一.二重积分计算的基本技巧

1. $D:\begin{cases} a \le x \le b \\ c \le y \le d \end{cases}, F(x,y) = f(x)\cdot g(y) \Rightarrow$

$$\iint_D F(x,y)d\sigma = \int_a^b f(x)dx \cdot \int_c^d g(y)dy$$

2.选坐标系

(1). D:矩形,三角形等直线形区域 $\Rightarrow$ 直角坐标系下积分

(2). $\begin{cases} D\text{:中心在原点的圆形,扇形,圆环域等} \\ F(x,y) = f(x^2+y^2) \end{cases} \Rightarrow$ 极坐标系下积分

(3). 直角坐标系下:二重积分 $\xrightarrow[\text{积分次序}]{\text{区域类型}}$ 二次积分

$$\begin{cases} X-\text{型区域:} \iint_D f(x,y)dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)dy \\ Y-\text{型区域:} \iint_D f(x,y)dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)dx \end{cases}$$

(4). 直角坐标系下: 两种定限方法 $\begin{cases} \text{几何定限法} \\ \text{代数定限法} \end{cases}$

I. 几何定限法: (以 X - 型区域为例)

(1) $D \xrightarrow{\text{投影}} x$轴 $\Rightarrow a \le x \le b$

(2) 在$[a,b]$内沿平行于y轴方向自下而上引线: $\begin{cases} \text{穿入为}y\text{下限} \\ \text{穿出为}y\text{上限} \end{cases}$

II. 代数定限法: (以 X - 型区域为例)

(1) D消去$y \Rightarrow \ a \le x \le b$

(2) $\forall x_0 \in (a,b)$, 计算$y_1(x_0), y_2(x_0)$:

若$y_1(x_0) \le y_2(x_0) \ \Rightarrow \ y_1(x) \le y \le y_2(x)$

(5). 极坐标系下: 两种定限方法 $\begin{cases}\text{几何定限法}\\ \text{代数定限法}\end{cases}$

I. 几何定限法:

(1) $D \xrightarrow{\text{投影}}$ 极点 $\Rightarrow \alpha \le \theta \le \beta$

(2) 在$[\alpha, \beta]$内,自极点向D引射线: $\begin{cases}\text{穿入为}r\text{下限}\\ \text{穿出为}r\text{上限}\end{cases}$

II. 代数定限法:

(1) D消去$r \Rightarrow \alpha \le \theta \le \beta$

(2) $\forall \theta_0 \in (\alpha, \beta)$,计算$r_1(\theta_0), r_2(\theta_0)$:

若$r_1(\theta_0) \le r_2(\theta_0) \Rightarrow r_1(\theta) \le r \le r_2(\theta)$

3.定积分次序

(1)原则:兼顾积分区域与被积函数,避免分区积分和对某个变量不可积的情形出现

(2) 难点:按某一积分次序积分比较困难,

不易计算 $\xrightarrow[\text{次序}]{\text{交换}}$ 易计算(积分函数发生变化)

4.奇偶性及对称性

(1)积分区域关于坐标轴等的对称性

(2) 被积函数关于相应变量的奇偶性

$\xrightarrow[\text{结合}]{\text{两者}}$ 化简积分计算

5.坐标变换

$$\boxed{x = x(u,v), y = y(u,v)}$$

$$\iint_{D_{xy}} f(x,y)dxdy = \iint_{D_{uv}} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

$$\boxed{x = r\cos\theta, y = r\sin\theta} \Rightarrow \boxed{\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r}$$

$$\iint_{D_{xy}} f(x,y)dxdy = \iint_{D_{r\theta}} f(x(r,\theta), y(r,\theta))\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta$$

