]^{\frac{1}{1-\epsilon}},$$

$$c_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} \frac{c_{H,t}}{\gamma},$$

$$(1-\gamma) \int_0^1 P_{F,t}(j^*) c_{F,t}(j^*) dj^* = c_{F,t} P_{F,t}, \quad P_{F,t} = \left[\int_0^1 P_{F,t}(j^*)^{1-\epsilon} dj^*\right]^{\frac{1}{1-\epsilon}},$$

$$c_{F,t}(j^*) = \left(\frac{P_{F,t}(j^*)}{P_{F,t}}\right)^{-\epsilon} \frac{c_{F,t}}{1-\gamma},$$

$$c_{H,t} P_{H,t} + c_{F,t} P_{F,t} = P_t c_t, \quad P_t = \left[\lambda P_{H,t}^{1-\frac{1}{\varsigma}} + (1-\lambda) P_{F,t}^{1-\frac{1}{\varsigma}}\right]^{\frac{1}{1-\frac{1}{\varsigma}}},$$

$$c_{H,t} = \left(\frac{P_{H,t}}{P_t}\right)^{-\frac{1}{\varsigma}} c_t \lambda, \quad c_{F,t} = \left(\frac{P_{F,t}}{P_t}\right)^{-\frac{1}{\varsigma}} c_t (1-\lambda),$$

$$c_{H,t}^* P_{H,t} + c_{F,t}^* P_{F,t} = P_t^* c_t, \quad P_t^* = \left[(1-\lambda^*) P_{H,t}^{1-\frac{1}{\varsigma}} + \lambda^* P_{F,t}^{1-\frac{1}{\varsigma}}\right]^{\frac{1}{1-\frac{1}{\varsigma}}},$$

$$c_{H,t}^* = \left(\frac{P_{H,t}}{P_t^*}\right)^{-\frac{1}{\varsigma}} c_t^* (1-\lambda^*), \quad c_{F,t}^* = \left(\frac{P_{F,t}}{P_t^*}\right)^{-\frac{1}{\varsigma}} c_t^* \lambda^*.$$

As usual with nested CES structures, the upper-most-level consumption c_t determines the consumption levels of all relevant lower levels, whence we can ignore them in the envelope program.

The envelope program of the domestic household finally is:

$$\max_{\{c_t,h_t,n_t,b_{t+1},k_t\}_{t\geq 0}} \quad E_0^{\mathscr{P}} \sum_{t=0}^{\infty} \beta^t u(c_t, h_t, n_t)$$
$$\text{s.t.} \quad c_t + \widetilde{P}_t^h(h_t - (1-\delta^h)h_{t-1}) + b_{t+1} + \Psi(b_{t+1}) + k_t =$$
$$w_t n_t + \frac{1+i_{t-1}}{\Pi_t} b_t + d(k_{t-\tau}, \xi_{t-\tau}) \cdot \widetilde{P}_t^h - T_t + \Sigma_t$$

The budget constraint is expressed in units of the country-H final composite consumption good, c, with $\Pi_t := P_t/P_{t-1}$. Σ_t are profits from domestic firms, which are owned evenly by all domestic households, w_t is the real wage, T_t are other transfers, and b_t is a one-period nominal zero-coupon bond (expressed here in units of country-H final composite) that pays i_{t-1} units of union-wide currency at date t. $\Psi : b \mapsto \frac{\psi_b}{2}(b - \underline{b})^2$ is a convex bond holding cost. Bond holding costs are a pure equilibrium selection device and will be chosen to be small (by $\psi_b \downarrow 0$).[1] $\widetilde{P}_t^h$ is the real house price, i.e. the price of one unit of housing in units of country-H final composite. k_t is the amount of consumption units dedicated to the production of new housing units, $d(k_{t-\tau}, \xi_{t-\tau}) = \xi_{k,t-\tau} \frac{k_{t-\tau}^{\eta}}{\eta}, \eta \in (0, 1)$, that can be retained to enjoy housing services, or sold. The time lag τ captures time to build: investing in housing production now may only generate new units after some time τ – the amount of new units produced is known at time of investment, though. The program for household F is symmetric, *mutatis mutandis*.

The problem, phrased with the following Lagrangian, yields the following first order conditions (ignoring the budget constraint temporarily):

$$\mathscr{L} \equiv E_0^{\mathscr{P}} \sum_{t=0}^{\infty} \beta^t \left[u(c_t, h_t, n_t) + \lambda_t \begin{pmatrix} w_t n_t + \frac{(1+i_{t-1})}{\Pi_t} b_t + d(k_{t-\tau}, \xi_{t-\tau}) \cdot \widetilde{P}_t^h - T_t + \Sigma \\ -c_t - \widetilde{P}_t^h (h_t - (1-\delta^h)h_{t-1}) - b_{t+1} - \Psi(b_{t+1}) - k_t \end{pmatrix} \right],$$

$$\forall t, \quad (c) \quad u_c'(c_t, h_t, n_t) =: u_{c,t}' = \lambda_t,$$

[1] Similar to small open economy models, the first order dynamics of our model feature a unit root, absent bond holding cost (cf. Schmitt-Grohé and Uribe, 2002, for a discussion). In contrast to models of small open economies, the non-stochastic steady state of our model uniquely exits even in the absence of bond holding costs. (This is because our economy is essentially closed, and steady-state bond holdings are therefore pinned down by resource constraints; another way to see this is to recall that precisely because of the absence of uncertainty, the non-stochastic steady-state of a currency-union-model with and one without Arrow-Securities must be the same; therefore, since the steady-state of a currency-union-model with Arrow-Securities uniquely exists, it also uniquely exists in an economy without AS.) Therefore, bond holding costs are only introduced to render the first order dynamics of the model stationary, and the anchor points of the bond holding cost function, $\underline{b}$, $\underline{b}^*$, are chosen so as to ensure existence of a steady state where bond holding costs are zero; see section 2.7.

$$
\begin{aligned}
&(h) \quad u'_{h,t} - \lambda_t \widetilde{P}^h_t + \beta(1-\delta^h)\mathbb{E}^{\mathcal{P}}_t \left\{\lambda_{t+1}\widetilde{P}^h_{t+1}\right\} = 0, \\
&(n) \quad u'_{n,t} = -\lambda_t w_t, \\
&(b) \quad \lambda_t(1+\Psi'(b_{t+1})) = \beta\mathbb{E}^{\mathcal{P}}_t \left\{\lambda_{t+1}\frac{1+i_t}{\Pi_{t+1}}\right\}, \\
&(k) \quad \beta^\tau \mathbb{E}^{\mathcal{P}}_t \left\{\lambda_{t+\tau}\widetilde{P}^h_{t+\tau}\right\} \cdot d'_k(k_t,\xi_t) = \lambda_t.
\end{aligned}
$$

After using (c) to eliminate the Lagrange-multipliers, and defining the risk-neutral house price $P^h_t := \widetilde{P}^h_t \cdot u'_{c,t}$ this becomes:

$$
\begin{aligned}
&(h) \quad P^h_t = u'_{h,t} + \beta(1-\delta^h)\mathbb{E}^{\mathcal{P}}_t \left\{P^h_{t+1}\right\}, \\
&(n) \quad -\frac{u'_{n,t}}{u'_{c,t}} = w_t, \\
&(b) \quad 1 = \beta\mathbb{E}^{\mathcal{P}}_t \left\{\frac{u'_{c,t+1}}{u'_{c,t}}\frac{1+i_t}{\Pi_{t+1}}(1+\Psi'(b_{t+1}))^{-1}\right\}, \\
&(k) \quad \beta^\tau \mathbb{E}^{\mathcal{P}}_t \left\{P^h_{t+\tau}\right\} \cdot d'_k(k_t,\xi_t) = u'_{c,t}.
\end{aligned}
\tag{1}
$$

After discussing the belief setup, we detail these equations further.

2.2 Subjective House Price Model

Denote any expectation that generalizes RE as $\mathbb{E}^{\mathcal{P}}$. Agents perceive risk-neutral house prices to follow a simple state-space model

$$
\begin{aligned}
\ln\frac{P^h_{t+1}}{P^h_t} &= \ln m_{t+1} + \ln\varepsilon_{t+1} \\
\ln m_{t+1} &= \ln m_t + \ln\nu_{t+1} \\
\begin{pmatrix}\ln\varepsilon_t & \ln\nu_t\end{pmatrix}' &\sim \mathcal{N}\left(\begin{pmatrix}-\frac{\sigma^2_\varepsilon}{2} & -\frac{\sigma^2_\nu}{2}\end{pmatrix}, \begin{pmatrix}\sigma^2_\varepsilon & 0 \\ 0 & \sigma^2_\nu\end{pmatrix}\right)
\end{aligned}
\tag{2}
$$

That is, house price growth rates are perceived as being the sum of a perfectly transitory and a perfectly persistent component. Crucially, $\ln\varepsilon_t$, $\ln\nu_t$ are not observable to the agents, rendering $\ln m_t$ un-observable. Agents, being optimizing given their perceived information and given their prior beliefs[2] in $t = 0$, apply the optimal Bayesian filter to arrive at the observable system,

$$\begin{aligned}\ln \frac{P^h_{t+1}}{P^h_t} &= \ln \overline{m}_t + \ln z_{t+1} \\ \ln \overline{m}_t &= \ln \overline{m}_{t-1} - \frac{\sigma_\nu^2}{2} + g \cdot \left(z_t + \frac{\sigma_\varepsilon^2 + \sigma_\nu^2}{2} \right)\end{aligned}$$

where $\ln \overline{m}_t := \mathbb{E}^{\mathcal{P}}_t(\ln m_t)$ is the posterior mean, $g = \frac{\sigma^2 + \sigma_\nu^2}{\sigma^2 + \sigma_\nu^2 + \sigma_\varepsilon^2}$ is the steady-state Kalman filter, $\sigma^2 = \frac{1}{2}[-\sigma_\nu^2 + \sqrt{\sigma_\nu^4 + 4\sigma_\nu^2 \sigma_\varepsilon^2}]$ is the steady-state Kalman filter uncertainty, and z_t is perceived to be a white noise process.

To avoid simultaneity in the house price[3] we modify the belief setup following Adam, Marcet & Beutel (2017),[4] to yield the same observable system but with lagged information being used in the posterior mean updating equation:

$$\ln \overline{m}_t = (1-g)\left(\ln \overline{m}_{t-1} - \frac{\sigma_\nu^2}{2} \right) + g \left(\ln \frac{P^h_{t-1}}{P^h_{t-2}} + \frac{\sigma_\varepsilon^2}{2} \right) \tag{3}$$

so that the posterior mean is pre-determined.

We may now derive the posterior mean on the τ-periods-ahead price:

$$\begin{aligned}\forall \tau \geq 1, \quad \mathbb{E}^{\mathcal{P}}_t P^h_{t+\tau} &= P^h_t \cdot \mathbb{E}^{\mathcal{P}}_t \frac{P^h_{t+\tau}}{P^h_t} = P^h_t \cdot \mathbb{E}^{\mathcal{P}}_t \prod_{s=1}^{\tau} \beta_{t+s} \varepsilon_{t+s} \\ &= P^h_t \cdot \mathbb{E}^{\mathcal{P}}_t \prod_{s=1}^{\tau} \left(m_t \prod_{k=1}^{s} \nu_{t+k} \right) \varepsilon_{t+s} \\ &= P^h_t \cdot \mathbb{E}^{\mathcal{P}}_t \left[m_t^\tau \prod_{s=1}^{\tau} \nu_{t+k}^{\tau+1-s} \right] \underbrace{\mathbb{E}^{\mathcal{P}}_t \prod_{s=1}^{\tau} \varepsilon_{t+s}}_{=1} \\ &= P^h_t \cdot \mathbb{E}^{\mathcal{P}}_t \left[\prod_{s=1}^{\tau} \nu_{t+s}^{\tau+1-s} \right] \underbrace{\mathbb{E}^{\mathcal{P}}_t (m_t^\tau)}_{= \mathbb{E}^{\mathcal{P}}_t(\exp(\tau \ln m_t)) = \exp\left(\tau \ln \overline{m}_t + \tau^2 \frac{\sigma^2}{2}\right)}\end{aligned}$$

[2] We assume agents' prior variance equals the steady-state Kalman variance.

[3] P^h_t appears twice: once in the forecast equation $\ln \frac{P^h_{t+1}}{P^h_t} = \ln \overline{m}_t + \ln z_{t+1}$ that is used together with the pricing agent's optimality condition to pin down P_t, and it appears in the Kalman-updating equation, $\ln \overline{m}_t = \ln \overline{m}_{t-1} - \frac{\sigma_\nu^2}{2} + g \cdot \left(z_t + \frac{\sigma_\varepsilon^2 + \sigma_\nu^2}{2} \right)$ through z_t. Since P_t depends on $_t$, but the latter also depends on the former, it is not assured that at any point an equilibrium asset price exists and whether it is unique. See Adam, Marcet & Beutel (2017) for the details.

