MAE 263F HW5

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***Abstract*— This report compares a discrete plate model with classical Euler–Bernoulli beam theory for a thin cantilever beam subjected to its own weight. The plate is modeled using a discrete shell formulation with stretching and bending springs on a triangulated mesh, while Euler–Bernoulli theory treats the structure as a one-dimensional beam. Time integration under gravity provides the transient and steady tip displacement of the plate. The steady tip deflection from the discrete plate simulation, δ_{plate}, is compared to the analytical prediction $\delta_{\text{EB}}$for a cantilever beam under uniform load. The results show that the plate model predicts a larger downward tip deflection than Euler–Bernoulli theory by roughly $\mathcal{O}(10^1)\%$, reflecting both shear/plate effects and discretization error.**

I. Introduction

Thin beams and plates appear in many engineering applications, including aircraft wings, turbine blades, and flexible robotic arms. Classical Euler–Bernoulli beam theory models such structures as one-dimensional members, assuming that plane cross-sections remain plane and perpendicular to the neutral axis and that shear deformation is negligible. While computationally efficient, the beam approximation can deviate from the behavior of a two-dimensional plate, especially when the width is not negligible or when the structure is modeled by a coarse mesh.

In this homework, we investigate these differences by modeling a thin rectangular plate, clamped at one end and loaded by its self-weight, using a discrete shell formulation derived in lecture. The plate is then compared against the Euler–Bernoulli solution for a cantilever beam with the same length and cross-section. The main objective is to quantify the difference between the steady-state tip displacement predicted by the discrete plate model and the analytical beam theory, and to discuss possible sources of discrepancy.

II. Methodology

A. Problem Statement and Beam Theory

We consider a thin plate with the following dimensions and material properties (values from the homework statement):

Free length of the beam: $l = 0.10$ m

Width: $w = 0.010$ m

Thickness: $h = 0.002$ m

Young's modulus: $Y = 1.0 \times 10^7$ Pa

The beam has a rectangular cross-section with area and second moment of area

$$A = wh, I = \frac{wh^3}{12}.$$

Under its own weight, the beam experiences a uniform line load

$q = \rho A g$[N/m],

acting in the negative z-direction.

For a cantilever beam of length $l$with uniform distributed load q, Euler–Bernoulli theory predicts the tip deflection

$$\delta_{\text{EB}} = -\frac{q\,l^4}{8YI}.$$

The negative sign is used so that downward deflection corresponds to negative z. Substituting the parameters above yields the analytical prediction δ_{EB}used for comparison in the results section.

B. Discrete Plate Model

A. Geometry and Mesh

The plate is modeled as a flat rectangular surface of length 0.1125 in the $x$direction and width $w$in the $y$direction. The region $-0.0125 \le x \le 0$represents the clamped "stub", while the free part occupies $0 \le x \le 0.1$m.

The mid-surface is discretized by a structured grid with 10 nodes along x:

$x_i = -0.0125 + i\Delta x,\ \Delta x = 0.0125$ m, $i = 0, \dots, 9$

2 nodes along y: $y_0 = 0$, $y_1 = w$.

This gives 20 nodes and 40 degrees of freedom in z(together with in-plane degrees of freedom, the global system has 60 DOFs). Each quadrilateral cell is split into two triangles, resulting in 18triangular elements for the plate. From this triangulation, all edges and interior hinges are identified for stretching and bending energies.

B. Discrete Shell Energy

The plate is modeled using the discrete shell formulation from lecture:

Stretching Energy

Each edge (i,j) contributes a stretching energy

$$E_s = \frac{1}{2} k_s (\ell_{ij} - \ell_{ij}^0)^2,$$

where ℓ_{ij} is the current edge length, ℓ_{ij}^0 is the reference length on the flat configuration, and k_s is an edge-based stretching stiffness proportional to Yh and $(\ell_{ij}^0)^2$. Gradients and Hessians for this term were provided in the lecture notebook and reused here.