$$= \iint_{D_{r\theta}} f(r\cos\theta, r\sin\theta) rdrd\theta$$

6.面积计算

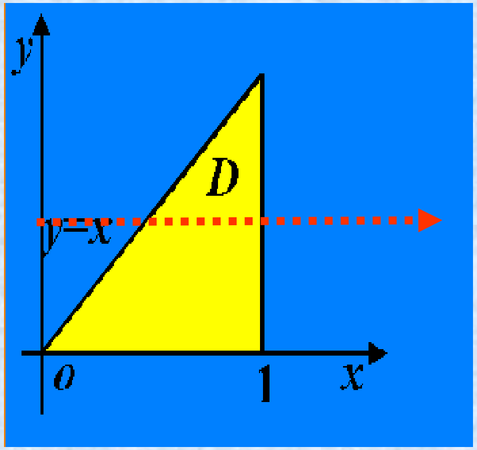
$$S_D = \iint_{D_{xy}} dxdy = \iint_{D_{uv}} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \Leftarrow \boxed{x = x(u,v), y = y(u,v)}$$

$$= \iint_{D_{r\theta}} rdrd\theta \qquad \Leftarrow \boxed{x = r\cos\theta, y = r\sin\theta}$$

二.利用换序计算二重积分

例1. 计算积分$\int_0^1 \mathrm{d}y\int_y^1 e^{-x^2}\mathrm{d}x.$

分析： 积分区域$D=\{(x,y)\,|\,y\le x\le 1, 0\le y\le 1\}.$

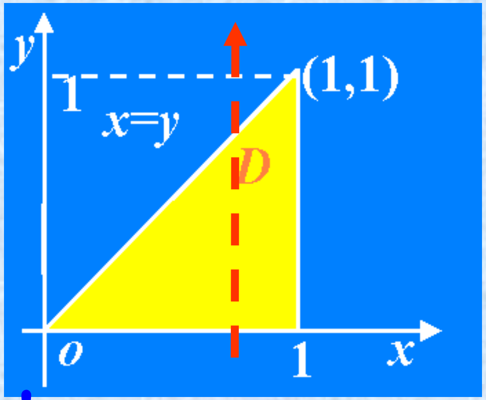


$\because e^{-x^2}$的原函数不是初等函数,

$\therefore \int_y^1 e^{-x^2}\mathrm{d}x$积分不出结果！！

$\therefore$ 改变积分顺序,先对y后对x积分

$$D=\{(x,y)\,|\,0\le y\le x, 0\le x\le 1\}.$$



$$\int_0^1 \mathrm{d}y\int_y^1 e^{-x^2}\mathrm{d}x=\int_0^1 \mathrm{d}x\int_0^x e^{-x^2}\mathrm{d}y$$

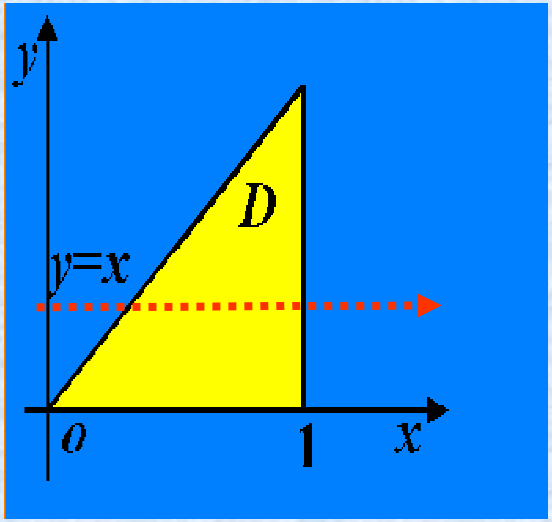
$$=\int_0^1 xe^{-x^2}\mathrm{d}x=\frac{1}{2}(1-e^{-1}).$$

注：适当的积分顺序决定了计算的成败！！

例1. 计算积分 $\int_0^1 \mathrm{d}y \int_y^1 e^{-x^2} \mathrm{d}x.$

$$\boxed{\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du}$$

分析: 积分区域 $D = \{(x, y) \mid y \le x \le 1, 0 \le y \le 1\}.$



$$\boxed{F(y) = \int_y^1 e^{-x^2} \mathrm{d}x = -\int_1^y e^{-x^2} \mathrm{d}x}$$

$$\int_0^1 \mathrm{d}y \int_y^1 e^{-x^2} \mathrm{d}x = \int_0^1 F(y)\mathrm{d}y$$