[4] The idea is to modify agents' perceived information setup in that they observe each period one component of the lagged transitory price growth.

$$
\begin{aligned}
&= P_t^h \cdot \exp\left(\tau \ln \overline{m}_t + \tau^2 \frac{\sigma^2}{2}\right) \cdot \mathbb{E}_t^{\mathcal{P}} \exp\left[\sum_{s=1}^{\tau} (\tau + 1 - s) \ln \nu_{t+s}\right] \\
\Longleftrightarrow \quad \mathbb{E}_t^{\mathcal{P}} P_{t+\tau}^h &= P_t^h \cdot \exp\left(\tau \ln \overline{m}_t + \varrho(\tau)\right)
\end{aligned}
$$

where $\varrho(\tau) = \tau^2 \frac{\sigma^2}{2} + (\tau - 1)\frac{\tau(\tau+1)}{3}\frac{\sigma_\nu^2}{2}$ is a time-invariant model parameter summarizing the adjustment of the posterior expectation to risk in the posterior estimate of the current realization of the persistent growth rate component ($\tau^2 \frac{\sigma^2}{2}$) and in the innovations to future persistent growth rate components ($(\tau - 1)\frac{\tau(\tau+1)}{3}\frac{\sigma_\nu^2}{2}$).[5] It can be made arbitrarily small by setting σ_ν small, as would be the case in a typical parametrization.

2.3 Explicit household FOC

Our functional form assumptions allow us to write the FOC (1) as:

$$
\begin{aligned}
&(h) \quad \widetilde{P}_t^h = \frac{\xi_{h,t} h_t^{-\nu}}{\xi_{c,t} c_t^{-\sigma}} + \beta(1 - \delta^h)\mathbb{E}_t^{\mathcal{P}} \left\{ \frac{\xi_{c,t+1}}{\xi_{c,t}} \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \widetilde{P}_{t+1}^h \right\}, \\
&(n) \quad \frac{\chi n_t^{\varphi}}{\xi_{c,t} c_t^{-\sigma}} = w_t, \\
&(b) \quad 1 + \psi_b(b_{t+1} - \underline{b}) = \beta \mathbb{E}_t^{\mathcal{P}} \left\{ \frac{\xi_{c,t+1}}{\xi_{c,t}} \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \frac{1 + i_t}{\Pi_{t+1}} \right\}, \\
&(k) \quad 1 = \beta^{\tau} \mathbb{E}_t^{\mathcal{P}} \left\{ \left(\frac{c_t}{c_{t+\tau}}\right)^{\sigma} \frac{\xi_{c,t+\tau}}{\xi_{c,t}} \cdot \widetilde{P}_{t+\tau}^h \right\} \cdot \xi_{k,t} k_t^{\eta - 1}
\end{aligned}
\tag{4}
$$

[5] The last line obtains by the following argument: (1) $\mathbb{E}_t^{\mathcal{P}} \exp\left[\sum_{s=1}^{\tau} (\tau + 1 - s) \ln \nu_{t+k}\right] = \exp\left[\sum_{s=1}^{\tau} (\tau + 1 - s)(-\frac{\sigma_\nu^2}{2}) + \frac{1}{2}\sum_{s=1}^{\tau} (\tau + 1 - s)^2 \sigma_\nu^2\right]$, since ν_{t+s} are i.i.d. log-normally distributed random variables; (2) $\sum_{s=1}^{\tau} (\tau + 1 - s) = \tau^2 + \tau - \frac{\tau(\tau+1)}{2} = \frac{\tau(\tau+1)}{2}$; (3) $\sum_{s=1}^{\tau} (\tau + 1 - s)^2 = \sum_{s=1}^{\tau} [(\tau + 1)^2 - 2(\tau + 1)s + s^2] = (\tau + 1)^2 \tau - 2(\tau + 1)\frac{\tau(\tau+1)}{2} + \sum_{s=1}^{\tau} s^2 = \frac{\tau(\tau+1)(2\tau+1)}{6}$; (4) $\frac{\tau(\tau+1)(2\tau+1)}{6} - \frac{\tau(\tau+1)}{2} = \tau(\tau + 1)\frac{2\tau+1-3}{6} = \tau(\tau + 1)\frac{2(\tau-1)}{6}$.

and the belief specification of the last section turns this into:

$$
\begin{aligned}
&(h) \quad P^h_t = \frac{\xi_{h,t} h_t^{-\nu}}{1 - \beta\,(1-\delta^h)\exp\left(\ln \overline{m}_t + \frac{\sigma^2}{2}\right)},\\
&(n) \quad \frac{\chi n_t^{\varphi}}{\xi_{c,t} c_t^{-\sigma}} = w_t,\\
&(b) \quad 1 + \psi_b(b_{t+1} - \underline{b}) = \beta \mathbb{E}^{\mathcal{P}}_t \left\{ \frac{\xi_{c,t+1}}{\xi_{c,t}} \left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \frac{1+i_t}{\Pi_{t+1}} \right\}, \qquad (5)\\
&(k) \quad k_t = \left[\beta^{\tau} P^h_t \cdot \exp\left(\tau \ln \overline{m}_t + \varrho(\tau)\right) \cdot \frac{\xi_{k,t}}{\xi_{c,t}} c_t^{\sigma}\right]^{\frac{1}{1-\eta}}\\
&\qquad\quad = \left[\beta^{\tau} \cdot \frac{\xi_{k,t}\xi_{h,t}}{\xi_{c,t}} \cdot \frac{c_t^{\sigma}}{h_t^{\nu}} \cdot \frac{\exp\left(\tau \ln \overline{m}_t + \varrho(\tau)\right)}{1 - \beta\,(1-\delta^h)\exp\left(\ln \overline{m}_t + \frac{\sigma^2}{2}\right)}\right]^{\frac{1}{1-\eta}}
\end{aligned}
$$

where $\overline{m}_t$ evolves according to (3). We can see from equation (5)-(k) that an increase in the time to build τ c.p. lowers the level of optimal capital investment via β^{τ} and increases it via $\exp\left(\tau \ln \overline{m}_t + \varrho(\tau)\right)$.

2.4 Consumption good production and price setting (Rotemberg)

All derivations are analogous for the j^*-producers.

Preliminaries: Relation between CES aggregator in HH section and production side
Note: This part largely builds on Bletzinger and von Thadden (2018, 2021). Firms face adjustment costs à la Rotemberg. If they adjust prices they need to pay an adjustment cost in the form of domestic goods which they buy from domestic firms according to the following demand function:

$$
\Phi_t(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} \Phi_t, \qquad \Phi_t = \frac{\kappa}{2}\left(\Pi_{H,t} - 1\right)^2 y_{H,t}
$$

Define, for notational convenience,

$$
x_{H,t}(j) := c_{H,t}(j) + k_{H,t}(j) + \Psi_H[b_{t+1}](j),
$$

as the total demand for good (H, j) from a H-consumer. (Analogously for the other variables.) Household optimality implies $k_{H,t}(j) = (\frac{P_{H,t}(j)}{P_{H,t}})^{-\epsilon} k_{H,t} \gamma^{-1}$, $k_{H,t} = (\frac{P_{H,t}}{P_t})^{-\frac{1}{\varsigma}} k_t \lambda$, and so on.

Goods market clearing for good j requires:

$$y_{H,t}(j) = \gamma x_{H,t}(j) + (1-\gamma)x^*_{H,t}(j) + \Phi_t(j)$$

Goods market clearing across all goods markets requires:

$$y_{H,t} := \int_0^1 y_{H,t}(j)dj = \gamma \int_0^1 x_{H,t}(j)dj + (1-\gamma)\int_0^1 x^*_{H,t}(j)dj + \int_0^1 \Phi_t(j)dj$$

As firms are symmetric and set the same price, we get:

$$y_{H,t} = \gamma^{-1}[\gamma x_{H,t} + (1-\gamma)x^*_{H,t} + \gamma\Phi_t] \tag{6}$$

And it follows that each variety-producer faces the same demand curve:

$$y_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} y_{H,t}$$

Hence, we can now apply Rotemberg in the usual manner.

Price adjustments
Firms solve the following problem:

$$\max_{P_{H,t}(j)} \mathbb{E}_0^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{P_t}\left[P_{H,t}(j)y_{H,t}(j) - W_t n_t(j) - P_{H,t}\frac{\kappa}{2}(\frac{P_{H,t}(j)}{P_{H,t-1}(j)} - 1)^2 y_{H,t}\right]$$
$$s.t. \quad y_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}}\right)^{-\epsilon} y_{H,t}$$

with that $y_{H,t}(j) = \xi_{a,t}n_t(j)$, and $\Lambda_t = u'_{c,t}/u'_{c,0}$. (We assume all private sector expectations to be identical; crucially this fact is not common knowledge among agents.)
In symmetric equilibrium ($P_{H,t}(j) = P_{H,t}\forall j$) and using the following definitions for real wage

and inflation: $w_t = \frac{W_t}{P_t}$, $\Pi_{H,t} = \frac{P_{H,t}}{P_{H,t-1}}$. One gets the following FOC:[6]

$$(1-\epsilon) + \epsilon \frac{w_t P_t}{\xi_{a,t} P_{H,t}} - \kappa(\Pi_{H,t} - 1)\Pi_{H,t} = -\mathbb{E}_t^{\mathcal{P}} \left[\beta \frac{\Lambda_{t+1} y_{H,t+1}}{\Lambda_t y_{H,t} \Pi_{t+1}} \kappa(\Pi_{H,t+1} - 1)\Pi_{H,t+1}^2 \right].$$

2.5 Market Clearing

Goods market clearing:

$$\left(1 - \frac{\kappa}{2}(\Pi_{H,t} - 1)^2\right) y_{H,t}\gamma = \gamma x_{H,t} + (1-\gamma)x_{H,t}^*$$
$$\left(1 - \frac{\kappa}{2}(\Pi_{F,t} - 1)^2\right) y_{F,t}^*(1-\gamma) = \gamma x_{F,t} + (1-\gamma)x_{F,t}^*$$

where we used the definition of Φ_t and equation (6) to plug in the Rotemberg-resource-cost. Moreover,

$$y_{H,t} = \xi_{a,t} n_t, \quad y_{F,t}^* = \xi_{a,t} n_t^*.$$

Bond market clearing:

$$\gamma P_t b_{t+1} + (1-\gamma) P_t^* b_{t+1}^* = 0.$$

The country-specific price indices enter because bonds are nominal, i.e. they make coupon payments in, and are cleared in units of union-wide currency (the model numéraire), but the variables b_t, b_t^* encode amounts of bond held in units of country-specific final composite *consumption good*. Since, in the face of home-bias, the two countries have differing final consumption aggregators, the consumption-unit representation of bonds has to be translated into the currency-unit representation through multiplication with the respective price level. Without home bias this distinction is immaterial, since both countries have the same final aggregator and the same price level.

Market clearing in the housing sectors is given by:

$$d(k_{t-\tau}, \xi_{t-\tau}) = (h_t - (1-\delta^h)h_{t-1}),$$
$$d(k_{t-\tau^*}^*, \xi_{t-\tau^*}^*) = (h_t^* - (1-\delta^h)h_{t-1}^*).$$

[6] Plug $y_{H,t}(j) = (\frac{P_{H,t}(j)}{P_{H,t}})^{-\epsilon} y_{H,t}$ and $y_{H,t}(j) = \xi_{a,t} n_t(j)$ into the profit function to receive $\mathbb{E}_0^{\mathcal{P}} \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{P_t} [P_{H,t}(j)^{1-\epsilon} P_{H,t}^{\epsilon} y_{H,t} - W_t \xi_{a,t}^{-1} P_{H,t}(j)^{-\epsilon} P_{H,t}^{\epsilon} y_{H,t} - \frac{\kappa}{2}(\frac{P_{H,t}(j)}{P_{H,t-1}(j)} - 1)^2 P_{H,t} y_{H,t}]$; then, take the ∂ with respect to $P_{H,t}(j)$ and obtain $\frac{\Lambda_t}{P_t}\left[(1-\epsilon)P_{H,t}(j)^{-\epsilon} P_{H,t}^{\epsilon} y_{H,t} + \epsilon W_t \xi_{a,t}^{-1} P_{H,t}(j)^{-\epsilon-1} P_{H,t}^{\epsilon} y_{H,t} - \kappa(\frac{P_{H,t}(j)}{P_{H,t-1}(j)} - 1)\frac{P_{H,t}(j)}{P_{H,t-1}(j)} y_{H,t}\right] = -\beta \mathbb{E}_t^{\mathcal{P}} \frac{\Lambda_{t+1}}{P_{t+1}} \left[\kappa(\frac{P_{H,t+1}(j)}{P_{H,t}(j)} - 1)\frac{P_{H,t+1}(j)}{P_{H,t}(j)^2} P_{H,t+1} y_{H,t}\right]$; Now, use $P_{H,t}(j) = P_{H,t} \forall j$, and $w_t = \frac{W_t}{P_t}, \Pi_{H,t} = \frac{P_{H,t}}{P_{H,t-1}}$: $(1-\epsilon)y_{H,t} + \epsilon w_t \xi_{a,t}^{-1} P_{H,t}^{-1} P_t y_{H,t} - \kappa(\Pi_{H,t} - 1)\Pi_{H,t} y_{H,t} = -\beta \mathbb{E}_t^{\mathcal{P}} \frac{\Lambda_{t+1}}{\Lambda_t \Pi_{t+1}} \left[\kappa(\Pi_{H,t+1} - 1)\Pi_{H,t+1}^2 y_{H,t+1}\right]$.

