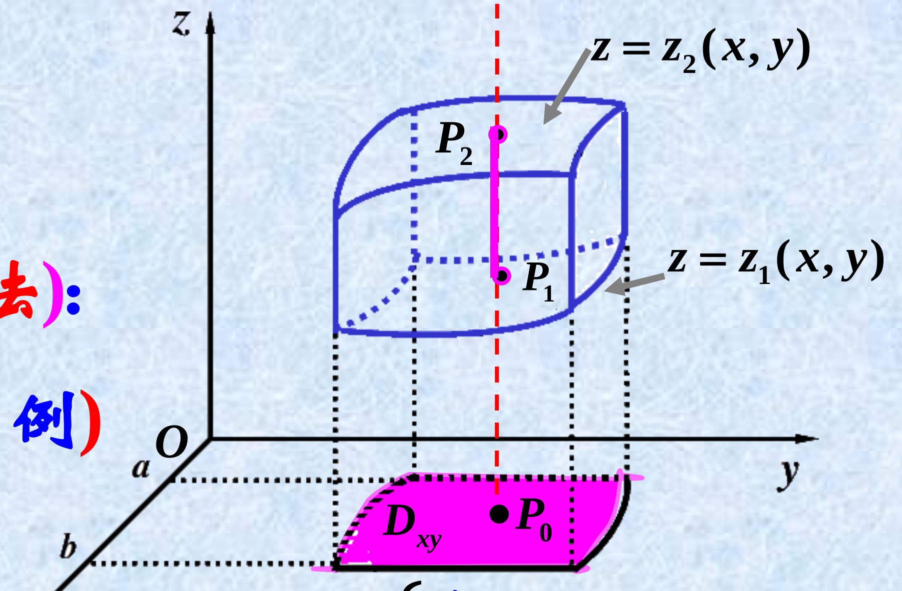
一.三重积分计算的基本技巧

1.直角坐标系下三重积分

(1) "先一后二"法(坐标面投影法):



I.几何定限法:(以xoy面投影为例)

(i) $V \xrightarrow{\text{投影}} xoy$面 $\Rightarrow D_{xy}$

(ii) 在D内沿平行于z轴方向自下而上引线: $\begin{cases} \text{穿入为z下限} \\ \text{穿出为z上限} \end{cases}$

II.代数定限法:

(i) V消去$z \Rightarrow D_{xy}$

(ii) $\forall (x_0, y_0) \in D$,计算$z_1(x_0, y_0), z_2(x_0, y_0)$:

若$z_1(x_0,y_0) \le z_2(x_0,y_0) \Rightarrow z_1(x,y) \le z \le z_2(x,y)$

$$\Rightarrow \iiint\limits_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint\limits_{D_{xy}} \mathrm{d}x\mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z$$

III. "先一后二"法(坐标面投影法)的应用

$$\iiint_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iint_{D_{xy}}\mathrm{d}x\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z$$

(i). $f(x,y,z)=F(x,y)$ ($V\xrightarrow{\text{投影}} xoy\text{面}\Rightarrow D_{xy}$)

$$\Rightarrow\iiint_V F(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iint_{D_{xy}}\mathrm{d}x\mathrm{d}y\int_{z_1(x,y)}^{z_2(x,y)} F(x,y)\mathrm{d}z$$

$$=\iint_{D_{xy}} F(x,y)[z_2(x,y)-z_1(x,y)]\mathrm{d}x\mathrm{d}y$$

(ii). $f(x,y,z)=G(x,z)$ ($V\xrightarrow{\text{投影}} xoz\text{面}\Rightarrow D_{xz}$)

$$\Rightarrow\iiint_V G(x,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z=\iint_{D_{xz}}\mathrm{d}x\mathrm{d}z\int_{y_1(x,z)}^{y_2(x,z)} G(x,z)\mathrm{d}y$$

$$=\iint_{D_{xz}} G(x,z)[y_2(x,z)-y_1(x,z)]\mathrm{d}x\mathrm{d}z$$

(iii). $f(x,y,z)=H(y,z)\Rightarrow$ ....