$$= yF(y)\Big|_0^1 - \int_0^1 y\,\mathrm{d}F(y)$$

$$= -\int_0^1 yF'(y)dy = -\int_0^1 y(-e^{-y^2})dy = \int_0^1 ye^{-y^2} dy$$

$$\boxed{F'(y) = -e^{-y^2}}$$

$$= -\frac{1}{2}\int_0^1 e^{-y^2} d(-y^2) = -\frac{1}{2} e^{-y^2}\Big|_0^1 = \frac{1}{2} e^{-y^2}\Big|_1^0$$

$$= \frac{1}{2}(1 - e^{-1})$$

注： $e^{-x^2}, \sin x^2, \sqrt{1+x^4}, \dfrac{\sin x}{x}, \dfrac{x}{\ln x}$等,

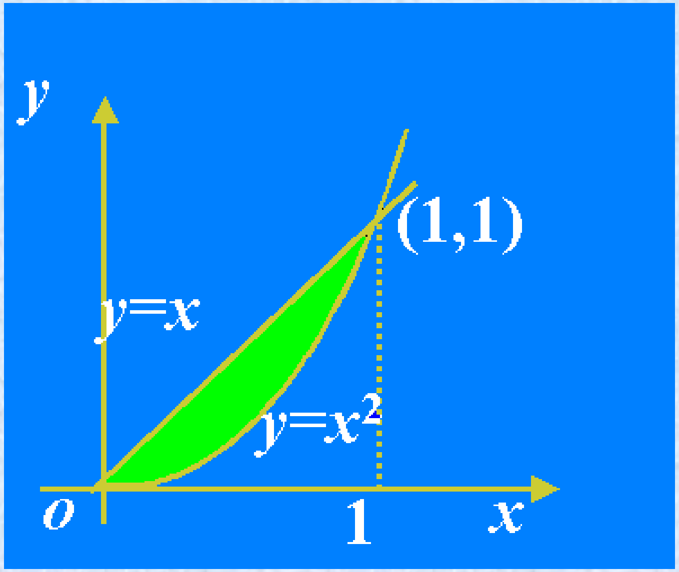
它们的原函数都不能用初等函数的有限形式表示.

例： 计算积分$I=\iint\limits_D \dfrac{\sin x}{x}dxdy$

$D: y=x, y=x^2$围成的闭区域.

$$D=\{(x,y)\,|\,x^2\le y\le x, 0\le x\le 1\}.$$

$$D=\{(x,y)\,|\,y\le x\le \sqrt{y}, 0\le y\le 1\}.$$



($Y-$型区域):

$$I=\iint_D \frac{\sin x}{x}dxdy=\int_0^1 \mathrm{d}y\int_y^{\sqrt{y}}\frac{\sin x}{x}\mathrm{d}x=?$$

($X-$型区域):

$$I=\iint_D \frac{\sin x}{x}dxdy=\int_0^1 \mathrm{d}x\int_x^{x^2}\frac{\sin x}{x}\mathrm{d}y=\int_0^1\frac{\sin x}{x}(x^2-x)\mathrm{d}x$$

$$=\int_0^1(\sin x-x\sin x)\mathrm{d}x=\int_0^1\sin x dx+\int_0^1 xd(\cos x)$$

$$=-\cos x\big|_0^1+x\cos x\big|_0^1-\int_0^1\cos x dx=\cos x\big|_1^0+\cos 1-\sin x\big|_0^1$$

$$=1-\cos 1+\cos 1-\sin 1=1-\sin 1$$

(Y－型区域):

$$\boxed{F(y)=\int_y^{\sqrt{y}}\frac{\sin x}{x}\mathrm{d}x}$$

$$I=\iint_D\frac{\sin x}{x}dxdy=\int_0^1\mathrm{d}y\int_y^{\sqrt{y}}\frac{\sin x}{x}\mathrm{d}x=\int_0^1F(y)\mathrm{d}y$$

$$=yF(y)\big|_0^1-\int_0^1 y\mathrm{d}F(y)=-\int_0^1 yF'(y)dy$$

$$\boxed{F'(y)=\frac{\sin\sqrt{y}}{2y}-\frac{\sin y}{y}}$$

$$=-\int_0^1 y\left(\frac{\sin\sqrt{y}}{2y}-\frac{\sin y}{y}\right)dy$$

$$=\int_0^1\left(\sin y-\frac{1}{2}\sin\sqrt{y}\right)dy$$

$$=-\cos y\big|_0^1-\frac{1}{2}\int_0^1\sin\sqrt{y}dy=\cos y\big|_1^0-\frac{1}{2}\int_0^1\sin xdx^2$$

