4.5 #4

For which n does the graph K_n contain an Euler circuit? Explain.

A graph K_n will have n vertices with $n-1$ edges for each vertex, so each vertex would have a degree of $n-1$. We also know that a graph has an Euler circuit if and only if the degree of every vertex is even. That is, $n-1$ must be even for K_n to have an Euler circuit. If $n-1$ is even then n must be odd. So n must be odd for the graph K_n to contain an Euler circuit. We have also defined a circuit to have nonzero length, so we know that K_1 cannot have a circuit, so all K_n with odd $n \geq 3$ will have an Euler circuit.

4.5 #5

For which m and n does the graph $K_{m,n}$ contain an Euler path? And Euler circuit? Explain.

A graph has an Euler path if at most 2 vertices have an odd degree. Since for a graph $K_{m,n}$, we know that m vertices have degree n and n vertices have degree m, so we can say that under these conditions, $K_{m,n}$ will contain an Euler path:

- m and n are both even. Then each vertex has an even degree, and the condition of at most 2 vertices with odd degree is met.
- m is odd and $n = 2$. Then only $n = 2$ vertices have odd degree m, and the other m vertices have even degree of $n = 2$, so the condition of at most 2 vertices with odd degree is met.
- $m = 2$ and n is odd. Then we have m vertices with even degree n and $m = 2$ vertices with odd degree n, so the condition of at most 2 vertices with odd degree is met.
- $m = n = 1$ has only two vertices, but each are of odd degree, so it contains an Euler path as well.

A graph has an Euler circuit if the degree of each vertex is even. For a graph $K_{m,n}$, the degree of each vertex is either m or n, so both m and n must be even.

4.5 #6

For which n does K_n contain a Hamilton path? A Hamilton cycle? Explain.

For all $n \geq 3$, K_n will contain a Hamilton cycle. We can prove this by thinking of K_n as a regular polygon with n vertices and therefore n sides. Since each vertex is connected to the other $n-1$ vertices, there is a path around the perimeter of the polygon that visits each vertex and arrives back at the starting vertex for all $n \geq 3$.

For all $n \geq 2$, K_n will contain a Hamilton path. We know that for $n \geq 3$, K_n will contain a Hamilton cycle, which is a Hamilton path that starts and ends at adjacent vertices. K_2 contains a Hamilton path: $\{1, 2\}$ or $\{2, 1\}$ for vertices 1 and 2. And K_1 does not contain a Hamilton path as a Hamilton path cannot have length 0.

4.5 #7

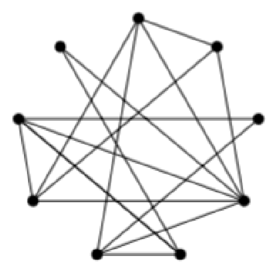
For which m and n does the graph $K_{m,n}$ contain a Hamilton path? A Hamilton cycle? Explain.

For a bipartite graph, we know any existing Hamilton cycle or path will have to alternate between vertices in V_1 and V_2. If we want to reach all the vertices and end at the same one, m must equal n (with both $m, n \neq 1$). If $m \neq n$, then we won't be able to start and end at the same vertex and alternate between vertices in V_1 and V_2 without repeating a vertex.

Since any bipartite graph $K_{m,n}$ with $m = n$ has a Hamiltonian cycle, we know that these graphs also have a Hamiltonian path. Additionally, without the restriction of starting and ending at the same vertex, we can allow either $m = n + 1$ or $n = m + 1$ and $m = n = 1$, since we can create a Hamiltonian path that alternates between vertices in V_1 and vertices in V_2 by starting at a vertex from the set that has more vertices than the other, so long as the difference between the number of vertices in each group is less than or equal to 1.

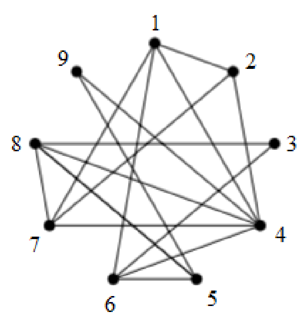
4.5 #9

Below is a graph representing friendships between a group of students (each vertex is a student and each edge is a friendship). Is it possible for the students to sit around a round table in such a way that every student sits between two friends? What does this question have to do with paths?



