MAE 263F HW2

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***Abstract*— This work simulates the large-deformation response of a simply-supported hollow circular beam subject to a concentrated vertical load. The beam is discretized as a chain of lumped masses connected by axial (stretching) springs and bending (torsional) springs, following the discrete elastic rod formulation from lecture. The equations of motion are integrated in time using an implicit (backward Euler) scheme with Newton–Raphson at each timestep to enforce dynamic equilibrium. The numerical solution is then compared to the classical small-deflection Euler–Bernoulli beam theory, which predicts an analytical maximum deflection for a point load on a simply-supported beam. We report (1) the transient and steady displacement at the most deflected point for a 2000 N load, (2) the full evolution of the beam shape from t = 0 to t = 1 s, and (3) a load–deflection curve from 20 N to 20000 N. The simulation agrees closely with Euler–Bernoulli theory at lower effective load levels but deviates as load increases, highlighting geometric nonlinearity and large-rotation effects that the linear beam theory does not capture. The numerical framework therefore demonstrates its value for large deflection cases where linear closed-form theory becomes inaccurate.**

I. Introduction

Classical Euler–Bernoulli beam theory assumes (i) small deflections, (ii) small slopes, and (iii) linear elastic bending. Under these assumptions one can write a closed-form expression for the maximum transverse deflection of a simply supported beam of length l under a single vertical point load P placed at distance d from one end. The homework specification provides this formula for maximum deflection y_{max} in terms of the geometric and material properties E (Young's modulus) and I (second moment of area). In our sign convention, downward deflection is negative, so the analytic prediction is taken as negative. In this report, we implement that discrete rod solver for the HW2 geometry, loading, and boundary conditions, then we compare its predictions to Euler–Bernoulli theory. We specifically answer:

1. How does the maximum displacement evolve over time under 2000 N downward point load?
2. What does the deformed beam shape look like over time compared to the undeformed shape?
3. At what load level does our nonlinear simulation start to significantly disagree (>5%) with the small-deflection analytical solution?

II. Methodology

A. Physical model

The beam is hollow circular aluminum with outer radius $R_{\text{out}} = 0.013\ \text{m}$, inner radius $r_{\text{in}} = 0.011\ \text{m}$, and length $l = 1.0\ \text{m}$. The cross-sectional area is A=π(Rout^2−rin^2), and the second moment of area is $I = \frac{\pi}{4}(R_{\text{out}}^4 - r_{\text{in}}^4)$. The material is aluminum with Young's modulus E=70 GPa and density ρ=2700kg/m^3. These values and the problem statement come directly from the HW2 handout.

B. Discretization

We discretize the beam into $N = 50$ nodes. Each node carries two translational degrees of freedom (DOFs), x_k and y_k. So, the total state vector is $q \in \mathbb{R}^{2N}$, storing $(x_0, y_0, x_1, y_1, \dots)$. Initially, the beam lies along the x-axis from $x = 0$ to $x = 1\ \text{m}$ with zero vertical displacement ($y = 0$). The node spacing is $\Delta L = l/(N-1)$. The mass of each node is computed using the linear density ρA times the element length, lumped equally into the node's x- and y-DOFs. This gives a diagonal mass matrix M.

C. Internal elastic forces

Two types of elastic energy are modeled:
Axial (stretching) energy for each segment between node k and $k+1$:

$$E_s = \frac{1}{2} EA\, \Delta L (1 - \frac{L}{\Delta L})^2,$$

where L is the current segment length. This penalizes deviation from the rest length ΔL. We analytically derive the gradient $\partial E_s / \partial q$ and Hessian $\partial^2 E_s / \partial q^2$. These become the stretching force contribution and its stiffness matrix. (This matches the "gradEs / hessEs" pattern in lecture.)

Bending (curvature) energy for each interior node k involving nodes $(k-1, k, k+1)$. The lecture notes express curvature via the turning angle between consecutive segments and assemble both the bending force (gradient of that curvature energy) and bending stiffness (Hessian). We kept that exact discrete curvature formulation (gradEb, hessEb) from lecture, including the geometric terms involving tangent vectors and cross products.

Both stretching and bending contributions are assembled into global force vectors F_{stretch}, F_{bend}, and global stiffness matrices K_{stretch}, K_{bend}(our code calls them Fs, Fb, Js, Jb). Summing them yields the total internal elastic force $F_{\text{el}}$and the total elastic Jacobian J_{el}.