$$=1-\cos1-\int_0^1 x\sin xdx=1-\cos1+\int_0^1 xd\cos x$$

$$=1-\cos1+x\cos x\big|_0^1-\int_0^1\cos xdx=1-\sin1$$

三.利用对称性计算二重积分

$$I=\iint_D f(x,y)d\sigma$$

$$D_1=\{(x,y)\,|\,(x,y)\in D, y\geq 0\}$$

$$D_2=\{(x,y)\,|\,(x,y)\in D, x\geq 0\}$$

$$D_3=\{(x,y)\,|\,(x,y)\in D, y\geq x\}$$

1. D关于x轴对称
$$\begin{cases}(1)\ f(x,-y)=-f(x,y) \Rightarrow I=0 \\ (2)\ f(x,-y)=f(x,y) \Rightarrow I=2\iint_{D_1} f(x,y)d\sigma\end{cases}$$

2. D关于y轴对称
$$\begin{cases}(1)\ f(-x,y)=-f(x,y) \Rightarrow I=0 \\ (2)\ f(-x,y)=f(x,y) \Rightarrow I=2\iint_{D_2} f(x,y)d\sigma\end{cases}$$

$$I=\iint_D f(x,y)d\sigma$$

$$D_1=\{(x,y)\,|\,(x,y)\in D, y\geq 0\}$$

$$D_2=\{(x,y)\,|\,(x,y)\in D, x\geq 0\}$$

$$D_3=\{(x,y)\,|\,(x,y)\in D, y\geq x\}$$

3. D关于原点对称

(1) $f(-x,-y)=-f(x,y)\Rightarrow I=0$

(2) $f(-x,-y)=f(x,y)\Rightarrow I=2\iint_{D_1} f(x,y)d\sigma$

4. D关于$y=x$对称

(1) $f(x,y)=-f(y,x)\Rightarrow I=0$

(2) $f(x,y)=f(y,x)\Rightarrow I=2\iint_{D_3} f(x,y)d\sigma$

4. 轮换对称性

$$D=\{(x,y)\mid G(x,y)\le c\}\xrightarrow[y\to x]{x\to y}D'=\{(x,y)\mid G(y,x)\le c\}$$

$$\Rightarrow \iint_D f(x,y)dxdy=\iint_{D'} f(y,x)dxdy$$

(i). 若 $D=D'$ ——— D具有轮换对称性

(ii). 若 $f(x,y)=f(y,x)$ ——— $f(x,y)$具有轮换对称性

(iii). 若 $f(x,y)=-f(y,x)$ ——— $f(x,y)$具有轮换反对称性

(1)若 $D=D'\Rightarrow \iint_D f(x,y)dxdy=\iint_D f(y,x)dxdy$

例： $D=\{(x,y)\mid x\ge 0, y\ge 0, x+y\le 1\}\xrightarrow[y\to x]{x\to y}$

$$D'=\{(x,y)\mid y\ge 0, x\ge 0, y+x\le 1\}=D$$

$$\Rightarrow \iint_D xdxdy=\iint_D ydxdy\ (f(x,y)=x\Rightarrow f(y,x)=y).$$

$$\Rightarrow \iint_D x\,dxdy = \frac{1}{2}\iint_D (x+y)\,dxdy$$

(2)若$D = D', f(x,y) = -f(y,x) \Rightarrow \iint_D f(x,y)\,dxdy = 0$

$$\iint_D f(x,y)\,dxdy = \iint_D f(y,x)\,dxdy = -\iint_D f(x,y)\,dxdy$$

例：$D = \{(x,y) \mid x^2 + y^2 \le 1\} \xrightarrow[y \to x]{x \to y} D' = \{(x,y) \mid y^2 + x^2 \le 1\} = D$

$$(f(x,y) = x^2 - y^2 \Rightarrow f(y,x) = y^2 - x^2).$$

$$\Rightarrow \iint_D (x^2 - y^2)\,dxdy = 0$$

(3) $f(x,y)=f(y,x) \Rightarrow \iint\limits_{D} f(x,y)dxdy=\frac{1}{2}\iint\limits_{D+D'} f(x,y)dxdy$