(2) "先二后一"法(坐标轴投影法)----截面法:

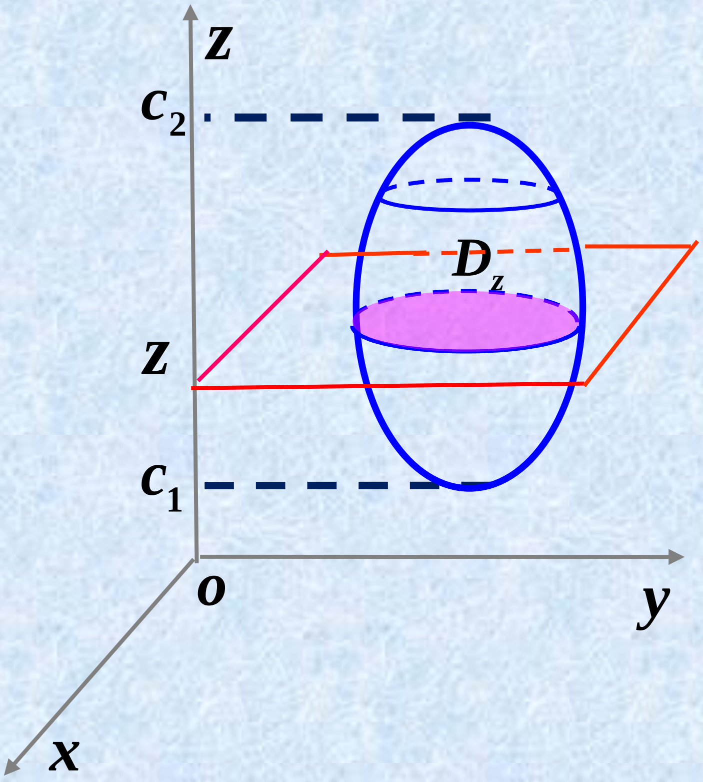
(以z轴投影为例)

(i) $V \xrightarrow{\text{投影}} z$轴 $\Rightarrow c_1 \le z \le c_2$

(ii) $\forall z \in [c_1, c_2]$,过z做垂直于z轴的平面与V相截 $\Rightarrow D_z$

$$\Rightarrow \iiint\limits_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \int_{c_1}^{c_2} \mathrm{d}z \iint\limits_{D_z} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y$$



(2) "先二后一"法(坐标轴投影法)----截面法的应用

$$\iiint\limits_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{c_1}^{c_2}\mathrm{d}z\iint\limits_{D_z} f(x,y,z)\mathrm{d}x\mathrm{d}y$$

(i). $f(x,y,z)=F(z)$ $\left(V \xrightarrow{\text{投影}} z\text{轴} \Rightarrow c_1 \le z \le c_2 \right)$

$$\Rightarrow \iiint\limits_V F(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{c_1}^{c_2}\mathrm{d}z\iint\limits_{D_z} F(z)\mathrm{d}x\mathrm{d}y = \int_{c_1}^{c_2} F(z)S(z)\mathrm{d}z$$

(ii). $f(x,y,z)=G(y)$ $\left(V \xrightarrow{\text{投影}} y\text{轴} \Rightarrow b_1 \le y \le b_2 \right)$

$$\Rightarrow \iiint\limits_V G(y)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{b_1}^{b_2}\mathrm{d}y\iint\limits_{D_y} G(y)\mathrm{d}x\mathrm{d}z = \int_{b_1}^{b_2} G(y)S(y)\mathrm{d}y$$

(iii). $f(x,y,z)=H(x)$ $\left(V \xrightarrow{\text{投影}} x\text{轴} \Rightarrow a_1 \le x \le a_2 \right)$

$$\Rightarrow \iiint\limits_V H(x)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{a_1}^{a_2}\mathrm{d}x\iint\limits_{D_x} H(x)\mathrm{d}y\mathrm{d}z = \int_{a_1}^{a_2} H(x)S(x)\mathrm{d}x$$

例 3　计算三重积分 $I=\iiint\limits_{\Omega}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 Ω 是椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leqslant 1$。

解： 本题也是用“先二后一”法计算最简单。由于被积函数 $f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$，所以可以把 $I=\iiint\limits_{\Omega}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ 分成 3 个三重积分来计算：

$$I=\iiint\limits_{\Omega}\frac{x^2}{a^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z+\iiint\limits_{\Omega}\frac{y^2}{b^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z+\iiint\limits_{\Omega}\frac{z^2}{c^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

其中 $\iiint\limits_{\Omega}\frac{z^2}{c^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_{-c}^{c}\frac{z^2}{c^2}\mathrm{d}z\iint\limits_{D_z}\mathrm{d}x\mathrm{d}y$，此时被积函数 $g(x,y,z)=\frac{z^2}{c^2}$，所以必须做垂直于 z 轴的平面截区域 Ω，得到椭圆截面 D_z，这里 D_z 表示椭圆面：$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1-\frac{z^2}{c^2}$。因为椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 的面积为 πab，所以截面 D_z 的面积为 $\pi\left[a\sqrt{1-\frac{z^2}{c^2}}\right]\left[b\sqrt{1-\frac{z^2}{c^2}}\right]=\pi ab\left(1-\frac{z^2}{c^2}\right)$。

