Solutions of Homework for Chapter (2)

First-Order Ordinary Differential Equations

Example (3):

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{6x^2 - 2x + 1}{\cos y + e^y}$.

Solution

* The given differential equation is

$$\frac{dy}{dx} = \frac{6x^2 - 2x + 1}{\cos y + e^y} .$$

* By separating variables , we obtain

$$\left(\cos y + e^y\right) dy = \left(6x^2 - 2x + 1\right) dx .$$

* Integrating both sides

$$\int \left(\cos y + e^y\right) dy = \int \left(6x^2 - 2x + 1\right) dx .$$

Remember that :

* $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad , n \neq -1$
* $\int \cos x \, dx = \sin x + C$
* $\int e^x \, dx = e^x + C$

* The general solution is

$$\boxed{\sin y + e^y = 2x^3 - x^2 + x + C}$$

where C is an arbitrary constant .

Example (4):

Solve the differential equation

$$3e^{x}\ \tan y\ dx + \left(1 - e^{x}\right) sec^{2}\ y\ dy\ = 0\ .$$

Solution

* The given differential equation is

$$3e^{x}\ \tan y\ dx + \left(1 - e^{x}\right) sec^{2}\ y\ dy\ = 0\ .$$

* By separating variables , we obtain

$$\left(1 - e^{x}\right) sec^{2}\ y\ dy\ = -3e^{x} \tan y\ dx\ .$$

* Multiplying both sides by $\dfrac{1}{\left(1 - e^{x}\right) \tan y}$

$$\frac{sec^{2}\ y}{\tan y}\ dy = \frac{-3\,e^{x}}{\left(1 - e^{x}\right)}\ dx$$

$$\frac{sec^{2}\ y}{\tan y}\ dy = \frac{3\,e^{x}}{\left(e^{x} - 1\right)}\ dx\ .$$

* Integrating both sides

$$\int \frac{sec^{2}\ y}{\tan y}\ dy = 3\int \frac{e^{x}}{e^{x} - 1}\ dx$$

Remember that :

$$* \quad \boxed{\int \frac{g'(x)}{g(x)}dx = \ln\left|g(x)\right| + C}$$

$$\ln\left|\tan y\right| = 3\ln\left|e^x - 1\right| + \ln\left|C\right|$$

where C is an arbitrary constant .

Remember that :

$$* \quad \boxed{\ln x^r = r\ln x}$$

$$* \quad \boxed{\ln(ab) = \ln a + \ln b}$$

$$\ln\left|\tan y\right| = \ln\left|\left(e^x - 1\right)^3\right| + \ln\left|C\right|$$

$$\ln\left|\tan y\right| = \ln\left|C\left(e^x - 1\right)^3\right|$$

$$e^{\ln\left|\tan y\right|} = e^{\ln\left|C\left(e^x - 1\right)^3\right|} .$$

Remember that :

$$* \quad \boxed{e^{\ln x} = x \,,\; x > 0}$$

* *The general solution is*

$$\boxed{\left|\tan y\right| = \left|C\left(e^x - 1\right)^3\right|} .$$

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Example (6):

Solve the initial-value problem (IVP)

$$y' = \left(1+y^2\right)\tan x \quad , \quad y\left(0\right)=\sqrt{3}\ .$$

Solution

* The given differential equation is

$$\frac{dy}{dx} = \left(1+y^2\right)\tan x\,.$$

* By separating variables , we obtain

$$\frac{1}{1+y^2}\,dy = \tan x\ dx.$$

* Integrating both sides

$$\int\frac{1}{1+y^2}\,dy = \int \tan x\ dx.$$

Remember that :

* $\int\frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

* $\int \tan x\ dx = \ln\left|\sec x\right| + C$

* The general solution is

$$\boxed{\tan^{-1} y = \ln\left|\sec x\right| + C}$$

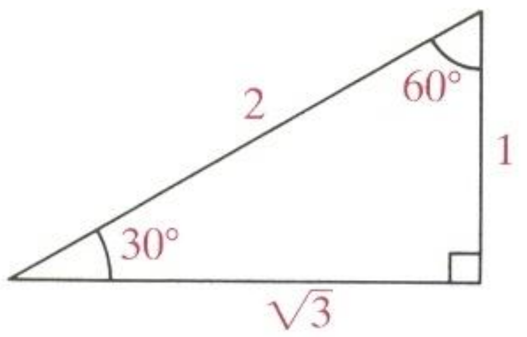
where C is an arbitrary constant .