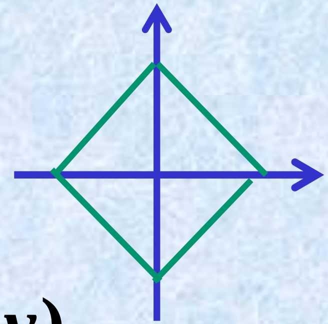
$$\iint\limits_{D} f(x,y)dxdy=\iint\limits_{D'} f(y,x)dxdy=\iint\limits_{D'} f(x,y)dxdy$$

$$\Rightarrow \iint\limits_{D} f(x,y)dxdy=\frac{1}{2}\iint\limits_{D+D'} f(x,y)dxdy$$

例2. 计算积分$I=\iint\limits_D(|x|+y)dxdy$

$D=\{(x,y)\ |\ |x|+|y|\le 1\}.$

分析： D关于x轴, y轴及原点, $y=x$均对称.



记$D_1=\{(x,y)|(x,y)\in D, x\ge 0, y\ge 0\}$

$$f(x,y)=|x|\Rightarrow\begin{cases}f(-x,y)=|-x|=|x|=f(x,y)\\ f(x,-y)=|x|=f(x,y)\end{cases}$$

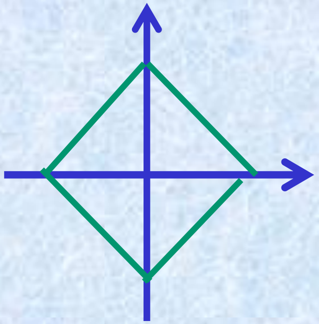
$$g(x,y)=y\Rightarrow\begin{cases}g(-x,y)=y=g(x,y)\\ g(x,-y)=-y=-g(x,y)\end{cases}$$

$$I=\iint\limits_D(|x|+y)dxdy=\iint\limits_D|x|dxdy+\iint\limits_D ydxdy=4\iint\limits_{D_1}|x|dxdy+0$$

$$=4\iint\limits_{D_1}xdxdy=4\int_0^1 x\mathrm{d}x\int_0^{1-x}\mathrm{d}y=4\int_0^1(x-x^2)dx=\frac{2}{3}$$

例3. 计算积分$I=\iint\limits_D \frac{y^2-x^2}{\sqrt{x^2+y^2+3}}dxdy$

$D=\{(x,y)\ |\ |x|+|y|\le 1\}.$



分析： D关于x轴, y轴及原点, $y=x$均对称.

$$f(x,y)=\frac{y^2-x^2}{\sqrt{x^2+y^2+3}}\Rightarrow$$

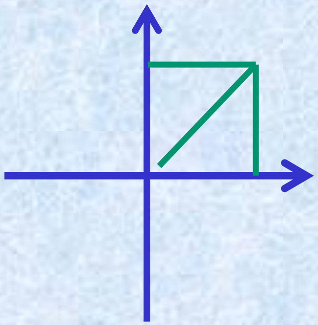
$$f(y,x)=\frac{x^2-y^2}{\sqrt{y^2+x^2+3}}=-\frac{y^2-x^2}{\sqrt{x^2+y^2+3}}=-f(x,y)$$

$$\Rightarrow I=0$$

例4. 设$\int_0^1 f(x)dx = a$,计算$\int_0^1 dx\int_x^1 f(x)f(y)dy$的值.