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于是 $\iiint\limits_{\Omega}\frac{z^2}{c^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\int_{-c}^{c}\frac{z^2}{c^2}\mathrm{d}z\iint\limits_{D_z}\mathrm{d}x\mathrm{d}y=\int_{-c}^{c}\frac{\pi ab}{c^2}z^2\left(1-\frac{z^2}{c^2}\right)\mathrm{d}z=\frac{4}{15}\pi abc$。

同理可得：$\iiint\limits_{\Omega}\frac{x^2}{a^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{4}{15}\pi abc$，$\iiint\limits_{\Omega}\frac{y^2}{b^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{4}{15}\pi abc$。

所以：$I=3\left(\frac{4}{15}\pi abc\right)=\frac{4}{5}\pi abc$。

2.柱坐标系下三重积分 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ $\left(\begin{array}{l} 0 \le r < +\infty \\ 0 \le \theta \le 2\pi \\ -\infty \le z \le +\infty \end{array}\right)$

$$dxdydz = rdrd\theta dz$$

(1) "先一后二"法(坐标面投影法):

$$V_{r\theta z} = \{(r,\theta,z) \mid z_1(r,\theta) \le z \le z_2(r,\theta), r_1(\theta) \le r \le r_2(\theta), \alpha \le \theta \le \beta\}$$

$$\iiint_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{D_{xy}} \mathrm{d}x\mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\mathrm{d}z$$

$$= \iint_{D_{r\theta}} r\mathrm{d}r\mathrm{d}\theta \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\mathrm{d}z$$

$$= \int_\alpha^\beta d\theta \int_{r_1(\theta)}^{r_2(\theta)} rdr \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\mathrm{d}z$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \begin{pmatrix} 0 \le r < +\infty \\ 0 \le \theta \le 2\pi \\ -\infty \le z \le +\infty \end{pmatrix} \quad dxdydz = rdrd\theta dz$$

(2) "先二后一"法(坐标轴投影法)———截面法

$$V_{r\theta z} = \{(r,\theta,z) \mid r_1(z,\theta) \le r \le r_2(z,\theta), \alpha \le \theta \le \beta, c_1 \le z \le c_2\}$$

$$\iiint\limits_V f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{c_1}^{c_2} \mathrm{d}z \iint\limits_{D_z} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{c_1}^{c_2} \mathrm{d}z \iint\limits_{D_{r\theta}} f(r\cos\theta, r\sin\theta, z) r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_{c_1}^{c_2} dz \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta,z)}^{r_2(\theta,z)} f(r\cos\theta, r\sin\theta, z) r dr$$

3.球坐标系下三重积分
$$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases} \quad \begin{pmatrix} 0\le\rho<+\infty \\ 0\le\varphi\le\pi \\ 0\le\theta\le 2\pi \end{pmatrix}$$

$$dxdydz=\rho^2\sin\varphi d\rho d\varphi d\theta$$

$$V_{\rho\varphi\theta}=\{(\rho,\varphi,\theta)\,|\,\rho_1(\varphi,\theta)\le\rho\le\rho_2(\varphi,\theta),\varphi_1(\theta)\le\varphi\le\varphi_2(\theta),\ \alpha\le\theta\le\beta\}$$

$$\iiint\limits_{V_{xyz}} f(x,y,z)dxdydz$$

$$=\iiint\limits_{V_{\rho\varphi\theta}} f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)\cdot\rho^2\sin\varphi d\rho d\varphi d\theta$$

$$=\int_\alpha^\beta d\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)}d\varphi\int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\rho\sin\varphi\cos\theta,\rho\sin\varphi\sin\theta,\rho\cos\varphi)\rho^2\sin\varphi d\rho$$

