一.曲线面积的基本方法

1. S: $z = z(x,y) \Rightarrow S = \iint\limits_{S} ds = \iint\limits_{D_{xy}} \sqrt{1+z_x^2+z_y^2}\,dxdy$

2. S: $y = y(x,z) \Rightarrow S = \iint\limits_{S} ds = \iint\limits_{D_{xz}} \sqrt{1+y_x^2+y_z^2}\,dxdz$

3. S: $x = x(y,z) \Rightarrow S = \iint\limits_{S} ds = \iint\limits_{D_{yz}} \sqrt{1+x_y^2+x_z^2}\,dydz$

二.第一类曲面积分的基本方法

1. $S: z = z(x,y),\ \ f(x,y,z) \in C(S)$

$$\Rightarrow \iint\limits_{S} f(x,y,z)ds = \iint\limits_{D_{xy}} f(x,y,z(x,y))\sqrt{1+z_x^2+z_y^2}dxdy$$

2. $S: y = y(x,z),\ \ f(x,y,z) \in C(S)$

$$\Rightarrow \iint\limits_{S} f(x,y,z)ds = \iint\limits_{D_{xz}} f(x,y(x,z),z)\sqrt{1+y_x^2+y_z^2}dxdz$$

3. $S:\ \ x = x(y,z),\ \ f(x,y,z) \in C(S)$

$$\Rightarrow \iint\limits_{S} f(x,y,z)ds = \iint\limits_{D_{yz}} f(x(y,z),y,z)\sqrt{1+x_y^2+x_z^2}dydz$$

三.利用对称性计算第一类曲面积分

1. S关于$x=0$(yoz面)对称, $S_1 = S \cap \{x \geq 0\} \Rightarrow$

$$\begin{cases} (1)\ f(-x,y,z) = -f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 0 \\ (2)\ f(-x,y,z) = f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 2\iint\limits_{S_1} f(x,y,z)ds \end{cases}$$

2. S关于$y=0$(xoz面)对称, $S_2 = S \cap \{y \geq 0\} \Rightarrow$

$$\begin{cases} (1)\ f(x,-y,z) = -f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 0 \\ (2)\ f(x,-y,z) = f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 2\iint\limits_{S_2} f(x,y,z)ds \end{cases}$$

3. S关于$z=0$(xoy面)对称, $S_3 = S \cap \{z \geq 0\} \Rightarrow$

$$\begin{cases} (1)\ f(x,y,-z) = -f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 0 \\ (2)\ f(x,y,-z) = f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 2\iint\limits_{S_3} f(x,y,z)ds \end{cases}$$

4. S关于原点中心对称，$S_1 = S \cap \{x \geq 0\} \Rightarrow$

$$\begin{cases} (1)\ f(-x,-y,-z) = -f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 0 \\ (2)\ f(-x,-y,-z) = f(x,y,z) \Rightarrow \iint\limits_S f(x,y,z)ds = 2\iint\limits_{S_1} f(x,y,z)ds \end{cases}$$

5. 轮换对称性：$S \xrightarrow{x \to y, y \to z, z \to x} S \Rightarrow$

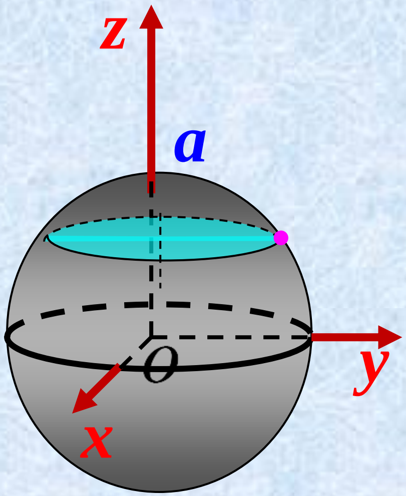
$$\iint\limits_S f(x,y,z)ds = \iint\limits_S f(y,z,x)ds = \iint\limits_S f(z,x,y)ds$$

特别：

$$\iint\limits_S f(x)ds = \iint\limits_S f(y)ds = \iint\limits_S f(z)ds$$

例. 半径为a的有质球面,其上任意点的面密度等于该点到球的一直径距离的平方,求球面的质量M.

解 建立如图所示坐标系,由题意知



曲面方程为 $x^2+y^2+z^2=a^2$且$\rho=x^2+y^2$

$$M=\iint_S (x^2+y^2)\mathrm{d}S=2\iint_{S_上}(x^2+y^2)\mathrm{d}S$$

$x^2+y^2+z^2=a^2$具有轮换对称性 $\Rightarrow$

$$\iint_S x^2\mathrm{d}S=\iint_S y^2\mathrm{d}S=\iint_S z^2\mathrm{d}S$$

$$\Rightarrow\quad M=\frac{2}{3}\iint_S (x^2+y^2+z^2)\mathrm{d}S=\frac{2}{3}a^2\iint_S \mathrm{d}S=\frac{2}{3}a^2\cdot 4\pi a^2=\frac{8}{3}\pi a^4.$$

例. 计算曲面积分$I=\iint\limits_S (x+y+z)\mathrm{d}S$,

其中S为上半球面: $z=\sqrt{1-x^2-y^2}$.

解 由对称性$\iint\limits_S x\mathrm{d}S=\iint\limits_S y\mathrm{d}S=0$.

$$\Rightarrow I=\iint\limits_S (x+y+z)\mathrm{d}S=\iint\limits_S z\mathrm{d}S$$

$$=\iint\limits_{D_{xy}} \sqrt{1-x^2-y^2}\sqrt{1+\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}}\mathrm{d}x\mathrm{d}y$$

$$=\iint\limits_{D_{xy}} \mathrm{d}x\mathrm{d}y=\pi.$$

例. 计算$\iint_S (xy+yz+xz)\mathrm{d}S$,

其中$S: z=\sqrt{x^2+y^2}$被柱面$x^2+y^2=2ax$所截下的部分.

解 由对称性$\iint_S xy\mathrm{d}S=\iint_S yz\mathrm{d}S=0 \quad \Rightarrow$

$$I=\iint_S (xy+yz+xz)\mathrm{d}S=\iint_S xz\mathrm{d}S =\iint_{D_{xy}} x\sqrt{x^2+y^2}\sqrt{2}\mathrm{d}x\mathrm{d}y$$

$$=2\sqrt{2}\int_0^{\frac{\pi}{2}} d\theta\int_0^{2a\cos\theta} r\cos\theta\cdot r\cdot rdr \ =\frac{64\sqrt{2}}{15}a^4$$