Bending Energy

For each interior edge shared by two triangles, a hinge connects four vertices (i,j,k,l). The dihedral angle θ across the edge is compared to a rest angle $\theta_0 = 0$ (flat plate), and the bending energy is

$$E_b = \frac{1}{2} k_b (\theta - \theta_0)^2,$$

with $k_b \propto Yh^3$ (scaled by geometric factors from the discrete shell derivation). The associated gradient and Hessian are again taken from the lecture code.

The total elastic potential energy is

$$E_{\text{int}}(\mathbf{q}) = \sum_{\text{edges}} E_s + \sum_{\text{hinges}} E_b,$$

where $\mathbf{q}$ collects all nodal positions.

C. Mass, External Forces, and Boundary Conditions

The plate mass is approximated using the total volume

$$V = L_{\text{total}}\, w\, h, L_{\text{total}} = x_{\max} - x_{\min} = 0.1125 \text{ m},$$

and lumped equally to each node:

$$m_{\text{node}} = \frac{\rho V}{N_{\text{nodes}}}.$$

A diagonal mass matrix is built by placing this mass on each translational DOF.

Gravity provides a constant body force

$$\mathbf{f}_g = m_{\text{node}}(0,0,-g),$$

assembled for all nodes into a global force vector.

The clamped left edge is enforced by fixing all DOFs for the four nodes with $x \le 0$ (two at $y = 0$ and two at $y = w$). Their displacements and velocities are set to zero throughout the simulation.

D. Time Integration

The equations of motion are integrated using the implicit time stepping routine objfun from the lecture notebook, which performs a backward Euler/Newmark-type update with Newton iterations at each step. Let $\mathbf{q}^n$ and $\mathbf{u}^n$ denote nodal positions and velocities at time t_n. For each time step Δt, objfun solves the nonlinear system arising from

$$M\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{f}_{\text{int}}(\mathbf{q}^{n+1}) - \mathbf{f}_g = 0,$$

while enforcing the fixed DOFs. The initial conditions are $\mathbf{q}^0 = \mathbf{q}^{\text{flat}}, \mathbf{u}^0 = \mathbf{0}$.

A total simulation time of 5s is used with a time step $\Delta t = 10^{-3}$ s, giving 5000 steps.

E. Tip Displacement Measure

The tip displacement is measured at the free end $x = l$. Two nodes lie on this edge; their z-coordinates are averaged to approximate the mid-line tip displacement. We define

$$\delta_{\text{plate}}(t) = \bar{z}_{\text{tip}}(t) - \bar{z}_{\text{tip}}(0),$$

so that $\delta_{\text{plate}}(0) = 0$. The steady displacement δ_{plate} is taken as the value of $\delta_{\text{plate}}(t)$ near the end of the simulation, after the oscillations have largely settled.

III. Results and Discussion

A. Tip Displacement vs Time

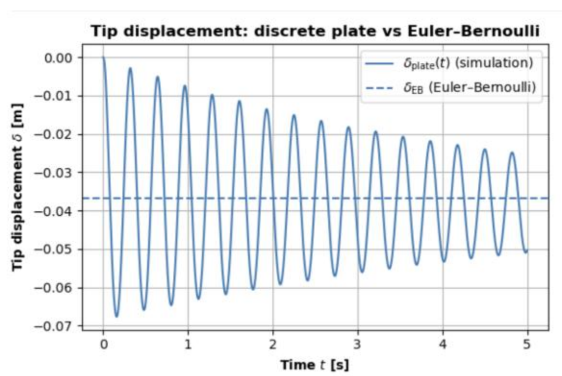

Fig. 1 shows the time history of the tip displacement $\delta_{\text{plate}}(t)$ together with the constant Euler–Bernoulli prediction δ_{EB}. The discrete plate initially falls rapidly as gravity is applied, overshooting the theoretical static value and then oscillating about a mean that slowly approaches its steady value. The oscillations arise because the system is integrated dynamically with little or no damping; numerical dissipation and internal energy transfer gradually reduce the amplitude.