4.坐标变换

$$\left.\begin{array}{l} V_{uvw} \xrightarrow[x_u, x_v, x_w, y_u, y_v, y_w, z_u, z_v, z_w \in C(V_{uvw})]{x=x(u,v,w),\, y=y(u,v,w),\, z=z(u,v,w)} V_{xyz} \\ \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} \neq 0 \\ f(x,y,z) \in C(V_{xyz}) \end{array}\right\} \Rightarrow$$

$$\iiint\limits_{V_{xyz}} f(x,y,z)\underbrace{dxdydz}_{} = \iiint\limits_{V_{uvw}} f[x(u,v,w), y(u,v,w), z(u,v,w)] \cdot \underbrace{|J|dudvdw}_{}$$

$$\Rightarrow$$

$$(1)\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \Longrightarrow J_{柱} = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r,$$

$$\Longrightarrow \iiint\limits_{V_{xyz}} f(x,y,z)\underbrace{dxdydz}_{} = \iiint\limits_{V_{r\theta z}} f(r\cos\theta, r\sin\theta, z) \cdot \underbrace{r\,dr\,d\theta\,dz}_{}$$

$$(2)\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases} \Longrightarrow$$

$$J_{\text{球}} = \frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix} = \rho^2\sin\varphi$$

$$\Longrightarrow$$

$$\iiint\limits_{V_{xyz}} f(x, y, z)\underbrace{\mathrm{d}x\mathrm{d}y\mathrm{d}z}$$

$$= \iiint\limits_{V_{\rho\varphi\theta}} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\cdot \underbrace{\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta}$$

$$
\textbf{(3)}\begin{cases} x = a\rho\sin\varphi\cos\theta \\ y = b\rho\sin\varphi\sin\theta \\ z = c\rho\cos\varphi \end{cases} \Longrightarrow
$$

$$
J_{\text{广义球}} = \frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} = \begin{vmatrix} a\sin\varphi\cos\theta & a\rho\cos\varphi\cos\theta & -a\rho\sin\varphi\sin\theta \\ b\sin\varphi\sin\theta & b\rho\cos\varphi\sin\theta & b\rho\sin\varphi\cos\theta \\ c\cos\varphi & -c\rho\sin\varphi & 0 \end{vmatrix} = abc\rho^2\sin\varphi
$$

$$
\Longrightarrow
$$

$$
\iiint\limits_{V_{xyz}} f(x, y, z)\underbrace{\mathrm{d}x\mathrm{d}y\mathrm{d}z}
$$

$$
= \iiint\limits_{V_{\rho\varphi\theta}} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\cdot\underbrace{abc\rho^2\sin\varphi\,\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta}
$$

5.体积计算

$$V_{体积}=\iiint\limits_{V_{xyz}}\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$=\iiint\limits_{V_{uvw}}\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w \Leftarrow \boxed{\begin{aligned}x&=x(u,v,w)\\y&=y(u,v,w)\\z&=z(u,v,w)\end{aligned}}$$

$$=\iiint\limits_{V_{r\theta z}} r\,\mathrm{d}r\,\mathrm{d}\theta\mathrm{d}z \Leftarrow \boxed{\begin{aligned}x&=r\cos\theta\\y&=r\sin\theta\\z&=z\end{aligned}}$$

$$=\iiint\limits_{V_{\rho\varphi\theta}}\rho^2\sin\varphi\mathrm{d}\rho\mathrm{d}\varphi\mathrm{d}\theta \Leftarrow \boxed{\begin{aligned}x&=\rho\sin\varphi\cos\theta\\y&=\rho\sin\varphi\sin\theta\\z&=\rho\cos\varphi\end{aligned}}$$

$$V_{体积} = \iiint\limits_{V_{xyz}} \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= \iiint\limits_{V_{\rho\varphi\theta}} abc\rho^2 \sin\varphi \, \mathrm{d}\rho \mathrm{d}\varphi \mathrm{d}\theta \quad \Leftarrow \quad \boxed{\begin{aligned} x &= a\rho\sin\varphi\cos\theta \\ y &= b\rho\sin\varphi\sin\theta \\ z &= c\rho\cos\varphi \end{aligned}}$$

6. 轮换对称性

$$V=\{(x,y,z)\mid G(x,y,z)\le c\}\xrightarrow[z\to x]{x\to y,y\to z}V'=\{(x,y,z)\mid G(y,z,x)\le c\}$$

$$\Rightarrow\iiint_V f(x,y,z)dxdydz=\iiint_{V'} f(y,z,x)dxdydz$$

(i). 若 $V=V'$ ---- V 具有轮换对称性

(ii). 若 $f(x,y,z)=f(y,z,x)$ ---- $f(x,y,z)$ 具有轮换对称性

(iii). 若 $f(x,y,z)=-f(y,z,x)$ -- $f(x,y,z)$ 具有轮换反对称性

(1)若 $V=V'\Rightarrow\iiint_V f(x,y,z)dxdydz=\iiint_V f(y,z,x)dxdydz$

例： $V=\{(x,y)\mid x\ge 0,y\ge 0,z\ge 0,x^2+y^2+z^2\le R^2\}\xrightarrow[z\to x]{x\to y,y\to z}$