Lastly, the balance-of-payments equation (together with the other market clearing conditions and Walras' law) ensures that both household budget constraints hold:

$$\gamma x_{F,t} P_{F,t} - P_{H,t}(1-\gamma)x^*_{H,t} + \gamma\left(P_t b_{t+1} - (1+i_{t-1})P_{t-1}b_t\right) = 0.$$

A derivation of the BOP equation together with a proof that Walras' law holds in our economy can be found in Appendices A.2 and A.1, respectively.

2.6 Nonlinear equilibrium conditions in MSV-form

As a starting point to solving the model, we collect all equilibrium conditions in a parsimonious fashion by performing light substitutions.

2.6.1 Expressing price levels with only inflation rates and terms of trade

Define the terms of trade as

$$s_t := \frac{P_{H,t}}{P_{F,t}}.$$

This entails

$$s_t = \frac{\Pi_{H,t}}{\Pi_{F,t}} s_{t-1},$$

and allows us to write

$$\begin{aligned}
\Pi_t &= \left[\left((\Pi_{H,t}^{1-1/\varsigma})^{-1} + \frac{1-\lambda}{\lambda}(s_t^{1-1/\varsigma})^{-1}\right)^{-1} + \left(\frac{\lambda}{1-\lambda}s_t^{1-1/\varsigma} + (\Pi_{F,t}^{1-1/\varsigma})^{-1}\right)^{-1}\right]^{\frac{1}{1-1/\varsigma}}, \\
\Pi^*_t &= \left[\left((\Pi_{H,t}^{1-1/\varsigma})^{-1} + \frac{\lambda^*}{1-\lambda^*}(s_t^{1-1/\varsigma})^{-1}\right)^{-1} + \left(\frac{1-\lambda^*}{\lambda^*}s_t^{1-1/\varsigma} + (\Pi_{F,t}^{1-1/\varsigma})^{-1}\right)^{-1}\right]^{\frac{1}{1-1/\varsigma}}, \\
\left(\frac{P_{H,t}}{P_t}\right)^{-\frac{1}{\varsigma}} &= \left[\lambda + (1-\lambda)s_t^{\frac{1}{\varsigma}-1}\right]^{\frac{1}{1/\varsigma-1}\cdot\left(-\frac{1}{\varsigma}\right)} =: p_H(s_t)^{-\frac{1}{\varsigma}}, \\
\left(\frac{P_{F,t}}{P_t}\right)^{-\frac{1}{\varsigma}} &= \left[\lambda s_t^{1-\frac{1}{\varsigma}} + (1-\lambda)\right]^{\frac{1}{1/\varsigma-1}\cdot\left(-\frac{1}{\varsigma}\right)} =: p_F(s_t)^{-\frac{1}{\varsigma}}, \\
\left(\frac{P_{H,t}}{P^*_t}\right)^{-\frac{1}{\varsigma}} &= \left[(1-\lambda^*) + \lambda^* s_t^{\frac{1}{\varsigma}-1}\right]^{\frac{1}{1/\varsigma-1}\cdot\left(-\frac{1}{\varsigma}\right)} =: p^*_H(s_t)^{-\frac{1}{\varsigma}}, \\
\left(\frac{P_{F,t}}{P^*_t}\right)^{-\frac{1}{\varsigma}} &= \left[(1-\lambda^*)s_t^{1-\frac{1}{\varsigma}} + \lambda^*\right]^{\frac{1}{1/\varsigma-1}\cdot\left(-\frac{1}{\varsigma}\right)} =: p^*_F(s_t)^{-\frac{1}{\varsigma}},
\end{aligned}$$

$$\frac{P_t}{P_t^*} = \left[\frac{1-\lambda^*}{\lambda + (1-\lambda)s_t^{\frac{1}{\varsigma}-1}} + \frac{\lambda^*}{\lambda s_t^{1-\frac{1}{\varsigma}} + (1-\lambda)}\right]^{\frac{1}{1-\frac{1}{\varsigma}}} =: \rho(s_t).$$

We have characterized every expression that involves any consumption price level in terms of the inflation rates and the terms of trade.

2.6.2 Condensing the set of market clearing conditions

Using the expressions above, and the demand schedules for varieties, we can rewrite the goods market clearing condition into

$$\left(1 - \frac{\kappa}{2}(\Pi_{H,t}-1)^2\right)\xi_{a,t}n_t\gamma = \gamma\lambda\rho_H(s_t)^{-\frac{1}{\varsigma}}x_t + (1-\gamma)(1-\lambda^*)\rho_H^*(s_t)^{-\frac{1}{\varsigma}}x_t^*,$$
$$\left(1 - \frac{\kappa}{2}(\Pi_{F,t}-1)^2\right)\xi_{a,t}n_t^*(1-\gamma) = \gamma(1-\lambda)\rho_F(s_t)^{-\frac{1}{\varsigma}}x_t + (1-\gamma)\lambda^*\rho_F^*(s_t)^{-\frac{1}{\varsigma}}x_t^*.$$

We can plug the demand schedules into the BOP equation, to receive

$$\gamma(1-\lambda)\rho_F(s_t)^{1-\frac{1}{\varsigma}}x_t - (1-\gamma)(1-\lambda^*)\rho_H^*(s_t)^{1-\frac{1}{\varsigma}}\rho(s_t)x_t^*$$
$$+ \gamma\left(b_{t+1} - (1+i_{t-1})\Pi_t^{-1}b_t\right) = 0,$$

where we have divided the equation by P_t. We are now ready to state the set of nonlinear equilibrium conditions.

2.6.3 Nonlinear equilibrium conditions

Using the short-hand notation $x_t := c_t + k_t + \Psi(b_{t+1})$ (analogously for *):

$$
\begin{aligned}
&\textbf{Household}\\
&(h) \quad \widetilde{P}_t^h = \frac{\xi_{h,t} h_t^{-\nu}}{\xi_{c,t} c_t^{-\sigma}} + \beta(1-\delta^h)\mathbb{E}_t^{\mathcal{P}}\left\{\frac{\xi_{c,t+1}}{\xi_{c,t}}\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\widetilde{P}_{t+1}^h\right\},\\
&(n) \quad \frac{\chi n_t^{\varphi}}{\xi_{c,t} c_t^{-\sigma}} = w_t,\\
&(b) \quad 1 + \psi_b(b_{t+1} - \underline{b}) = \beta\mathbb{E}_t^{\mathcal{P}}\left\{\frac{\xi_{c,t+1}}{\xi_{c,t}}\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\frac{1+i_t}{\Pi_{t+1}}\right\},\\
&(k) \quad 1 = \beta^{\tau}\mathbb{E}_t^{\mathcal{P}}\left\{\left(\frac{c_t}{c_{t+\tau}}\right)^{\sigma}\frac{\xi_{c,t+\tau}}{\xi_{c,t}}\cdot\widetilde{P}_{t+\tau}^h\right\}\cdot\xi_{k,t}k_t^{\eta-1},\\
&\textbf{Household*}\\
&(h^*) \quad \widetilde{P}_t^{h,*} = \frac{\xi_{h,t}(h_t^*)^{-\nu}}{\xi_{c,t}(c_t^*)^{-\sigma}} + \beta(1-\delta^h)\mathbb{E}_t^{\mathcal{P}}\left\{\frac{\xi_{c,t+1}}{\xi_{c,t}}\left(\frac{c_t^*}{c_{t+1}^*}\right)^{\sigma}\widetilde{P}_{t+1}^{h,*}\right\},\\
&(n^*) \quad \frac{\chi(n_t^*)^{\varphi}}{\xi_{c,t}(c_t^*)^{-\sigma}} = w_t^*,\\
&(b^*) \quad 1 + \psi_b(b_{t+1}^* - \underline{b}^*) = \beta\mathbb{E}_t^{\mathcal{P}}\left\{\frac{\xi_{c,t+1}}{\xi_{c,t}}\left(\frac{c_t^*}{c_{t+1}^*}\right)^{\sigma}\frac{1+i_t}{\Pi_{t+1}}\right\},\\
&(k^*) \quad 1 = \beta^{\tau^*}\mathbb{E}_t^{\mathcal{P}}\left\{\left(\frac{c_t^*}{c_{t+\tau^*}^*}\right)^{\sigma}\frac{\xi_{c,t+\tau^*}}{\xi_{c,t}}\cdot\widetilde{P}_{t+\tau^*}^{h,*}\right\}\cdot\xi_{k,t}(k_t^*)^{\eta-1},\\
&\textbf{Firm}\\
&(PC) \quad \kappa(\Pi_{H,t}-1)\Pi_{H,t} - (1-\epsilon) - \epsilon\frac{w_t}{\xi_{a,t}}p_H(s_t)^{-1} =\\
&\qquad \mathbb{E}_t^{\mathcal{P}}\left\{\beta\frac{\xi_{c,t+1}}{\xi_{c,t}}\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\frac{a_{t+1}n_{t+1}}{\xi_{a,t}n_t\Pi_{t+1}}\kappa(\Pi_{H,t+1}-1)\Pi_{H,t+1}^2\right\},\\
&\textbf{Firm*}\\
&(PC^*) \quad \kappa(\Pi_{F,t}-1)\Pi_{F,t} - (1-\epsilon) - \epsilon\frac{w_t^*}{\xi_{a,t}}p_F^*(s_t)^{-1} =\\
&\qquad \mathbb{E}_t^{\mathcal{P}}\left\{\beta\frac{\xi_{c,t+1}}{\xi_{c,t}}\left(\frac{c_t^*}{c_{t+1}^*}\right)^{\sigma}\frac{a_{t+1}n_{t+1}^*}{\xi_{a,t}n_t^*\Pi_{t+1}^*}\kappa(\Pi_{F,t+1}-1)\Pi_{F,t+1}^2\right\},\\
&\textbf{Bond market clearing}\\
&(B) \quad \gamma b_{t+1} + (1-\gamma)p(s_t)b_{t+1}^* = 0,
\end{aligned}
\tag{7}
$$

Goods market clearing

$$(GMC)\quad \left(1-\frac{\kappa}{2}(\Pi_{H,t}-1)^2\right)\xi_{a,t}n_t\gamma = \gamma\lambda p_H(s_t)^{-\frac{1}{\varsigma}}x_t + (1-\gamma)(1-\lambda^*)p_H^*(s_t)^{-\frac{1}{\varsigma}}x_t^*,$$

$$(GMC^*)\quad \left(1-\frac{\kappa}{2}(\Pi_{F,t}-1)^2\right)\xi_{a,t}n_t^*(1-\gamma) = \gamma(1-\lambda)p_F(s_t)^{-\frac{1}{\varsigma}}x_t + (1-\gamma)\lambda^* p_F^*(s_t)^{-\frac{1}{\varsigma}}x_t^*,$$

$$(BOP)\quad \gamma(1-\lambda)p_F(s_t)^{1-\frac{1}{\varsigma}}x_t-(1-\gamma)(1-\lambda^*)p_H^*(s_t)^{1-\frac{1}{\varsigma}}p(s_t)x_t^* + \gamma\left(b_{t+1}-(1+i_{t-1})\Pi_t^{-1}b_t\right)=0,$$

Housing market clearing

$$(HMC)\quad \eta^{-1}\xi_{t-\tau}(k_{t-\tau})^{\eta} = h_t-(1-\delta^h)h_{t-1},$$

$$(HMC^*)\quad \eta^{-1}\xi_{t-\tau^*}(k^*_{t-\tau^*})^{\eta} = h_t^*-(1-\delta^h)h_{t-1}^*,$$

Price indices

$$(s)\quad s_t = \frac{\Pi_{H,t}}{\Pi_{F,t}}s_{t-1},$$

$$(\Pi)\quad \Pi_t = \left[\left((\Pi_{H,t}^{1-1/\varsigma})^{-1}+\frac{1-\lambda}{\lambda}(s_t^{1-1/\varsigma})^{-1}\right)^{-1}+\left(\frac{\lambda}{1-\lambda}s_t^{1-1/\varsigma}+(\Pi_{F,t}^{1-1/\varsigma})^{-1}\right)^{-1}\right]^{\frac{1}{1-1/\varsigma}},$$

$$(\Pi^*)\quad \Pi_t^* = \left[\left((\Pi_{H,t}^{1-1/\varsigma})^{-1}+\frac{\lambda^*}{1-\lambda^*}(s_t^{1-1/\varsigma})^{-1}\right)^{-1}+\left(\frac{1-\lambda^*}{\lambda^*}s_t^{1-1/\varsigma}+(\Pi_{F,t}^{1-1/\varsigma})^{-1}\right)^{-1}\right]^{\frac{1}{1-1/\varsigma}}.$$

with the shocks $(\xi_t)_{t\geq 0}$, the allocation variables $(c_t, c_t^*, b_t, b_t^*, h_t, h_t^*, k_t, k_t^*, n_t, n_t^*)_{t\geq 0}$ and the price variables $(w_t, w_t^*, \widetilde{P}_t^h, \widetilde{P}_t^{h,*}, i_t, \Pi_t, \Pi_t^*, \Pi_{H,t}, \Pi_{F,t}, s_t)_{t\geq 0}$