分析： $D = \{(x,y)|0 \le x \le 1, x \le y \le 1\}$

$D' = \{(x,y)|0 \le y \le 1, y \le x \le 1\}$



$$F(x,y) = f(x)f(y) \Rightarrow F(y,x) = f(y)f(x) = F(x,y)$$

$$\Rightarrow \int_0^1 dx\int_x^1 f(x)f(y)dy = \iint_D F(x,y)dxdy = \iint_{D'} F(x,y)dxdy$$

$$= \int_0^1 dx\int_0^x f(x)f(y)dy$$

$$\int_0^1 dx\int_x^1 f(x)f(y)dy = \frac{1}{2}\iint_{D+D'} F(x,y)dxdy$$

$$= \frac{1}{2}[\int_0^1 dx\int_0^1 f(x)f(y)dy] = \frac{1}{2}[\int_0^1 f(x)dx\int_0^1 f(y)dy] = \frac{1}{2}a^2$$

例5. 计算积分$I=\iint\limits_{D}\dfrac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}dxdy$

$$D=\{(x,y)\mid x^2+y^2\le 4, x\ge 0, y\ge 0\}.$$

分析：$F(x,y)=\dfrac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}$

$\Rightarrow$

D关于$y=x$对称.

$$\iint\limits_{D}F(x,y)dxdy=\frac{1}{2}\iint\limits_{D}[F(x,y)+F(y,x)]dxdy$$

$$I=\iint\limits_{D}\frac{a\sqrt{f(x)}+b\sqrt{f(y)}}{\sqrt{f(x)}+\sqrt{f(y)}}dxdy=\frac{1}{2}\iint\limits_{D}(a+b)dxdy$$

$$=\frac{a+b}{2}\cdot\frac{1}{4}\pi\cdot 2^2=\frac{a+b}{2}\pi$$

四.利用变量代换计算二重积分

例4. 计算积分$I=\iint\limits_D x^2y^2dxdy$

$D: xy=1, xy=2, y=x, y=4x$围成的第一象限内的闭区域.

分析: 令: $u=xy, v=\dfrac{y}{x} \Leftrightarrow x=\sqrt{\dfrac{u}{v}}, y=\sqrt{uv}$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \dfrac{1}{2\sqrt{uv}} & \dfrac{-\sqrt{u}}{2\sqrt{v^3}} \\ \dfrac{\sqrt{v}}{2\sqrt{u}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix}=\frac{1}{2v}$$

$$D_{xy} \Leftrightarrow D_{uv}: 1\le u\le 2, 1\le v\le 4$$

$$\Rightarrow I=\iint\limits_{D_{xy}} x^2y^2dxdy=\iint\limits_{D_{uv}} u^2\frac{1}{2v}dudv=\frac{1}{2}\int_1^2 u^2du\int_1^4\frac{1}{v}dv=\frac{7}{3}\ln 2$$

例5. 计算积分$I=\iint\limits_{D} e^{\frac{y}{x+y}}dxdy$

$D: x$轴, y轴, $x+y=1$围成的闭区域.

分析： 令: $u=x+y, v=\dfrac{y}{x+y} \Leftrightarrow x=u-uv, y=uv$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}=u$$

$$\begin{cases} x+y=1 \leftrightarrow u=1 \\ x=0 \leftrightarrow u(1-v)=0 \leftrightarrow u=0\text{或}v=1 \\ y=0 \leftrightarrow uv=0 \leftrightarrow u=0\text{或}v=0 \end{cases} \Rightarrow D_{xy} \Leftrightarrow D_{uv}:\begin{cases} 0\le u\le 1 \\ 0\le v\le 1 \end{cases}$$

$$\Rightarrow I=\iint\limits_{D_{xy}} e^{\frac{y}{x+y}}dxdy=\iint\limits_{D_{uv}} e^{v}ududv=\int_0^1 udu\int_0^1 e^{v}dv=\frac{1}{2}(e-1)$$

例6. 计算积分$I=\iint\limits_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy \quad \left(D:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right)$

分析： 令：$x=a\rho\cos\theta, y=b\rho\sin\theta \Rightarrow \frac{\partial(x,y)}{\partial(\rho,\theta)}=ab\rho$

$$D_{xy}:\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1 \Leftrightarrow D_{\rho\theta}:0\le\rho\le 1, 0\le\theta\le 2\pi$$

$$\Rightarrow I=\iint\limits_D\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy=\iint\limits_{D_{\rho\theta}}\rho^2 ab\rho d\rho d\theta$$

$$=ab\int_0^{2\pi}\mathrm{d}\theta\int_0^1\rho^3\mathrm{d}\rho=\frac{\pi}{2}ab$$