$$V'=\{(x,y,z)\mid y\ge 0,z\ge 0,x\ge 0,y^2+z^2+x^2\le R^2\}=V$$

$$(f(x,y,z)=x^2\Rightarrow f(y,z,x)=y^2,f(z,x,y)=z^2).$$

$$\Rightarrow \iiint\limits_V x^2 dxdydz = \iiint\limits_V y^2 dxdydz = \iiint\limits_V z^2 dxdydz$$

例：$V = \{(x,y,z) \mid x \ge 0, y \ge 0, z \ge 0, x^2 + y^2 + z^2 \le 1\} \xrightarrow[z \to x]{x \to y,\, y \to z}$

$$V' = \{(x,y,z) \mid x \ge 0, y \ge 0, z \ge 0, y^2 + z^2 + x^2 \le 1\} = V$$

$$\Rightarrow \iiint\limits_V y(x-z)dxdydz = 0$$

$$\left(f(x,y,z) = y(x-z) = xy - yz \Rightarrow \begin{cases} f(y,z,x) = yz - zx \\ f(z,x,y) = zx - xy \end{cases} \right).$$

$$\Rightarrow \iiint\limits_V y(x-z)dxdydz = \iiint\limits_V z(y-x)dxdydz = \iiint\limits_V x(z-y)dxdydz$$

$$\iiint\limits_V y(x-z)dxdydz$$

$$= \frac{1}{3}\left[\iiint\limits_V y(x-z)dxdydz + \iiint\limits_V z(y-x)dxdydz + \iiint\limits_V x(z-y)dxdydz\right]$$

$$= \frac{1}{3}\left[\iiint\limits_V \underbrace{[y(x-z) + z(y-x) + x(z-y)]}_{0} dxdydz\right] = 0$$

(2)若$V=V', f(x,y,z)=-f(y,z,x) \Rightarrow \iiint\limits_V f(x,y,z)dxdydz=0$

$$\iiint\limits_V f(x,y,z)dxdydz=\iiint\limits_V f(y,z,x)dxdydz=-\iiint\limits_V f(x,y,z)dxdydz$$

例题 求$\iiint\limits_{\Omega} x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z$，其中 Ω：$x^2+y^2+z^2\leqslant R^2(R>0)$，$z\geqslant 0$.

解 由轮换对称性知：

$$\iiint\limits_{\Omega} x^2\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{1}{2}\iiint\limits_{x^2+y^2+z^2\leqslant R^2} x^2\mathrm{d}x\mathrm{d}z\mathrm{d}y=$$

$$\frac{1}{6}\iiint\limits_{x^2+y^2+z^2\leqslant R^2}(x^2+y^2+z^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z=$$

$$\frac{1}{6}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^{R}\rho^2\cdot\rho^2\sin\varphi\mathrm{d}\rho=\frac{2}{15}\pi R^5.$$

例： 计算由$x+y+z=0, x+y+z=2, x-y-z=-1, x-y-z=2,$

$x+z=0, x+z=2$所围六面体体积.

解： $\begin{cases} u=x+y+z \\ v=x-y-z \\ w=x+z \end{cases} \Leftrightarrow \begin{cases} x=\frac{1}{2}u+\frac{1}{2}v \\ y=u-w \\ z=-\frac{1}{2}u-\frac{1}{2}v+w \end{cases}$

$$V_{xyz} \leftrightarrow V_{uvw}: \begin{cases} 0\le u\le 2 \\ -1\le v\le 2 \\ 0\le w\le 2 \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{vmatrix}=-\frac{1}{2}$$

$$V_{体积} = \iiint\limits_{V_{xyz}} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_{V_{uvw}} \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \mathrm{d}u\mathrm{d}v\mathrm{d}w$$

$$= \iiint\limits_{V_{uvw}} \frac{1}{2} \mathrm{d}u\mathrm{d}v\mathrm{d}w = \int_0^2 du \int_{-1}^2 dv \int_0^2 \frac{1}{2} dw = 6$$

$$\begin{cases} u = x + y + z \\ v = x - y - z \\ w = x + z \end{cases} \Rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -2$$

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{\dfrac{\partial(u, v, w)}{\partial(x, y, z)}} = -\frac{1}{2}$$