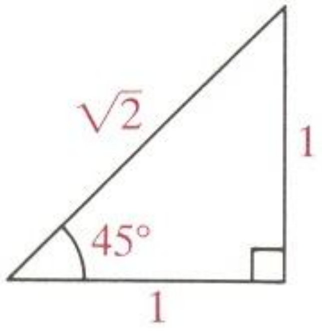
* To find the value of the constant C , we use the initial condition $y\left(0\right)=\sqrt{3}$, i.e. $y=\sqrt{x}$ when $x=0$

$$\tan^{-1}\sqrt{3} = \ln\left|\sec 0\right| + C$$

Remember that :

$$* \quad \sec 0 = 1 \quad , \quad \ln 1 = 0 \quad , \quad \tan^{-1}\sqrt{3} = 60^{\circ} = \frac{\pi}{3}$$





$$\frac{\pi}{3} = 0 + C \quad \Rightarrow \quad C = \frac{\pi}{3}.$$

* Then the particular solution is

$$\boxed{\tan^{-1} y = \ln|\sec x| + \frac{\pi}{3}}.$$

Equations Reducible to Separable :

Procedure for Solving Equations Reducible to Separable (2.4) :

Example (9):

Solve the differential equation $y' = (8x + 2y - 1)^2$.

Solution

* The given differential equation is

$$y' = (8x + 2y - 1)^2.$$

* Put $\boxed{z = 8x + 2y - 1}$, then

$$\frac{dz}{dx} = 8 + 2\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\left(\frac{dz}{dx} - 8\right).$$

* *Substituting in the given equation*

$$\frac{1}{2}\left(\frac{dz}{dx} - 8\right) = z^2$$

$$\frac{dz}{dx} = 2z^2 + 8.$$

* *By separating variables , we obtain*

$$\frac{dz}{2z^2 + 8} = dx.$$

* *Integrating both sides*

$$\frac{1}{2}\int \frac{1}{z^2 + 4}dz = \int dx$$

Remember that :

* $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad , n \neq 1$

* $\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

$$\frac{1}{4}\tan^{-1}\left(\frac{z}{2}\right) = x + C_1$$

where C_1 is an arbitrary constant .

$$\tan^{-1}\left(\frac{z}{2}\right) = 4x + C.$$

* *Put* $\boxed{z = 8x + 2y - 1}$ *, then the general solution is*

$$\boxed{\tan^{-1}\left(\frac{8x+2y-1}{2}\right)=4x+C}$$

where C is an arbitrary constant .

2.2 Homogeneous Equations :

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Example (11):

Solve the differential equation

$$\left(x^3+y^2\sqrt{y^2+x^2}\right)dx-xy\sqrt{x^2+y^2}\,dy=0\ .$$

Solution

* The given differential equation is

$$\left(x^3+y^2\sqrt{y^2+x^2}\right)dx-xy\sqrt{x^2+y^2}\,dy=0\,.$$

* Since the two functions

$$M(x,y)=x^3+y^2\sqrt{y^2+x^2}\quad,\quad N(x,y)=-x\,y\sqrt{x^2+y^2}$$

homogeneous of the same degree $n=3$, then the given differential equation is homogeneous , and we can put it in the form

$$xy\sqrt{x^2+y^2}\,dy=\left(x^3+y^2\sqrt{y^2+x^2}\right)dx$$

$$\frac{dy}{dx}=\frac{x^3+y^2\sqrt{y^2+x^2}}{xy\sqrt{x^2+y^2}}\,. \qquad (1)$$

* Put $\boxed{\dfrac{y}{x}=v}$, then

$$y=vx \quad \Rightarrow \quad \frac{dy}{dx}=v+x\frac{dv}{dx} .$$

* Substituting in equation (1)

$$v+x\frac{dv}{dx}=\frac{x^3+v^2x^2\sqrt{v^2x^2+x^2}}{xvx\sqrt{x^2+v^2x^2}}$$

$$=\frac{1+v^2\sqrt{v^2+1}}{v\sqrt{v^2+1}}$$

$$=\frac{1}{v\sqrt{v^2+1}}+v \;\; .$$

* By separating the two variables x and v, we obtain

$$x\frac{dv}{dx}=\frac{1}{v\sqrt{v^2+1}}$$

$$v\sqrt{v^2+1}\;dv=\frac{1}{x}\;dx\;.$$

* Integrating both sides

$$\frac{1}{2}\int 2v\sqrt{v^2+1}\;dv=\int\frac{1}{x}\;dx$$

<u>**Remember that :**</u>

* $$\int\left[g(x)\right]^n.g'(x)\;dx=\frac{\left[g(x)\right]^{n+1}}{n+1}+C\;,\;n\neq-1$$

