2.4.1 Proposition : Let $C$ be the collection of all intervals of the form $(-\infty, a]$ in $\mathbb{R}$. Then

$\sigma(C) = \mathrm{Bor}(\mathbb{R})$.

$\boxed{\text{PROOF}}$ : We will prove this by showing $\mathrm{Bor}(\mathbb{R}) \subseteq \mathrm{Bor}'(\mathbb{R})$ and $\mathrm{Bor}'(\mathbb{R}) \subseteq \mathrm{Bor}(\mathbb{R})$ and thus $\mathrm{Bor}'(\mathbb{R}) = \mathrm{Bor}(\mathbb{R})$; $\mathrm{Bor}'(\mathbb{R})$ is $\sigma(C)$ where $C$ is the collection of all intervals of the form $(-\infty, a]$

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1) $\mathrm{Bor}(\mathbb{R}) \subseteq \mathrm{Bor}'(\mathbb{R})$ :

given $(a, b) \in \mathrm{Bor}(\mathbb{R})$ we construct

$$(a, b] = (-\infty, b] \cap \underbrace{\left((-\infty, a]\right)^C}_{\substack{\text{in } \mathrm{Bor}'(\mathbb{R}) \\ \text{because closedness} \\ \text{under complement}}} = (-\infty, b] \cap (a, \infty) =$$

$$= (a, b] \in \mathrm{Bor}'(\mathbb{R})$$

We can now construct $(a, b)$ ~~from its~~ as

$$(a, b) = \bigcup_{n \in \mathbb{N}} (a, b - \tfrac{1}{n}]$$ where each $(a, b - \frac{1}{n}] \in \mathrm{Bor}'(\mathbb{R})$

2) $\underline{Bor'(\mathbb{R}) \subseteq Bor(\mathbb{R})}$:

given ~~a~~ $(-\infty, a]$ we can construct:

$$(a, b) \in Bor(\mathbb{R}) \Rightarrow (a, b)^c = (-\infty, a] \cup [b, \infty) \in Bor(\mathbb{R})$$

$$\Rightarrow (-\infty, a] \in Bor(\mathbb{R})$$

$$1) + 2) \Rightarrow Bor'(\mathbb{R}) = Bor(\mathbb{R})$$

$\square$