2.7 Non-stochastic steady state with zero net inflation

The non-stochastic steady state with zero net inflation (ZIRSS) obtains by setting $\mathrm{Var}[\xi_t] = 0, \Pi_{ss} = \Pi^*_{ss} = \Pi_{H,ss} = \Pi_{F,ss} = 1$, which imply $s_{ss} = 1$ under $s_{-1} = 1$ and equation (7)-(s). Moreover, we can use that $p_H(1) = p^*_H(1) = p_F(1) = p^*_F(1) = p(1) = 1$. The ZIRSS is therefore characterized by the following equations:

$$
\begin{aligned}
&\textbf{Household}\\
&(h) \quad \widetilde{P}^h_{ss} = \frac{\xi_{h,ss} h_{ss}^{-\nu}}{\xi_{c,ss} c_{ss}^{-\sigma}} + \beta(1-\delta^h)\widetilde{P}^h_{ss},\\
&(n) \quad \frac{\chi n_{ss}^{\varphi}}{\xi_{c,ss} c_{ss}^{-\sigma}} = w_{ss},\\
&(b) \quad 1 + \psi_b(b_{ss} - \underline{b}) = \beta(1+i_{ss}),\\
&(k) \quad 1 = \beta^{\tau}\widetilde{P}^h_{ss} \cdot \xi_{k,ss} k_{ss}^{\eta-1},\\
&\textbf{Household*}\\
&(h^*) \quad \widetilde{P}^{h,*}_{ss} = \frac{\xi_{h,ss} (h^*_{ss})^{-\nu}}{\xi_{c,ss} (c^*_{ss})^{-\sigma}} + \beta(1-\delta^h)\widetilde{P}^{h,*}_{ss},\\
&(n^*) \quad \frac{\chi (n^*_{ss})^{\varphi}}{\xi_{c,ss} (c^*_{ss})^{-\sigma}} = w^*_{ss},\\
&(b^*) \quad 1 + \psi_b(b^*_{ss} - \underline{b}^*) = \beta(1+i_{ss}),\\
&(k^*) \quad 1 = \beta^{\tau^*}\widetilde{P}^{h,*}_{ss} \cdot \xi_{k,ss} (k^*_{ss})^{\eta-1},\\
&\textbf{Firm}\\
&(PC) \quad w_{ss} = \frac{\epsilon-1}{\epsilon}\xi_{a,ss},\\
&\textbf{Firm*}\\
&(PC^*) \quad w^*_{ss} = \frac{\epsilon-1}{\epsilon}\xi_{a,ss},\\
&\textbf{Bond market clearing}\\
&(B) \quad \gamma b_{ss} + (1-\gamma) b^*_{ss} = 0,
\end{aligned}
\tag{8}
$$

Goods market clearing

$(GMC)\quad \xi_{a,ss} n_{ss}\gamma = \gamma\lambda x_{ss} + (1-\gamma)(1-\lambda^*)x^*_{ss},$

$(GMC^*)\quad \xi_{a,ss} n^*_{ss}(1-\gamma) = \gamma(1-\lambda)x_{ss} + (1-\gamma)\lambda^* x^*_{ss},$

$(BOP)\quad \gamma(1-\lambda)x_{ss} - (1-\gamma)(1-\lambda^*)x^*_{ss} - \gamma i_{ss} b_{ss} = 0,$

Housing market clearing

$(HMC)\quad \eta^{-1}\xi_{k,ss}(k_{ss})^{\eta} = \delta^h h_{ss},$

$(HMC^*)\quad \eta^{-1}\xi_{k,ss}(k^*_{ss})^{\eta} = \delta^h h^*_{ss}.$

where $\underline{b}, \underline{b}^*$ are model parameters chosen so as to ensure equilibrium existence. The first requirement is that $(\S 1)\ \gamma\underline{b} + (1-\gamma)\underline{b}^* = 0$.

We solve for the ZIRSS in 4 steps.

1. First, we solve a number of equations explicitly, thus substituting out a number of variables:

   (a) (b) and (b^*) together with (§1) and (B) imply $(b_{ss} - \underline{b}) = (b^*_{ss} - \underline{b}^*) = 0$ and $i_{ss} = \beta^{-1} - 1$ whence it follows $x_{ss} = c_{ss} + k_{ss}$ and analogously for *. (§1) then implies $b_{ss} = -b^*_{ss} \cdot (1-\gamma)/\gamma$ where b^*_{ss} is not pinned down yet. We will solve for it in the very last step.

   (b) (PC) and (PC^*) imply $w_{ss} = w^*_{ss} = \xi_{a,ss}\frac{\epsilon-1}{\epsilon}$;

   (c) together with (n) and (n^*) this implies $n_{ss} = \left(\xi_{a,ss}\frac{\varepsilon-1}{\varepsilon}\xi^{-1}_{c,ss}\chi^{-1}\right)^{1/\varphi} \cdot c_{ss}^{-\sigma/\varphi} =: \phi(c_{ss})$ with $\phi' < 0$ and analogously for n^*_{ss} with the *same* function ϕ;

   (d) (HMC) and (HMC^*) imply $h_{ss} = (\delta^h)^{-1}\xi_{k,ss}\eta^{-1}k^{\eta}_{ss}$ and analogously for h^*_{ss};

   (e) (h) and (h^*) imply

$$\widetilde{P}^h_{ss} = \left(1-\beta\left(1-\delta^h\right)\right)^{-1}\xi_{h,ss}\left[\left(\delta^h\right)^{-1}\xi_{k,ss}\eta^{-1}k^{\eta}_{ss}\right]^{-\nu}\xi^{-1}_{c,ss}c^{\sigma}_{ss}$$

$$\widetilde{P}^{h,*}_{ss} = \left(1-\beta\left(1-\delta^h\right)\right)^{-1}\xi_{h,ss}\left[\left(\delta^h\right)^{-1}\xi_{k,ss}\eta^{-1}\left(k^*_{ss}\right)^{\eta}\right]^{-\nu}\xi^{-1}_{c,ss}(c^*_{ss})^{\sigma}$$

   (f) (BOP) now reads $\gamma(1-\lambda)x_{ss} = (1-\gamma)(1-\lambda^*)x^*_{ss} - (1-\gamma)(\beta^{-1}-1)b^*_{ss}$, and the symmetric (BOP^*) which is redundant by Walras' law reads $(1-\gamma)(1-\lambda^*)x^*_{ss} = \gamma(1-\lambda)x_{ss} + (1-\gamma)(\beta^{-1}-1)b^*_{ss}$; using this in (GMC), (GMC^*) produces

$$(GMC)\quad \xi_{a,ss}\phi(c_{ss}) = x_{ss} + (1-\gamma)/\gamma\cdot(\beta^{-1}-1)b^*_{ss},$$

$$(GMC^*)\quad \xi_{a,ss}\phi(c^*_{ss}) = x^*_{ss} - (\beta^{-1}-1)b^*_{ss},$$

2. The remaining equations are (k), (k^*) and (GMC), (GMC^*), (BOP) with unknowns $k_{ss}, k^*_{ss}, c_{ss}, c^*_{ss}, b^*_{ss}$. In this step, we show there are strictly increasing functions that yield k_{ss}, k^*_{ss} given c_{ss}, c^*_{ss} respectively:

$$\beta^{\tau}\left(1-\beta\left(1-\delta^{h}\right)\right)^{-1}\xi_{h,ss}\left[\left(\delta^{h}\right)^{-1}\xi_{k,ss}\eta^{-1}k_{ss}^{\eta}\right]^{-\nu}\xi_{k,ss}k_{ss}^{\eta-1}\xi_{c,ss}^{-1}c_{ss}^{\sigma}=1$$

(the equation for * is symmetric.) Now since $\eta \in (0,1)$ and $\nu > 0$, the expression on the left-hand-side is a strictly decreasing function of k_{ss} for any c_{ss}. Moreover, for $k_{ss} \to 0$, the $LHS \to +\infty$ and for $k_{ss} \to \infty$, the $LHS \to 0$, whence Bolzano's intermediate value theorem (and continuity) ensures that for each c_{ss} there exists a unique k_{ss}. Call this implicitly defined mapping $k_{ss} = \psi(c_{ss})$. As the implicit function theorem shows, $\eta, \nu, \sigma > 0$ imply $\psi' > 0$. Analogous arguments hold for *. Lastly, since the LHS is decreasing in τ (because $\beta \in (0,1)$), it is that k_{ss} decreases strictly in τ. Thus, assuming w.l.o.g. that $\tau > \tau^*$ we have $\psi(c) < \psi^*(c), \forall c$, $\psi', \psi^{*\prime} > 0$.

3. We now insert our previous findings into the only remaining equations:

$$\begin{aligned}
(GMC) \quad & \xi_{a,ss}\phi(c_{ss}) - c_{ss} - \psi(c_{ss}) - (1-\gamma)/\gamma\cdot(\beta^{-1}-1)b^*_{ss} =: \zeta(c_{ss}, b^*_{ss}) = 0,\\
(GMC^*) \quad & \xi_{a,ss}\phi(c^*_{ss}) - c^*_{ss} - \psi^*(c^*_{ss}) + (\beta^{-1}-1)b^*_{ss} =: \zeta^*(c^*_{ss}, b^*_{ss}) = 0,
\end{aligned}$$

Observe now $c_{ss} \mapsto \zeta$ is continuous and strictly decreasing with $\lim_{c\to 0}\zeta = +\infty$ (by $\lim_{c\to 0}\phi = +\infty$ and $\lim_{c\to 0}\psi < +\infty$) and $\lim_{c\to+\infty}\zeta = -\infty$ (by $\lim_{c\to+\infty} c, \psi = +\infty$ and $\lim_{c\to\infty}\phi = 0$). Therefore, Bolzano's intermediate value theorem ensures there exists a unique c_{ss} for each b^*_{ss}. The exactly analogous argument ensures existence and uniqueness of c^*_{ss}. Call these mappings $\varpi : b^*_{ss} \mapsto c_{ss}, \varpi^* : b^*_{ss} \mapsto c^*_{ss}$. The implicit function theorem now yields:

$$\partial\varpi/\partial b^*_{ss} < 0 \text{ and } \partial\varpi^*/\partial b^*_{ss} > 0.$$

4. Finally, only one equation remains, (BOP), with only one variable, b^*_{ss}:

$$\begin{aligned}
&\gamma(1-\lambda)[\varpi(b^*_{ss}) + \psi(\varpi(b^*_{ss}))] - (1-\gamma)(1-\lambda^*)[\varpi^*(b^*_{ss}) + \psi^*(\varpi^*(b^*_{ss}))]\\
&\quad + (1-\gamma)(\beta^{-1}-1)b^*_{ss} =: \hbar(b^*_{ss}) = 0
\end{aligned}$$

with $b^*_{ss} \mapsto \hbar$ continuous. It also holds that $\hbar(0) < 0$.[7] On the other hand, as $b^*_{ss} \to +\infty$,

[7] Proof: (1) (GMC)&(GMC^*) imply $\frac{\partial c}{\partial \tau} = -\frac{\partial\zeta/\partial\tau(+)}{\partial\zeta/\partial c(-)} > 0$ Ulrich: @ Hannes: please cross-check this! ; (2) Now $\tau > \tau^*$ implies that $\varpi(0) > \varpi^*(0)$; (3) using (GMC)&(GMC^*) and the definitions of ϖ, ϖ^* to substitute into (BOP), and using that we have symmetric home-bias, $\gamma(1-\lambda) = (1-\gamma)(1-\lambda^*)$, produces $\hbar(0) = \xi_{a,ss}\gamma(1-\lambda)[\phi\circ\varpi(0) - \phi\circ\varpi^*(0)]$; (4) Now fact 2 together with $\phi' < 0$ implies that the last expression

$\hbar \to +\infty$.[8] Thus, Bolzano's intermediate value theorem ensures **existence** of $b^*_{ss} > 0$.