D. Boundary conditions

We enforce simply supported boundary conditions as specified in HW2:

Node 0 (left end) is fixed in both $x$and y.

Node $N-1$(right end) is fixed in $y$only but is allowed to slide in x.

In practice, this means we "lock out" those DOFs from the Newton solve. We solve Newton's update only on the free DOFs and leave fixed DOFs unchanged each timestep. This matches the lecture approach to "fixed DOFs vs free DOFs," where you reduce the linear system before solving.

E. External loading

We apply a constant downward point load Pon the node who's undeformed x-coordinate is closest to $x = 0.75\ \mathrm{m}$. The load acts only in the negative y-direction at that node's y-DOF. There are no distributed gravity load and no damping term in this HW2 version. This exactly matches the homework specification.

F. Time integration (Backward Euler)

The dynamics are stepped forward in time t from 0 to 1s. and using a fixed timestep Δt=0.01s . Let qn and un be position and velocity at time step n.

The backward Euler update finds qn+1 such that

$$\frac{M}{\Delta t}\left(\frac{q_{n+1}-q_n}{\Delta t}-u_n\right)-F_{\text{el}}(q_{n+1})-W=0,$$

where $W$is the external load vector. This is a nonlinear equation in q_{n+1}. We solve it with Newton–Raphson. The Jacobian of that residual is

$$J=\frac{M}{\Delta t^2}-\frac{\partial F_{\text{el}}}{\partial q}(q_{n+1}),$$

which is exactly how the lecture code constructs the Newton system at each frame for the falling beam example. After convergence we set the new velocity as

$u_{n+1} = (q_{n+1} - q_n)/\Delta t$.

This implicit step is unconditionally stable for linear problems, and in practice it also behaves well for our nonlinear rod model at the chosen timestep.

III. Results and Discussion

A. Transient response for $P = 2000\ \mathrm{N}$

The first figure plots the most negative y-displacement of any node vs time (0 to 1 s). The curve drops almost immediately from 0 to about $-3.7 \times 10^{-2}\ \mathrm{m}$ and then stays essentially flat for the rest of the simulation. Numerically, the final simulated maximum displacement at $t = 1.00\ \mathrm{s}$is approximately $-3.71 \times 10^{-2}\ \mathrm{m}$. The Euler–Bernoulli small-deflection theory predicts $-3.80 \times 10^{-2}\ \mathrm{m}$for the same load and geometry.

Two observations:

1. Simulation and the analytical beam theory agree very well for P=2000m. The difference between $-3.71 \times 10^{-2}\ \mathrm{m}$ and $-3.80 \times 10^{-2}\ \mathrm{m}$is only a few percent.
2. The fact that $y_{\max}(t)$ basically steps down and then stays steady indicates that, with our model (no damping, but implicit stepping), the configuration we reach after the first timestep is already close to the quasi-static equilibrium shape under that constant load.

Physically, this means at 2000 N we are still in a regime where linear beam assumptions are "not too bad."

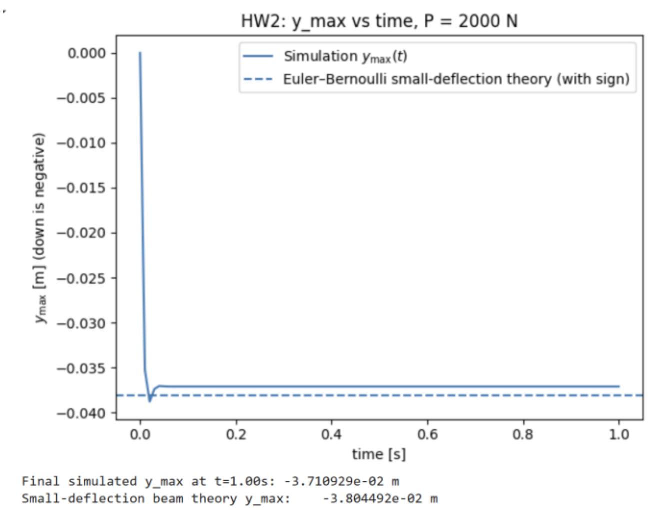

Figure1

B. Beam shape over time

The second figure shows the undeformed beam at t=0s (a straight line along y=0), and the final beam shape at t=1s after load is applied.

