The problem computing the Hale–Tee(HT) transform:

$$w(q) = 2\arcsin\left[\chi_c \frac{\mathrm{sn}\,(k_0 q, k)}{\mathrm{dn}\,(k_0 q, k)}\right] \tag{1}$$

where χ_c is the transformation parameter, k_0 is the base wavenumber:

$$k_0 = \frac{\mathcal{K}(\sqrt{1-\chi_c^2})}{\pi} \tag{2}$$

and $\mathcal{K}$ denotes the complete Jacobi integral of the first kind:

$$\mathcal{K}(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{3}$$

The inaccuracies stem from the catastrophic loss of precision in evaluating the Jacobi elliptic functions $\mathrm{sn}(u,k)$ and $\mathrm{dn}(u,k)$ when the value of u is close to $\frac{\pi}{2}$, and k is near 1.

0.1 Direct Approach

We Taylor expand sn and dn for small χ_c and evaluate the series, we found that for $\chi_c = 0.01$, we need 10 terms in the expansion to reach 32 digit accuracy. The Taylor expansion at $\chi_c = 0$ is given by:

$$\begin{aligned}
&\mathrm{sn}(q, 1-\chi_c^2) = && (4)\\
&\quad \mathrm{O}(1): \ \tanh(q) + && (5)\\
&\quad \mathrm{O}(\chi_c^2): \ -\frac{1}{4}\left(q\,\mathrm{sech}^2 q - \tanh q\right) && (6)\\
&\quad \mathrm{O}(\chi_c^4): \ -\frac{1}{64}\left(4q^2\tanh q\,\mathrm{sech}^2 q + \sinh 2q - 11\tanh Q + 9q\,\mathrm{sech}^2 q\right) && (7)\\
& && (8)
\end{aligned}$$

0.2 Indirect Approach

The elliptic functions can be expressed in terms of incomplete Jacobi elliptic integral given:

$$u = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{9}$$ {incEllipK}

then:

$$\begin{aligned}
\mathrm{sn}\,u &= \sin\phi, && (10)\\
\mathrm{cn}\,u &= \cos\phi, && (11)\\
\mathrm{dn}\,u &= \sqrt{1-k^2\sin^2\phi}, && (12)
\end{aligned}$$

We Taylor expand the equation (9) in powers χ_c^2 to order p and solve the resulting differential equation by separation of variables, to have:

$$u_\xi = -\frac{1}{\sqrt{1+\xi^2}\sqrt{1+(\chi_c\xi)^2}} \tag{13}$$

where

$$\xi = \frac{\tan\left(\frac{\pi}{2}-\phi\right)}{\chi_c} \tag{14}$$

and the resulting expression for u_ξ is:

$$u_\xi = -\frac{1}{\sqrt{1+\xi^2}\sqrt{1+(\chi_c\xi)^2}} = -\frac{1}{\sqrt{1+\xi^2}}\left(1-\frac{\chi_c^2}{2}\xi^2+\mathrm{O}(\xi_c^4)\right) \tag{15}$$

$$u(\xi) = -\operatorname{arcsinh}(\xi)+\mathrm{O}(\xi^2) \tag{16}$$