Uniqueness of the steady state can be shown by establishing strict positive monotonicity of $\hbar$: (we suppress arguments for brevity)

$$\begin{aligned}
\frac{\partial \hbar}{\partial b^*_{ss}} &= \gamma(1-\lambda)(1+\psi')\varpi' - (1-\gamma)(1-\lambda^*)(1+\psi^{*\prime})\varpi^{*\prime} \\
&\quad + (1-\gamma)(\beta^{-1}-1) \\
\text{step 2, cf. notes below:} \quad &> -(1-\gamma)(1-\lambda)(\beta^{-1}-1) - (1-\gamma)(1-\lambda^*)(\beta^{-1}-1) \\
&\quad + (1-\gamma)(\beta^{-1}-1) \\
&= (1-\gamma)(\beta^{-1}-1)[1-1+\lambda-1+\lambda^*] \\
&\overset{\text{sign}}{=} \lambda + \lambda^* - 1 \\
\text{step 3, cf. notes below:} \quad &= \lambda + 1 - \frac{\gamma}{1-\gamma}(1-\lambda) - 1 \\
&\overset{\text{sign}}{=} (1-\gamma)\lambda - \gamma(1-\lambda) \\
&= \lambda - \gamma \\
&\geq 0 \text{ by assumption.}
\end{aligned}$$

Step 3 follows by symmetric home bias, i.e. $\gamma(1-\lambda) = (1-\gamma)(1-\lambda^*) \iff \lambda^* = 1 - \frac{\gamma}{1-\gamma}(1-\lambda)$; Step 2 requires slightly more work: First, use the implicit function theorem on (GMC)&(GMC^*), respectively, to obtain

$$\begin{aligned}
\varpi' &= -\frac{\partial \zeta / \partial b^*_{ss}}{\partial \zeta / \partial c_{ss}} = \frac{\frac{1-\gamma}{\gamma}(\beta^{-1}-1)}{\phi' - (1+\psi')} < 0, \\
\varpi^{*\prime} &= -\frac{\partial \zeta^* / \partial b^*_{ss}}{\partial \zeta^* / \partial c^*_{ss}} = \frac{-(\beta^{-1}-1)}{\phi' - (1+\psi^{*\prime})} > 0;
\end{aligned}$$

Second, recognize that since $\phi' < 0$ it is

$$\frac{1+\psi'}{1+\psi'-\phi'} < 1 \iff \frac{1+\psi'}{-(1+\psi')+\phi'} > -1$$

and symmetrically for *. This shows that $\frac{\partial \hbar}{\partial b^*_{ss}} > 0$, and the ZIRSS is unique.

in 3 is negative. ■

[8] Proof: (1) $\lim_{b^*_{ss} \to +\infty} \gamma(1-\lambda)[\varpi(b^*_{ss}) + \psi(\varpi(b^*_{ss}))] \geq 0$ by non-negativity of consumption & housing investment; (2) $-(1-\lambda^*)[\varpi^*(b^*_{ss}) + \psi^*(\varpi^*(b^*_{ss}))] + (\beta^{-1}-1)b^*_{ss} = -(1-\lambda^*)[\xi_{a,ss}\phi(c^*_{ss}) + (\beta^{-1}-1)b^*_{ss}] + (\beta^{-1}-1)b^*_{ss} = -(1-\lambda^*)\xi_{a,ss}\phi(c^*_{ss}) + \lambda^*(\beta^{-1}-1)b^*_{ss}$ where the substitution is made using the definition of ϖ^*; (3) $\lim_{b^*_{ss} \to +\infty} c^*_{ss} = +\infty$ (assuming the contrary produces a contradiction with (GMC^*)); (4) Fact 3 and $\lim_{c \to +\infty} \phi(c) = 0$ implies $-(1-\lambda^*)\xi_{a,ss}\phi(c^*_{ss}) + \lambda^*(\beta^{-1}-1)b^*_{ss} \to +\infty$. ■

This completes the proof. ■

We see that in any ZIRSS featuring $\tau > \tau^*$ it must be $b^*_{ss} > 0$, i.e. country F lends to country H. The ordering of consumption in turn is not independent of parameters and may go either way. The ordering of investment is parameter-independent, though: If $c_{ss} < c^*_{ss}$ then, monotonicity of ψ, ψ^* as well as $\psi < \psi^*$ imply that $k^*_{ss} = \psi^*(c^*_{ss}) > \psi(c_{ss}) = k_{ss}$, i.e. country F invests more; If, on the other hand, $c_{ss} > c^*_{ss}$, the only way to guarantee $\hbar(b^*_{ss}) = 0 \iff 0 > -(1-\gamma)(\beta^{-1}-1)b^*_{ss} = \gamma(1-\lambda)[\varpi(b^*_{ss}) - \varpi^*(b^*_{ss}) + \psi(\varpi(b^*_{ss})) - \psi^*(\varpi^*(b^*_{ss}))]$ is to have $k^*_{ss} = \psi^*(c^*_{ss}) > \psi(c_{ss}) = k_{ss}$, i.e. country F invests more.

2.7.1 ZIRSS with subjective beliefs

The subjective house price expectation dynamics are fully characterized by

$$\begin{aligned}
&\mathbb{E}^{\mathcal{P}}_t P^h_{t+\tau} = P^h_t \cdot \exp\left(\tau \ln \overline{m}_t + \varrho(\tau)\right), \qquad \varrho(\tau) = \tau^2 \frac{\sigma^2}{2} + (\tau - 1)\frac{\tau(\tau+1)}{3}\frac{\sigma^2_\nu}{2}, \\
&\ln \overline{m}_t = (1-g)\left(\ln \overline{m}_{t-1} - \frac{\sigma^2_\nu}{2}\right) + g\left(\ln \frac{P^h_{t-1}}{P^h_{t-2}} + \frac{\sigma^2_\varepsilon}{2}\right), \\
&\qquad + \text{equations determining equilibrium-evolution of price level, } P^h_t.
\end{aligned}$$

In order to align this system with the non-stochastic steady state, we require the price expectation equation to be consistent with a constant price level, i.e. $\exp(\tau \ln \overline{m}_t + \varrho(\tau)) = 1$ in steady state. To achieve this, it is convenient to think of the subjectively perceived house price model as being parameterized by the scale of perceived shocks, ω. That is, the variance of the shocks that households perceive are $\omega \cdot \sigma^2_\nu$, $\omega \cdot \sigma^2_\varepsilon$, whence it immediately follows that the time-invariant Kalman-filter uncertainty changes to $\omega \cdot \sigma^2$; the Kalman-gain is unaffected by this scaling procedure. Now we can think of the non-stochastic steady state as the *outcome* of the nonlinear model with learning, if the variance of the exogeneous shocks vanishes, $\text{Var}[\xi_t] = 0$, as well as does the variance of the perceived shocks: $\omega \to 0$.
Upon $\omega \to 0$, we have that all variances vanish, and so $\varrho(\tau) \to 0$. The unique time-invariant posterior mean belief about the price growth rate in the perceived model with $\omega \to 0$ is found by using this limit, and $P^h_t = const.$, $\forall t$, in the updating equation:

$$\ln \overline{m}_{ss} = (1-g)\ln \overline{m}_{ss}.$$

Since the Kalman-gain is unchanged by taking the limit, $g \neq 1$, the unique solution is $\ln \overline{m}_{ss} = 0$, which delivers exactly $\lim_{\omega\to 0} \exp(\tau \ln \overline{m}_t + \varrho(\tau)) = 1$.
Finally, in simultaneity to perturbing the equilibrium law of motion with a small $\{\xi_t\}_{t\geq 0}$, we perturb it with a small $\omega = O(|\{\xi_t\}_{t\geq 0}|^2)$, in order to receive a properly defined filtering problem

in the face of time-varying house prices.

2.8 Log linear equations under RE

Define for $X \in \{\Pi, \Pi_H, \Pi_F\}$ the variable $x_t := X_t - 1$.

Notice that H's steady-state bond level is negative, $b_{ss} < 0$. Thus, the log-deviation, $\widehat{b_t} = \log(b_t) - \log(b_{ss})$ is not defined. For this case, we re-define $\widehat{b_t} := \frac{b_t - b_{ss}}{|b_{ss}|}$ which is of course still a valid first-order-approximation. With this definition, $\widehat{b_t}$ is symmetric in interpretation to the other $\widehat{\cdot}$-variables: it gives the deviation of H's bond level relative to the size of its steady-state-level.[9] All other $\widehat{\cdot}$-variables remain unaffected by this.

Household block, home: (linearized in level-price $\widetilde{P}_t^h$)

$$
\begin{aligned}
&(h) \quad \widehat{\widetilde{p}}_t^h - \sigma \widehat{c}_t + \widehat{\xi}_{c,t} = \frac{h_{ss}^{-\nu} \xi_{h,ss}}{c_{ss}^{-\sigma} \xi_{c,ss} \widetilde{P}_{ss}^h} (\widehat{\xi}_{h,t} - \nu \widehat{h}_t) + \beta(1 - \delta^h)(\mathbb{E}_t \widehat{\widetilde{p}}_{t+1}^h - \sigma \mathbb{E}_t \widehat{c}_{t+1} + \mathbb{E}_t \widehat{\xi}_{c,t+1}) \\
&(n) \quad \varphi \widehat{n}_t + \sigma \widehat{c}_t - \widehat{\xi}_{c,t} = \widehat{w}_t \\
&(b) \quad \widehat{c}_t = \mathbb{E}_t \widehat{c}_{t+1} - \frac{1}{\sigma}(i_t + \log \beta - \mathbb{E}_t \pi_{t+1} + \mathbb{E}_t \widehat{\xi}_{c,t+1} - \widehat{\xi}_{c,t} - \psi_b |b_{ss}| \widehat{b}_{t+1}) \\
&(k) \quad \mathbb{E}_t \widehat{\widetilde{p}}_{t+\tau}^h - \sigma \mathbb{E}_t \widehat{c}_{t+\tau} + \widehat{\xi}_{k,t} + (\eta - 1)\widehat{k}_t - \widehat{\xi}_{c,t} + \mathbb{E}_t \widehat{\xi}_{c,t+\tau} = -\sigma \widehat{c}_t
\end{aligned}
$$

Firm block, home:

$$
\begin{aligned}
&(PC) \quad \pi_{H,t} = \beta \mathbb{E}_t \pi_{H,t+1} + \frac{\epsilon - 1}{\kappa}(\widehat{w}_t - \widehat{a}_t - (1 - \lambda)\widehat{s}_t) \\
&(y) \quad \widehat{y}_{H,t} = \widehat{a}_t + \widehat{n}_t
\end{aligned}
$$

Household block, foreign: (linearized in level-price $\widetilde{P}_t^{*h}$)

$$
\begin{aligned}
&(n^*) \quad \varphi \widehat{n}_t^* + \sigma \widehat{c}_t^* - \widehat{\xi}_{c,t} = \widehat{w}_t^* \\
&(b^*) \quad \widehat{c}_t^* = \mathbb{E}_t \widehat{c}_{t+1}^* - \frac{1}{\sigma}(i_t + \log \beta - \mathbb{E}_t \pi_{t+1}^* + \mathbb{E}_t \widehat{\xi}_{c,t+1} - \widehat{\xi}_{c,t} - \psi_b b_{ss}^* \widehat{b}_{t+1}^*) \\
&(h^*) \quad \widehat{\widetilde{p}}_t^{*h} - \sigma \widehat{c}_t^* + \widehat{\xi}_{c,t} = \frac{h_{ss}^{*-\nu} \xi_{h,ss}}{c_{ss}^{*-\sigma} \xi_{c,ss} \widetilde{P}_{ss}^{*h}} (\widehat{\xi}_{h,t} - \nu \widehat{h}_t^*) + \beta(1 - \delta^h)(\mathbb{E}_t \widehat{\widetilde{p}}_{t+1}^{*h} - \sigma \mathbb{E}_t \widehat{c}_{t+1}^* + \mathbb{E}_t \widehat{\xi}_{c,t+1}) \\
&(k^*) \quad \mathbb{E}_t \widehat{\widetilde{p}}_{t+\tau^*}^{*h} - \sigma \mathbb{E}_t \widehat{c}_{t+\tau^*}^* + \widehat{\xi}_{k,t} + (\eta - 1)\widehat{k}_t^* - \widehat{\xi}_{c,t} + \mathbb{E}_t \widehat{\xi}_{c,t+\tau^*} = -\sigma \widehat{c}_t^*
\end{aligned}
$$

[9] E.g. if in steady-state, H borrows 200 units, $b_{ss} = -200$, and reduces its borrowing in t by 10 units, then $\widehat{b_t}$ is -5%.