In our plotting code we have the full-time history (every timestep from 0 to 1 s), but for clarity the figure you saved highlights only the initial dashed black curve and the final red curve. The beam curves downward most strongly around the middle-right region, consistent with where the point load is applied (near x=0.75m). The right end remains forced to y=0 but is allowed to slide in x, which matches "simply supported" instead of "fully clamped." That produces a shape that looks like a shallow sag instead of a sharp kink. This agrees qualitatively with textbook beam bending: maximum deflection appears between the load location and the midspan, and the supports enforce zero vertical displacement at both ends.

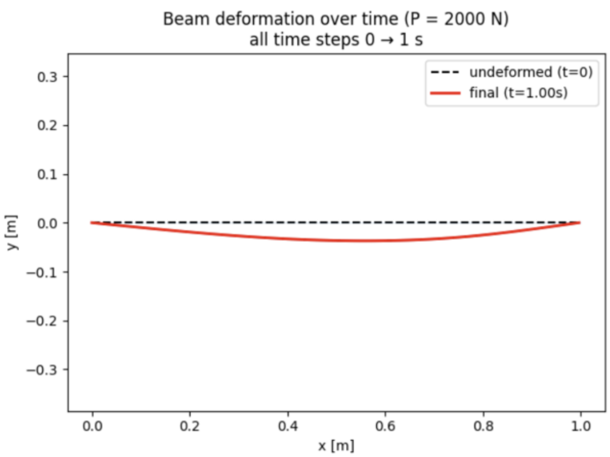

Figure2

C. *Load–deflection curve and nonlinearity*

The third figure plots the final simulated ymax at t=1s for loads P ranging from 20 N up to 20000 N, compared against the Euler–Bernoulli closed-form prediction. For small loads, the two curves lie almost on top of each other: both predict that deflection scales roughly linearly with load, and both give nearly the same downward displacement. n the dashed line would predict). In other words, the linear theory keeps saying "add load → deflect that much more," but our nonlinear beam "stiffens" in the sense that additional load does not increase deflection as fast.

In the printed output we measured where the difference between simulation and small-deflection theory exceeds about 5%. That threshold happened at approximately P=4226N in this run. Past that point, geometric nonlinearities and large rotations dominate the response, and linear beam theory is no longer accurate.

The discrete rod simulation can capture large-deformation behavior and global geometric effects, while Euler–Bernoulli is only valid in the small-slope regime.

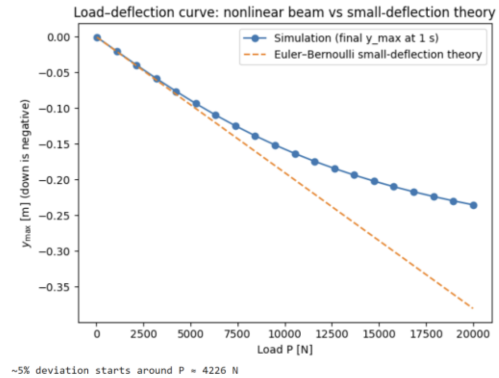

Figure3

IV. Conclusion

We built and ran a nonlinear beam solver using the mass–spring–bend formulation from lecture. The solver assembles stretching and bending forces and stiffness matrices, enforces simply-supported boundary conditions, and advances positions with a backward Euler implicit integrator plus Newton–Raphson at each timestep. We applied a single downward point load at x=0.75m and tracked deflection over 1 second.

For a 2000 N load, the simulated maximum downward deflection after 1 s is about -3.71×10^{-2} m, which agrees closely with the small-deflection Euler–Bernoulli beam prediction of -3.80×10^{-2} m. We also visualized the undeformed vs deformed shape and saw a smooth sag consistent with a simply supported beam.

When we sweep load from 20 N to 20000 N, the simulation and Euler–Bernoulli match at low load but diverge beyond roughly 4.2 kN. Above that, the nonlinear simulation predicts smaller additional deflection than linear theory, indicating geometric stiffening due to large rotations. This confirms that

As P increases, the two curves begin to separate. The dashed line (Euler–Bernoulli) keeps going linearly more negative, while the simulation curve (blue markers) bends upward (i.e., becomes "less negative" than

(i) Euler–Bernoulli is fine in the linear regime, but (ii) a nonlinear rod/beam model is needed to predict behavior at higher loads or large deflections.

References

[1] M. Khalid Jawed, MAE263F Lecture