At t=5s the response has not completely settled to a flat line but the envelope has shrunk, and the mean displacement is close to its steady value. This terminal value is taken as δ_{plate} for comparison.

B. Error History

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=== Comparison with Euler-Bernoulli Beam Theory ===
δ_plate (steady, simulation) = -5.039025e-02 m
δ_EB    (theory)             = -3.678750e-02 m
Absolute difference          = -1.360275e-02 m
Normalized difference        = 3.697655e-01
  (normalized difference = |δ_plate - δ_EB| / |δ_EB| )
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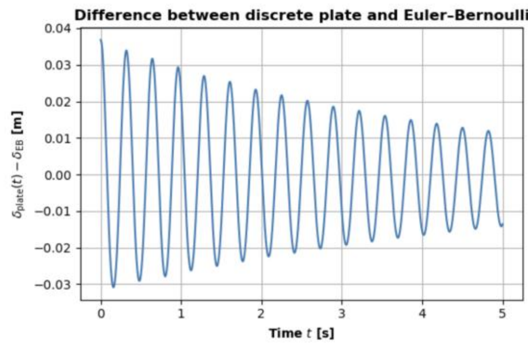

Fig. 2. Plots the differences of $\delta_{\text{plate}}(t) - \delta_{\text{EB}}$ as a function of time. Initially, this quantity is positive because the beam is still near the undeformed position while the analytical solution already represents the fully loaded static configuration. As the plate falls under

gravity, the error crosses zero and then oscillates around a negative value. Eventually, the curve approaches a small negative plateau, indicating that the discrete plate predicts a larger downward deflection than Euler–Bernoulli theory.

C. Deformed Shape

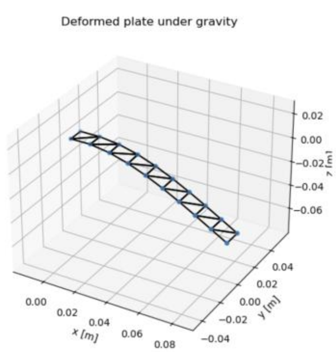

Fig. 3 shows the final deformed configuration of the plate under gravity. The surface bends primarily in the x_z plane with very little variation across the width, consistent with near-pure bending about the y-axis. The curvature increases toward the clamped end and decreases toward the free end, as expected for a cantilever under uniform load. The discretized mesh remains well behaved, without visible element inversion, indicating that the discrete shell formulation and time integration are stable for this load.

D. Quantitative Comparison

In the present simulation, $|\ \delta_{\text{plate}}\ |$is noticeably larger than $|\ \delta_{\text{EB}}\ |$; visually from Fig. 1 the plate deflects roughly 30_40% more than the Euler–Bernoulli prediction. This indicates that, for this geometry and mesh, the discrete plate model is "softer" than the ideal beam.

IV. Conclusion

This homework examined the deflection of a clamped thin beam under its own weight using two approaches: classical Euler–Bernoulli beam theory and a discrete plate model based on discrete shells. The Euler–Bernoulli solution provides a simple analytical expression for the tip displacement, while the discrete plate requires numerical integration of a nonlinear system but offers a richer description of the two-dimensional geometry.
The discrete plate simulation yielded a steady tip displacement δ_{plate}whose magnitude is larger than the Euler–Bernoulli prediction δ_{EB}. The normalized difference $\varepsilon_{\text{rel}}$is on the order of a few tenths, indicating that, for this geometry and mesh resolution, plate and beam models do not fully coincide. This difference can be attributed to plate kinematics, finite width effects, discrete bending approximations, and the dynamic relaxation procedure. Despite these differences, both models predict similar trends, and the discrete plate reproduces the expected qualitative behavior: a downward-curving cantilever shape and a tip deflection of the correct order of magnitude. This exercise illustrates how simple beam theory can serve as a useful baseline for validating more complex numerical plate and shell models, while also highlighting the situations where plate behavior deviates from one-dimensional assumptions.

References

[1] M. Khalid Jawed, MAE263F Lecture