Firm block, foreign:

$$(PC^*) \quad \pi_{F,t} = \beta \mathbb{E}_t \pi_{F,t+1} + \frac{\epsilon - 1}{\kappa}(\widehat{w}_t^* - \widehat{a}_t + (1-\lambda^*)\widehat{s}_t)$$

$$(y^*) \quad \widehat{y}_{F,t} = \widehat{a}_t + \widehat{n}_t^*$$

Prices:

$$(\Pi) \quad \pi_t = \lambda \pi_{H,t} + (1-\lambda)\pi_{F,t}$$

$$(\Pi^*) \quad \pi_t^* = \lambda^* \pi_{F,t} + (1-\lambda^*)\pi_{H,t}$$

$$(\Pi^{cu}) \quad \pi_t^{cu} = \gamma \pi_t + (1-\gamma)\pi_t^*$$

$$(s) \quad \widehat{s}_t = \widehat{s}_{t-1} + \pi_{H,t} - \pi_{F,t}$$

Market Clearing:

$$(MC) \quad \widehat{y}_t = \lambda(\frac{c_{ss}}{y_{ss}}\widehat{c}_t + \frac{k_{ss}}{y_{ss}}\widehat{k}_t) + \frac{1-\gamma}{\gamma}(1-\lambda^*)(\frac{c_{ss}^*}{y_{ss}}\widehat{c}_t^* + \frac{k_{ss}^*}{y_{ss}}\widehat{k}_t^*) - \frac{1}{\varsigma y_{ss}}[\lambda(1-\lambda)(c_{ss}+k_{ss}) + \frac{1-\gamma}{\gamma}\lambda^*(1-\lambda^*)(c_{ss}^*+k_{ss}^*)]\widehat{s}_t$$

$$(MC^*) \quad \widehat{y}_t^* = \lambda^*(\frac{c_{ss}^*}{y_{ss}^*}\widehat{c}_t^* + \frac{k_{ss}^*}{y_{ss}^*}\widehat{k}_t^*) + \frac{\gamma}{1-\gamma}(1-\lambda)(\frac{c_{ss}}{y_{ss}^*}\widehat{c}_t + \frac{k_{ss}}{y_{ss}^*}\widehat{k}_t) + \frac{1}{\varsigma y_{ss}^*}[\lambda^*(1-\lambda^*)(c_{ss}^*+k_{ss}^*) + \frac{\gamma}{1-\gamma}\lambda(1-\lambda)(c_{ss}+k_{ss})]\widehat{s}_t$$

$$(HMC) \quad \delta^h(\widehat{\xi}_{k,t-\tau} + \eta\widehat{k}_{t-\tau}) = \widehat{h}_t - (1-\delta^h)\widehat{h}_{t-1}$$

$$(HMC^*) \quad \delta^h(\widehat{\xi}_{k,t-\tau^*} + \eta\widehat{k}_{k,t-\tau^*}^*) = \widehat{h}_t^* - (1-\delta^h)\widehat{h}_{t-1}^*$$

$$(B) \quad \gamma|b_{ss}|\widehat{b}_{t+1} = -(1-\gamma)b_{ss}^*[(1-\lambda)(1-\lambda^*) - \lambda\lambda^*]\widehat{s}_t - (1-\gamma)b_{ss}^*\widehat{b}_{t+1}^*$$

$$
\begin{aligned}
(BOP)\quad & \gamma(1-\lambda)(c_{ss}\hat{c}_t + k_{ss}\hat{k}_t) - (1-\gamma)(1-\lambda^*)(c^*_{ss}\hat{c}^*_t + k^*_{ss}\hat{k}^*_t) \\
& - \hat{s}_t[\gamma(1-\lambda)(c_{ss}+k_{ss})(1-1/\varsigma)\lambda + \\
& \quad (1-\gamma)(1-\lambda^*)(c^*_{ss}+k^*_{ss})((1-\lambda^*)(1-\lambda) - \lambda^*\lambda + (1-1/\varsigma)\lambda^*)] \\
& - (1-\gamma)b^*_{ss}(\hat{b}^*_{t+1} - \beta^{-1}[\hat{b}^*_t + i_{t-1} + \log\beta - \pi_t]) \\
& - (1-\gamma)b^*_{ss}[(1-\lambda^*)(1-\lambda) - \lambda^*\lambda][\hat{s}_t - \beta^{-1}\hat{s}_{t-1}] = 0
\end{aligned}
$$

2.9 Diagnostic expectations

Re-write the HH problem as follows:

$$
\begin{aligned}
\max_{\{c_t,h_t,n_t,b_{t+1},k_t\}_{t\geq 0}} \quad & \frac{\xi_{c,0}c_0^{1-\sigma}}{1-\sigma} + \frac{\xi_{h,0}h_0^{1-\nu}}{1-\nu} - \frac{\chi n_0^{1+\varphi}}{1+\varphi} + \mathbb{E}_0^\theta \sum_{t=1}^{\infty} \beta^t \frac{\xi_{c,t}c_t^{1-\sigma}}{1-\sigma} + \frac{\xi_{h,t}h_t^{1-\nu}}{1-\nu} - \frac{\chi n_t^{1+\varphi}}{1+\varphi} \\
\text{s.t.} \quad & c_t + \widetilde{P}_t^h(h_t - (1-\delta^h)h_{t-1}) + b_{t+1} + \Psi(b_{t+1}) + k_t = \\
& w_t n_t + \frac{1+i_{t-1}}{\Pi_t} b_t + d(k_{t-\tau}, \xi_{t-\tau}) \cdot \widetilde{P}_t^h - T_t + \Sigma_t
\end{aligned}
$$

The FOCs are given by:

$$
\begin{aligned}
(h) \quad & \xi_{h,t}h_t^{-\nu} = \widetilde{P}_t^h \xi_{c,t}c_t^{-\sigma} + \beta(1-\delta^h)\mathbb{E}_t^\theta\left\{\xi_{c,t+1}c_{t+1}^{-\sigma}\widetilde{P}_{t+1}^h\right\}, \\
(n) \quad & \frac{\chi n_t^{\varphi}}{\xi_{c,t}c_t^{-\sigma}} = w_t, \\
(b) \quad & \xi_{c,t}c_t^{-\sigma}\left(1 + \psi_b(b_{t+1} - \underline{b})\right) = \beta\mathbb{E}_t^\theta\left\{\xi_{c,t+1}c_{t+1}^{-\sigma}\frac{1+i_t}{\Pi_{t+1}}\right\}, \\
(k) \quad & \xi_{c,t}c_t^{-\sigma} = \beta^\tau \mathbb{E}_t^\theta\left\{c_{t+\tau}^{-\sigma}\xi_{c,t+\tau}\widetilde{P}_{t+\tau}^h \xi_{k,t}k_t^{\eta-1}\right\}
\end{aligned}
$$

Follwoing L'Hullier et al. (22) one can re-write the diagnostic expectations operator in log-linearized form as follows: $\mathbb{E}_t^\theta x_t = (1+\theta)\mathbb{E}_t x_t - \theta\mathbb{E}_{t-1}x_t$. Where $\mathbb{E}_t$ is the rational expectations operator. Note that therefore it was not possible to exclude i_t, k_t, $\xi_{k,t}$ from the diagnostic expectation operators above even though they are only contemporaneous variables. Log-linearizing and applying above's relation to the diagnostic expectations gives:

$$
\begin{aligned}
(h) \quad & \widehat{\widetilde{p}}^h_t - \sigma \widehat{c}_t + \widehat{\xi}_{c,t} = \frac{h_{ss}^{-\nu} \xi_{h,ss}}{c_{ss}^{-\sigma} \xi_{c,ss} \widetilde{P}_{ss}^h} (\widehat{\xi}_{h,t} - \nu \widehat{h}_t) \\
& \quad + \beta (1 - \delta^h) \left((1+\theta)(\mathbb{E}_t \widehat{\widetilde{p}}^h_{t+1} - \sigma \mathbb{E}_t \widehat{c}_{t+1} + \mathbb{E}_t \widehat{\xi}_{c,t+1}) - \theta (\mathbb{E}_{t-1} \widehat{\widetilde{p}}^h_{t+1} - \sigma \mathbb{E}_{t-1} \widehat{c}_{t+1} + \mathbb{E}_{t-1} \widehat{\xi}_{c,t+1}) \right) \\
(n) \quad & \varphi \widehat{n}_t + \sigma \widehat{c}_t - \widehat{\xi}_{c,t} = \widehat{w}_t \\
(b) \quad & \widehat{c}_t = (1+\theta) \mathbb{E}_t \widehat{c}_{t+1} - \theta \mathbb{E}_{t-1} \widehat{c}_{t+1} - \frac{1+\theta}{\sigma} \left(i_t - \mathbb{E}_t \pi_{t+1} + \mathbb{E}_t \widehat{\xi}_{c,t+1} - \widehat{\xi}_{c,t} \right) \\
& \quad + \frac{\theta}{\sigma} \left(\mathbb{E}_{t-1} i_t - \mathbb{E}_{t-1} \pi_{t+1} + \mathbb{E}_{t-1} \widehat{\xi}_{c,t+1} - \mathbb{E}_{t-1} \widehat{\xi}_{c,t} \right) + \frac{1}{\sigma} \psi_b |b_{ss}| \widehat{b}_{t+1} \\
(k) \quad & (1+\theta) \left(\mathbb{E}_t \widehat{\widetilde{p}}^h_{t+\tau} - \sigma \mathbb{E}_t \widehat{c}_{t+\tau} + \widehat{\xi}_{k,t} + (\eta - 1) \widehat{k}_t + \mathbb{E}_t \widehat{\xi}_{c,t+\tau} \right) \\
& \quad - \theta \left(\mathbb{E}_{t-1} \widehat{\widetilde{p}}^h_{t+\tau} - \sigma \mathbb{E}_{t-1} \widehat{c}_{t+\tau} + \mathbb{E}_{t-1} \widehat{\xi}_{k,t} + (\eta - 1) \mathbb{E}_{t-1} \widehat{k}_t + \mathbb{E}_{t-1} \widehat{\xi}_{c,t+\tau} \right) - \widehat{\xi}_{c,t} = -\sigma \widehat{c}_t
\end{aligned}
$$

Firms solve the following problem:

$$
\begin{aligned}
& \max_{P_{H,t}(j)} D_t + \mathbb{E}_0^\theta \sum_{t=0}^{\infty} \beta^t \Delta_t D_t \\
& s.t. \quad y_{H,t}(j) = \left(\frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} y_{H,t} \\
& with: \quad D_t = P_{H,t}(j) y_{H,t}(j) - W_t n_t(j) - P_{H,t} \frac{\kappa}{2} \left(\frac{P_{H,t}(j)}{P_{H,t-1}(j)} - 1 \right)^2 y_{H,t}
\end{aligned}
$$

Following similar steps as for the derivation of the PC in the RE model and applying above's rule to diagnostic expectation gives the following log-linearized PC:

$$
\pi_{H,t} = (1+\theta) \beta \mathbb{E}_t \pi_{H,t+1} - \theta \mathbb{E}_{t-1} \pi_{H,t+1} + \frac{\epsilon - 1}{\kappa} (\widehat{w}_t - \widehat{a}_t - (1 - \lambda) \widehat{s}_t)
$$

The foreign country is equivalent and the remaining equations remain unchanged compared to the RE model.

Appendix

A Proofs

Accounting for home bias in consumption is done by introducing the parameters λ and λ^*, respectively. Compared to the model without home bias only two equations change in the household set-up. We now have:

$$
\begin{aligned}
c_t &= [\lambda^{\varsigma} c_{H,t}^{1-\varsigma} + (1-\lambda)^{\varsigma} c_{F,t}^{1-\varsigma}]^{\frac{1}{1-\varsigma}} \\
P_t &= [\lambda P_{H,t}^{1-\frac{1}{\varsigma}} + (1-\lambda) P_{F,t}^{1-\frac{1}{\varsigma}}]^{\frac{1}{1-\frac{1}{\varsigma}}} \\
c_t^* &= [(1-\lambda^*)^{\varsigma} c_{H,t}^{1-\varsigma} + (\lambda^*)^{\varsigma} c_{F,t}^{1-\varsigma}]^{\frac{1}{1-\varsigma}} \\
P_t^* &= [(1-\lambda^*) P_{H,t}^{1-\frac{1}{\varsigma}} + \lambda^* P_{F,t}^{1-\frac{1}{\varsigma}}]^{\frac{1}{1-\frac{1}{\varsigma}}}
\end{aligned}
$$

And equivalent for the foreign economy (with λ^* taking the place of λ). Ulrich: Since the H, and F weights do not appear in the envelope program of the hh, the hh-FOC remain the same – with exception of the demand schedules, of course, which now read:

$$
\gamma \int_0^1 P_{H,t}(j) c_{H,t}(j) dj = c_{H,t} P_{H,t}, \quad P_{H,t} = [\int_0^1 P_{H,t}(j)^{1-\epsilon} dj]^{\frac{1}{1-\epsilon}}, \quad c_{H,t}(j) = (\frac{P_{H,t}(j)}{P_{H,t}})^{-\epsilon} \frac{c_{H,t}}{\gamma},
$$

$$
(1-\gamma) \int_0^1 P_{F,t}(j^*) c_{F,t}(j^*) dj^* = c_{F,t} P_{F,t}, \quad P_{F,t} = [\int_0^1 P_{F,t}(j^*)^{1-\epsilon} dj^*]^{\frac{1}{1-\epsilon}}, \quad c_{F,t}(j^*) = (\frac{P_{F,t}(j^*)}{P_{F,t}})^{-\epsilon} \frac{c_{F,t}}{1-\gamma},
$$

$$
c_{H,t} P_{H,t} + c_{F,t} P_{F,t} = P_t c_t, \quad P_t = [\lambda P_{H,t}^{1-\frac{1}{\varsigma}} + (1-\lambda) P_{F,t}^{1-\frac{1}{\varsigma}}]^{\frac{1}{1-\frac{1}{\varsigma}}},
$$

$$
c_{H,t} = (\frac{P_{H,t}}{P_t})^{-\frac{1}{\varsigma}} c_t \lambda, \quad c_{F,t} = (\frac{P_{F,t}}{P_t})^{-\frac{1}{\varsigma}} c_t (1-\lambda).
$$

Following the same derivations as before one can note that the log-linear system is mostly unchanged in equations: (h), (n), (b), (k), (y), (HMC), (h^*), (n^*), (b^*), (k^*), (y^*), (HMC^*), (s). But instead for π_t in the foreign economy, we know have π_t^* as consumer prices differ across countries. Further, $c_t \neq c_t^*$. In the following blocks we have some changes and need to add several equations.