This question is essentially asking if the graph has a Hamilton cycle, as arranging the students along the outside of round table in the order of a Hamilton cycle would be the way to sit each student at the table between two friends.

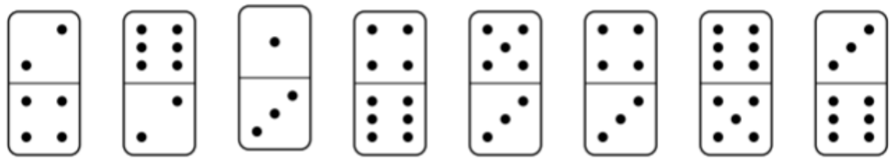
Yes, it can be done:



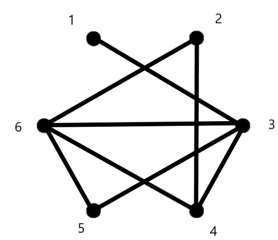
The path {1, 2, 4, 9, 5, 6, 3, 8, 7, 1 } is a Hamilton cycle.

4.5 #10

On the table rest 8 dominoes, as shown below. If you were to line them up in a single row, so that any two sides touching had matching numbers, what would the sum of the two end numbers be?



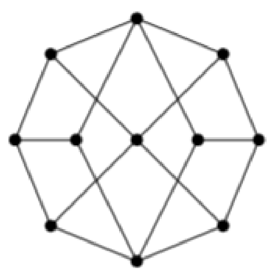
We can think of these dominoes as edges connecting vertices labeled 1, 2, 3, 4, 5 and 6. For example, the first domino is an edge connecting vertex 2 to vertex 4. We can draw a graph of this:



And we know that the endpoints of an Euler path of this graph will be the two end numbers of the line of dominoes. Since 1 and 4 are the only vertices with odd degree, they 4 must be the endpoints of the path, and the sum of the two end numbers is 5.

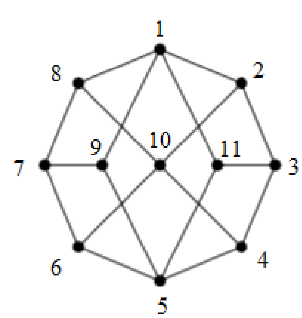
4.5 #12

Consider the following graph:



(a) Find a Hamilton path. Can your path be extended to a Hamilton cycle?

There are many correct answers for a Hamilton path on this graph. If we label each vertex like this:

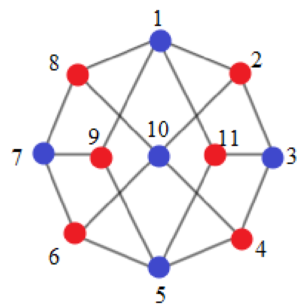


An example of one Hamilton path on this graph is {2, 1, 11, 3, 4, 10, 8, 7, 9, 5, 6}.

This path cannot be extended to a Hamilton cycle.

(b) Is the graph bipartite? If so, how many vertices are in each "part"?

Yes, the graph is bipartite:



V_1 ={1, 3, 5, 7, 10} and V_2 ={2, 4, 6, 8, 9, 11}. There are 5 and 6 vertices in each part.

(c) Use your answer to part (b) to prove that the graph has no Hamilton cycle.

We know that any bipartite graph that has a Hamiltonian cycle must have the same number of vertices in V_1 and V_2, otherwise it will be impossible to create a cycle that alternates between vertices in V_1 and V_2 that starts and ends at the same vertex. We have $|V_1| = 5 \neq |V_2| = 6$, so we know there can't be a Hamilton cycle in this graph.

(d) Suppose you have a bipartite graph G in which one part has at least two more vertices than the other. Prove that G does not have a Hamilton path.

If we have a bipartite graph G, then we have $|V_1| = m$ and $|V_2| = n$. If G is complete, then $G = K_{m,n}$. If G is not complete, then G is a subset of $K_{m,n}$, and it suffices to show that if a complete graph with $|m - n| \geq 2$ does not have a Hamilton path, then any G with $|V_1| = m$ and $|V_2| = n$ cannot have a Hamilton path, either, since if $K_{m,n}$ has no Hamiltonian path, then certainly no subset of $K_{m,n}$ does, either.

So let $K_{m,n}$ be a complete graph where $|m - n| \geq 2$.