Firm block, home: Ulrich: agree

$$
(PC) \quad \pi_{H,t} = \beta \mathbb{E}_t \pi_{H,t+1} + \frac{\epsilon - 1}{\kappa} (\widehat{w}_t - \widehat{a}_t - (1-\lambda) \widehat{s}_t)
$$

Firm block, foreign: Ulrich: agree

$$(PC^*) \quad \pi_{F,t} = \beta \mathbb{E}_t \pi_{F,t+1} + \frac{\epsilon - 1}{\kappa}(\widehat{w}_t^* - \widehat{a}_t + (1-\lambda^*)\widehat{s}_t)$$

Prices: Ulrich: agree

$$(\Pi) \quad \pi_t = \lambda \pi_{H,t} + (1-\lambda)\pi_{F,t}$$
$$(\Pi^*) \quad \pi_t^* = \lambda^* \pi_{F,t} + (1-\lambda^*)\pi_{H,t}$$
$$(\Pi^{cu}) \quad \pi_t^{cu} = \gamma \pi_t + (1-\gamma)\pi_t^*$$

Of course π_t^{cu} now shows up in the Taylor rule. The most severe changes occur in the market clearing conditions for goods.
Market Clearing: Ulrich: agree

$$\begin{aligned}
\widehat{y}_t &= \lambda(\frac{c_{ss}}{y_{ss}}\widehat{c}_t + \frac{k_{ss}}{y_{ss}}\widehat{k}_t) + \frac{1-\gamma}{\gamma}(1-\lambda^*)(\frac{c_{ss}^*}{y_{ss}}\widehat{c}_t^* + \frac{k_{ss}^*}{y_{ss}}\widehat{k}_t^*) \\
&-\frac{1}{\varsigma y_{ss}}[\lambda(1-\lambda)(c_{ss}+k_{ss}) + \frac{1-\gamma}{\gamma}\lambda^*(1-\lambda^*)(c_{ss}^*+k_{ss}^*)]\widehat{s}_t \\
\widehat{y}_t^* &= \lambda^*(\frac{c_{ss}^*}{y_{ss}^*}\widehat{c}_t^* + \frac{k_{ss}^*}{y_{ss}^*}\widehat{k}_t^*) + \frac{\gamma}{1-\gamma}(1-\lambda)(\frac{c_{ss}}{y_{ss}^*}\widehat{c}_t + \frac{k_{ss}}{y_{ss}^*}\widehat{k}_t) \\
&+\frac{1}{\varsigma y_{ss}^*}[\lambda^*(1-\lambda^*)(c_{ss}^*+k_{ss}^*) + \frac{\gamma}{1-\gamma}\lambda(1-\lambda)(c_{ss}+k_{ss})]\widehat{s}_t
\end{aligned}$$

A.1 Derivation of Walras' law

To make sure the economics of the model with home bias checks out, we prove that Walras' law holds in our model economy.

We start by providing a list of all market clearing conditions, household budget constraints, and relevant variable definitions (assuming $T_t = T_t^* = 0$), with equations involving more than one good being in nominal terms (i.e. units of union-wide currency):

$$
\begin{aligned}
(GMC) \quad & \left(1-\frac{\kappa}{2}\pi_{H,t}^2\right) y_{H,t}\gamma = \gamma x_{H,t} + (1-\gamma)x_{H,t}^*, \\
(GMC^*) \quad & \left(1-\frac{\kappa}{2}\pi_{F,t}^2\right) y_{F,t}^*(1-\gamma) = \gamma x_{F,t} + (1-\gamma)x_{F,t}^*, \\
(HMC) \quad & d(k_{t-\tau}, \xi_{t-\tau}) = h_t - (1-\delta^h)h_{t-1}, \\
(HMC^*) \quad & d(k_{t-\tau}^*, \xi_{t-\tau}) = h_t^* - (1-\delta^h)h_{t-1}^*, \\
(B) \quad & \gamma P_t b_{t+1} + (1-\gamma)P_t^* b_{t+1}^* = 0, \\
(BC) \quad & x_{H,t}P_{H,t} + x_{F,t}P_{F,t} + P_t\widetilde{P}_t^h(h_t - (1-\delta^h)h_{t-1}) + P_t b_{t+1} = W_t n_t \\
& \quad + (1+i_{t-1})P_{t-1}b_t + d(k_{t-\tau}, \xi_{t-\tau}) \cdot P_t\widetilde{P}_t^h + P_t\Sigma_t, \\
(BC^*) \quad & x_{H,t}^*P_{H,t} + x_{F,t}^*P_{F,t} + P_t^*\widetilde{P}_t^{h,*}(h_t^* - (1-\delta^h)h_{t-1}^*) + P_t^* b_{t+1}^* = \\
& \quad W_t^* n_t^* + (1+i_{t-1})P_{t-1}^* b_t^* + d(k_{t-\tau}^*, \xi_{t-\tau}) \cdot P_t^*\widetilde{P}_t^{h,*} + P_t^*\Sigma_t^*, \\
(\Sigma) \quad & P_t\Sigma_t = P_{H,t}\left(1-\frac{\kappa}{2}\pi_{H,t}^2\right) y_t - W_t n_t, \\
(\Sigma^*) \quad & P_t^*\Sigma_t^* = P_{F,t}\left(1-\frac{\kappa}{2}\pi_{F,t}^2\right) y_t^* - W_t^* n_t^*.
\end{aligned} \tag{9}
$$

These are 9 conditions – Walras' law now says that any one of these nine conditions should be obtainable through the summation of the remaining eight conditions. We show that the collective of all equations, except for (B), implies equation (B).

First, plug (HMC) and (Σ) into (BC) (and analogously for *) to get

$$
\begin{aligned}
(HMC\&\Sigma\&BC) \quad & x_{H,t}P_{H,t} + x_{F,t}P_{F,t} + P_t b_{t+1} = (1+i_{t-1})P_{t-1}b_t + P_{H,t}\left(1-\frac{\kappa}{2}\pi_{H,t}^2\right) y_t \\
(HMC^*\&\Sigma^*\&BC^*) \quad & x_{H,t}^*P_{H,t} + x_{F,t}^*P_{F,t} + P_t^* b_{t+1}^* = (1+i_{t-1})P_{t-1}^* b_t^* + P_{F,t}\left(1-\frac{\kappa}{2}\pi_{F,t}^2\right) y_t^* \\
& \iff \\
(HMC\&\Sigma\&BC) \quad & \gamma\left(x_{H,t}P_{H,t} + x_{F,t}P_{F,t} + P_t b_{t+1} - (1+i_{t-1})P_{t-1}b_t\right) \\
& \quad = P_{H,t}\left(1-\frac{\kappa}{2}\pi_{H,t}^2\right) y_t\gamma \\
(HMC^*\&\Sigma^*\&BC^*) \quad & (1-\gamma)\left(x_{H,t}^*P_{H,t} + x_{F,t}^*P_{F,t} + P_t^* b_{t+1}^* - (1+i_{t-1})P_{t-1}^* b_t^*\right) \\
& \quad = P_{F,t}\left(1-\frac{\kappa}{2}\pi_{F,t}^2\right) y_t^*(1-\gamma)
\end{aligned}
$$

Now use (GMC) and (GMC^*):

$$
\begin{aligned}
(HMC\&\Sigma\&BC\&GMC) \quad & \gamma\left(x_{H,t}P_{H,t} + x_{F,t}P_{F,t} + P_t b_{t+1} - (1+i_{t-1})P_{t-1}b_t\right) \\
& \quad = P_{H,t}\left(\gamma x_{H,t} + (1-\gamma)x_{H,t}^*\right)
\end{aligned}
$$

$$(HMC^*\&\Sigma^*\&BC^*\&GMC^*) \quad (1-\gamma)\left(x^*_{H,t}P_{H,t} + x^*_{F,t}P_{F,t} + P^*_t b^*_{t+1} - (1+i_{t-1})P^*_{t-1}b^*_t\right)$$
$$= P_{F,t}\left(\gamma x_{F,t} + (1-\gamma)x^*_{F,t}\right)$$

and adding the two equations gives:

$$(HMC\&\Sigma\&BC\&GMC) + (HMC^*\&\Sigma^*\&BC^*\&GMC^*) \quad [\gamma P_t b_{t+1} + (1-\gamma)P^*_t b^*_{t+1}] - (1+i_{t-1})[\gamma P_{t-1}b_t + (1-\gamma)P^*_{t-1}b^*_t] = 0.$$

Since b_0, b^*_0 are model-parameters and chosen s.th. $\gamma P_{-1}b_0 + (1-\gamma)P^*_{-1}b^*_0 = 0$, a simple induction-argument establishes (B), $\forall t$. ■

A.2 Derivation of the balance-of-payments-equation (BOP)

Start with

$$(HMC\&\Sigma\&BC\&GMC) \quad \gamma\left(x_{H,t}P_{H,t} + x_{F,t}P_{F,t} + P_t b_{t+1} - (1+i_{t-1})P_{t-1}b_t\right)$$
$$= P_{H,t}\left(\gamma(c_{H,t} + k_{H,t}) + (1-\gamma)x^*_{H,t}\right)$$
$$(HMC^*\&\Sigma^*\&BC^*\&GMC^*) \quad (1-\gamma)\left(x^*_{H,t}P_{H,t} + x^*_{F,t}P_{F,t} + P^*_t b^*_{t+1} - (1+i_{t-1})P^*_{t-1}b^*_t\right)$$
$$= P_{F,t}\left(\gamma x_{F,t} + (1-\gamma)(c^*_{F,t} + k^*_{F,t})\right)$$
$$\Longleftrightarrow$$
$$(HMC\&\Sigma\&BC\&GMC) \quad \gamma x_{F,t}P_{F,t} - P_{H,t}(1-\gamma)x^*_{H,t} + \gamma\left(P_t b_{t+1} - (1+i_{t-1})P_{t-1}b_t\right)$$
$$= 0$$
$$(HMC^*\&\Sigma^*\&BC^*\&GMC^*) \quad (1-\gamma)x^*_{H,t}P_{H,t} - P_{F,t}\gamma x_{F,t} + (1-\gamma)\left(P^*_t b^*_{t+1} - (1+i_{t-1})P^*_{t-1}b^*_t\right)$$
$$= 0$$

Now both equations dictate that the value of net imports in the respective country (imports less of exports) be covered by an equal-sized increase in the level of debt. By Walras' law (see above) equation $(HMC\&\Sigma\&BC)$ complemented by the bond market clearing condition implies equation $(HMC^*\&\Sigma^*\&BC^*)$. (Just grant the two conditions, undo the last transformation on $(HMC\&\Sigma\&BC)$, insert the bond market clearing condition solved for $\gamma P^*_t b^*_{t+1}$, and cancel terms.) Therefore, the whole list of market clearing conditions, budget constraints and relevant variable definitions, (9), is summarized in the balance of payments equation

$$(BOP) \quad \gamma(1-\lambda)\left(\frac{P_{F,t}}{P_t}\right)^{1-\frac{1}{\varsigma}}x_t - (1-\gamma)(1-\lambda^*)\left(\frac{P_{H,t}}{P^*_t}\right)^{1-\frac{1}{\varsigma}}\frac{P^*_t}{P_t}x^*_t$$
$$- (1-\gamma)\left(\frac{P^*_t}{P_t}b^*_{t+1} - (1+i_{t-1})\Pi_t^{-1}\frac{P^*_{t-1}}{P_{t-1}}b^*_t\right) = 0.$$

where we have used the demand schedules to substitute out the H and F good variables that do not feature in the MSV-representation of the model, and we have divided by P_t to get the representation in units of country H's final consumption good composite.

A.3 Symmetry across countries of non-housing impulse responses, under RE, no home-bias and perfect consumption risk-sharing

Recall the of the log-linearized version of the model under RE as detailed in section 2.8. Now admit the additional assumption that there is no home bias, effectively equalizing the consumption aggregators in both countries, and the additional assumption that there is perfect consumption risk-sharing, $c_t = \Xi c_t^*$ for some $\Xi > 0$ that depends on initial endowments. (We set $\Xi = 1$.)[10] In this section we use the minimum state variable representation of the above system to show that the equilibrium reaction of all country-specific variables, with exception of housing stock and house prices, in response to a symmetric shock other than housing taste, ξ_h, is itself symmetric between both countries – in signs:

Proposition 1. $\lambda = \gamma,\ \lambda^* = 1-\gamma, c_t = \Xi c_t^* \implies (\xi_{h,t} \equiv 0 \implies x_t = x_t^*,\ \forall x \notin \{\widehat{\widetilde{p}}_t^h, \widehat{\widetilde{p}}_t^{h*}, \widehat{h}_t, \widehat{h}_t^*\})$.

The key intuition here is that the house price is the sufficient statistic that guides investment decisions. Given that, since (i) house prices map investment decisions into prices in exactly the same manner in both countries, and since (ii) the additional discounting that features in the nonlinear investment-equations (7)-(k), (k^*) is of second order, the housing investment decisions are identical in both countries.

Proof. To this end, we perform various substitutions on the system from 2.8 to arrive at the MSV representation.

1. (n) into (MC);

2. For the risk-adjusted house price $z_t := \widehat{\widetilde{p}}_t^h - \sigma\widehat{c}_t$, solve ($h$) forward to get $z_t = \mathbb{E}_t \sum_{j\geq 0} \beta^j (1-\delta^h)^j [A_h(\widehat{\xi}_{h,t+j} - \nu\widehat{h}_{t+j}) - \widehat{\xi}_{c,t+j} + \beta(1-\delta^h)\widehat{\xi}_{c,t+j+1}]$, where $A_h = \frac{h_{ss}^{-\nu}\xi_{h,ss}}{c_{ss}^{-\sigma}\xi_{c,ss}\widetilde{P}_{ss}^h}$;

3. Solve (HMC) forward in terms of to get $\widehat{h}_t = (1-\delta^h)^{t-\tau}\widehat{h}_{\tau-1} + \mathbb{1}\{t \geq \tau\} \cdot \sum_{j=0}^{t-\tau}(1-\delta^h)^j\delta^h(\widehat{\xi}_{k,t-\tau-j} + \eta\widehat{k}_{t-\tau-j})$ where $\widehat{h}_{\tau-1} = 0$ since we assume that shocks hit out of steady state in period 0;

[10] As detailed in Benigno (2004), the assumption of consumption risk-sharing, i.e. the existence of AD-securities in the union-wide consumption composite, is redundant if the H- and F-consumption goods are Cobb-Douglas-aggregated, $\varsigma = 1$.

4. Substitute this into z_t to obtain

$$
\begin{aligned}
z_t &= \mathbb{E}_t \sum_{j\geq 0} \beta^j (1-\delta^h)^j \Bigg[A_h \Bigg(\widehat{\xi}_{h,t+j} \\
&\quad -\mathbb{1}\{t+j \geq \tau\} \cdot \nu \sum_{i=0}^{t+j-\tau} (1-\delta^h)^i \delta^h (\widehat{\xi}_{k,t+j-\tau-i} + \eta \widehat{k}_{t+j-\tau-i}) \Bigg) - \widehat{\xi}_{c,t+j} + \beta(1-\delta^h)\widehat{\xi}_{c,t+j+1} \Bigg] \\
&= \mathbb{E}_t \sum_{j\geq 0} \beta^j (1-\delta^h)^j \left[A_h \widehat{\xi}_{h,t+j} - \widehat{\xi}_{c,t+j} + \beta(1-\delta^h)\widehat{\xi}_{c,t+j+1} \right] \\
&\quad - \mathbb{E}_t \sum_{j\geq 0} \left(\mathbb{1}\{j \geq \tau - t\} \frac{\beta^j (1-\delta^h)^j}{1-\beta(1-\delta^h)^2} + \mathbb{1}\{j < \tau - t\} \frac{\beta^{\tau-t}(1-\delta^h)^{2(\tau-t)-j}}{1-\beta(1-\delta^h)^2} \right) \cdot \\
&\qquad A_h \nu \delta^h (\widehat{\xi}_{k,t-\tau+j} + \eta \widehat{k}_{t-\tau+j}) \\
&\quad - \sum_{j=0}^{t-\tau-1} \left(\mathbb{1}\{\tau - t \leq 0\} \frac{(1-\delta^h)^{t-\tau-j}}{1-\beta(1-\delta^h)^2} + \mathbb{1}\{\tau - t > 0\} \frac{\beta^{\tau-t}(1-\delta^h)^{\tau-t-j}}{1-\beta(1-\delta^h)^2} \right) A_h \nu \delta^h (\widehat{\xi}_{k,j} + \eta \widehat{k}_j) \\
&= \mathbb{E}_t \sum_{j\geq 0} \beta^j (1-\delta^h)^j \left[A_h \widehat{\xi}_{h,t+j} - \widehat{\xi}_{c,t+j} + \beta(1-\delta^h)\widehat{\xi}_{c,t+j+1} \right] \\
&\quad - \mathbb{E}_t \sum_{j\geq 0} \frac{\beta^j (1-\delta^h)^j}{1-\beta(1-\delta^h)^2} A_h \nu \delta^h (\widehat{\xi}_{k,t-\tau+j} + \eta \widehat{k}_{t-\tau+j}) \\
&\quad - \sum_{j=0}^{t-\tau-1} \frac{(1-\delta^h)^{t-\tau-j}}{1-\beta(1-\delta^h)^2} A_h \nu \delta^h (\widehat{\xi}_{k,j} + \eta \widehat{k}_j), \quad \text{if } t \geq \tau;
\end{aligned}
$$

5. All substitutions are made analogously for the *-variables.

In the end we get: (leaving z_t, z_t^* implicit for notational convenience)

$$
\begin{aligned}
&(b) \quad \widehat{c}_t = \mathbb{E}_t \widehat{c}_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} + \mathbb{E}_t \widehat{\xi}_{c,t+1} - \widehat{\xi}_{c,t}) \\
&(k) \quad \mathbb{E}_t z_{t+\tau} + \widehat{\xi}_{k,t} + (\eta - 1)\widehat{k}_t - \widehat{\xi}_{c,t} + \mathbb{E}_t \widehat{\xi}_{c,t+\tau} = -\sigma \widehat{c}_t, \\
&(k^*) \quad \mathbb{E}_t z^*_{t+\tau^*} + \widehat{\xi}_{k,t} + (\eta - 1)\widehat{k}^*_t - \widehat{\xi}_{c,t} + \mathbb{E}_t \widehat{\xi}_{c,t+\tau^*} = -\sigma \widehat{c}_t, \\
&(PC) \quad \pi_{H,t} = \beta \mathbb{E}_t \pi_{H,t+1} + \frac{\epsilon - 1}{\kappa}(\widehat{w}_t - \widehat{a}_t - (1-\gamma)\widehat{s}_t), \\
&(PC^*) \quad \pi_{F,t} = \beta \mathbb{E}_t \pi_{F,t+1} + \frac{\epsilon - 1}{\kappa}(\widehat{w}^*_t - \widehat{a}_t + \gamma \widehat{s}_t), \\
&(MC) \quad \widehat{a}_t + \varphi^{-1}[\widehat{w}_t - \sigma \widehat{c}_t + \widehat{\xi}_{c,t}] = \frac{c_{ss}}{y_{H,ss}}\widehat{c}_t + \gamma \frac{k_{ss}}{y_{H,ss}}\widehat{k}_t + (1-\gamma)\frac{k^*_{ss}}{y_{H,ss}}\widehat{k}^*_t - \frac{1-\gamma}{\varsigma}\widehat{s}_t, \\
&(MC^*) \quad \widehat{a}_t + \varphi^{-1}[\widehat{w}^*_t - \sigma \widehat{c}_t + \widehat{\xi}_{c,t}] = \frac{c_{ss}}{y_{F,ss}}\widehat{c}_t + \gamma \frac{k_{ss}}{y_{F,ss}}\widehat{k}_t + (1-\gamma)\frac{k^*_{ss}}{y_{F,ss}}\widehat{k}^*_t + \frac{\gamma}{\varsigma}\widehat{s}_t,
\end{aligned}
$$

$(s)\quad \pi_t = \gamma\pi_{H,t} + (1-\gamma)\pi_{F,t},$

$(\Pi)\quad \widehat{s}_t = \widehat{s}_{t-1} + \pi_{H,t} - \pi_{F,t},$

$(i)\quad i_t = \phi_i i_{t-1} + \phi_\pi \pi_t + \xi_{i,t}.$

Now defining $\Delta x_t := x_t - x_t^*$ (and $\Delta\pi_t := \pi_{H,t} - \pi_{F,t}$) we can subtract each F-equation from its corresponding H-equation, while preserving the currency-union level equations. This returns a 3-equation system in 3 variables, $\Delta\widehat{k}_t, \Delta\pi_t, \Delta\widehat{w}_t$ (recall we characterized $z_{t+\tau}, z^*_{t+\tau^*}$ purely in terms of $\widehat{k}_t, \widehat{k}_t^*$):

$$(\Delta k)\quad \mathbb{E}_t\{z_{t+\tau} - z^*_{t+\tau^*}\} + (\eta-1)\Delta\widehat{k}_t + \mathbb{E}_t\{\widehat{\xi}_{c,t+\tau} - \widehat{\xi}_{c,t+\tau^*}\} = 0,$$

$$(\Delta PC)\quad \Delta\pi_t = \beta\mathbb{E}_t\Delta\pi_{t+1} + \frac{\epsilon-1}{\kappa}\left(\Delta\widehat{w}_t - \sum_{j=0}^{t}\Delta\pi_t\right),$$

$$(\Delta MC)\quad \varphi^{-1}\Delta\widehat{w}_t = \left(\gamma\frac{k_{ss}}{y_{H,ss}} + (1-\gamma)\frac{k^*_{ss}}{y_{H,ss}}\right)\Delta\widehat{k}_t - \frac{1}{\varsigma}\sum_{j=0}^{t}\Delta\pi_t,$$

where we have used $y_{H,ss} = y_{F,ss}$ and $\widehat{s}_t = \widehat{s}_{t-1} + \Delta\pi_t$. Now using the result of step 4 above, and assuming that the exogenous process is diagonal Markov, $\widehat{\xi}_{x,t} = \rho_x\widehat{\xi}_{x,t-1} + \varepsilon_{x,t} \implies \mathbb{E}_t\widehat{\xi}_{x,t+j} = \rho_x^j\widehat{\xi}_{x,t}$, we can see

$$\begin{aligned}
\mathbb{E}_t\{z_{t+\tau} - z^*_{t+\tau^*}\} &= \mathbb{E}_t\sum_{j\geq 0}\beta^j(1-\delta^h)^j\Big[A_h(\widehat{\xi}_{h,t+\tau+j} - \widehat{\xi}_{h,t+\tau^*+j}) - (\widehat{\xi}_{c,t+\tau+j} - \widehat{\xi}_{c,t+\tau^*+j})\\
&\quad +\beta(1-\delta^h)(\widehat{\xi}_{c,t+\tau+j+1} - \widehat{\xi}_{c,t+\tau^*+j+1})\Big] - \sum_{j=0}^{t-1}\underbrace{\frac{(1-\delta^h)^{t-j}}{1-\beta(1-\delta^h)^2}A_h\nu\delta^h\eta}_{=:A_{k,j}}\,\Delta\widehat{k}_j\\
&\quad -\sum_{j\geq 0}\underbrace{\frac{\beta^j(1-\delta^h)^j}{1-\beta(1-\delta^h)^2}A_h\nu\delta^h\eta}_{=:B_{k,j}}\,\mathbb{E}_t\Delta\widehat{k}_{t+j}\\
&= \sum_{j\geq 0}\beta^j(1-\delta^h)^j\Big[A_h(\rho_h^{t+\tau+j} - \rho_h^{t+\tau^*+j})\widehat{\xi}_{h,t} - (\rho_c^{t+\tau+j} - \rho_c^{t+\tau^*+j})\widehat{\xi}_{c,t}\\
&\quad +\beta(1-\delta^h)(\rho_c^{t+\tau+j+1} - \rho_c^{t+\tau^*+j+1})\widehat{\xi}_{c,t}\Big] + \ldots\\
&= \frac{A_h(\rho_h^{t+\tau} - \rho_h^{t+\tau^*})}{1-\beta(1-\delta^h)}\widehat{\xi}_{h,t} - (\rho_c^{t+\tau} - \rho_c^{t+\tau^*})\widehat{\xi}_{c,t} - \sum_{j=0}^{t-1}A_{k,j}\Delta\widehat{k}_j - \sum_{j\geq 0}B_{k,j}\mathbb{E}_t\Delta\widehat{k}_{t+j}.
\end{aligned}$$

(We have repeatedly used the fact that $\sum_{j\geq 0} ab^j = \frac{a}{1-b}$ for $b\in(0,1)$.)

At this stage it is already visible that only a housing taste shock can drive a wedge between the housing investment rates of the two countries – the only other shock appearing in equation (Δk) is the consumption taste shock, which cancels out. Suppose $\xi_{h,t} = 0$, a.s.$\forall t$. Then the system reads

$$(\Delta k) \quad (\eta - 1)\Delta\widehat{k}_t - \sum_{j=0}^{t-1} A_{k,j}\Delta\widehat{k}_j - \sum_{j\geq 0} B_{k,j}\mathbb{E}_t\Delta\widehat{k}_{t+j} = 0,$$

$$(\Delta PC) \quad \Delta\pi_t = \beta\mathbb{E}_t\Delta\pi_{t+1} + \frac{\epsilon - 1}{\kappa}\Big(\Delta\widehat{w}_t - \sum_{j=0}^{t}\Delta\pi_t\Big),$$

$$(\Delta MC) \quad \varphi^{-1}\Delta\widehat{w}_t = \left(\gamma\frac{k_{ss}}{y_{H,ss}} + (1-\gamma)\frac{k^*_{ss}}{y_{H,ss}}\right)\Delta\widehat{k}_t - \frac{1}{\varsigma}\sum_{j=0}^{t}\Delta\pi_t.$$

Given determinacy of the original MSV-system, a simple substitution using the above 3 equations shows that the unique solution is $\Delta\widehat{k}_t = 0, \Delta\pi_t = 0, \Delta\widehat{w}_t = 0$. (That is, if the original stochastic processes are unique, then their difference must be unique, i.e. there can be no sunspot shocks.) ■