* $$\int\frac{g'(x)}{g(x)}\;dx=\ln\left|\,g(x)\,\right|+C$$

$$\frac{1}{2}\,\frac{\left(v^2+1\right)^{3/2}}{\frac{3}{2}}=\ln|x|+\ln|C|$$

where C is an arbitrary constant .

Remember that :

$* \quad \ln(ab)=\ln a+\ln b$

$$\frac{1}{3}\left(v^2+1\right)^{3/2}=\ln|C\,x|\,.$$

** Put $\boxed{v=\frac{y}{x}}$, then the general solution is*

$$\boxed{\left[\left(\frac{y}{x}\right)+1\right]^{3/2}=3\ln|C\,x|}\,.$$

Example (13):

Solve the initial-value problem (IVP)

$$y'=\sec\left(\frac{y}{x}\right)+\frac{y}{x}\quad,\quad y(1)=\frac{\pi}{2}\,.$$

Solution

** The given differential equation is*

$$\frac{dy}{dx}=\sec\left(\frac{y}{x}\right)+\frac{y}{x}\,. \tag{1}$$

** Equation (1) is homogeneous .*

** Put $\boxed{\frac{y}{x}=v}$, then*

$$y = v\,x \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}\ .$$

* *Substituting in equation* (1)

$$v + x\frac{dv}{dx} = \sec v + v\ .$$

* *By separating the two variables* x *and* v*, we obtain*

$$x\frac{dv}{dx} = \sec v$$

$$\frac{dv}{\sec v} = \frac{dx}{x}$$

$$\cos v\ dv = \frac{1}{x}\ dx\ .$$

* *Integrating both sides*

$$\int \cos v\ dv = \int \frac{1}{x}\ dx$$

Remember that :

* $\int \cos x\,dx = \sin v + C$
* $\int \frac{1}{x}dx = \ln|x| + C$

$$\sin v = \ln|x| + C$$

where C *is an arbitrary constant .*

* *Put* $\boxed{v = \frac{y}{x}}$ *, then the general solution is*

$$\boxed{\sin\left(\frac{y}{x}\right) = \ln|x| + C}\ .$$

* To find the value of the constant C, we use the initial condition $y(1)=\frac{\pi}{2}$, i.e. $y=\frac{\pi}{2}$ when $x=1$.

$$\sin\left(\frac{\pi/2}{1}\right)=\ln|1|+C$$

Remember that :

* $\boxed{\sin\frac{\pi}{2}=1 \quad , \quad \ln 1=0}$

$$1=0+C \quad \Rightarrow \quad C=1.$$

* Then the particular solution is

$$\boxed{\sin\left(\frac{y}{x}\right)=\ln|x|+1} .$$

Equations Reducible to Either Homogeneous or Separable :

Example (17):

Solve the differential equation

$$\left(2xy^4+\sin y\right)dx+\left(4x^2y^3+x\cos y\right)dy=0 .$$

Solution

* The given differential equation is

$$\left(2xy^4+\sin y\right)dx+\left(4x^2y^3+x\cos y\right)dy=0 .$$

* Comparing the given equation with the equation

$$\boxed{M(x,y)\,dx+N(x,y)\,dy=0} .$$

* Then

$$M(x,y)=2xy^4+\sin y \quad , \quad N(x,y)=4x^2y^3+x\cos y .$$

* Since

$$\boxed{\left(\frac{\partial M(x,y)}{\partial y}=8xy^3+\cos y\right)=\left(\frac{\partial N(x,y)}{\partial x}=8xy^3+\cos y\right)} ,$$

then the given equation is exact.

* The solution of given equation can be put in the form $F(x,y)=C$, where

$$F(x,y)=\int M(x,y)\,dx+g(y)$$
$$=\int\left(2xy^4+\sin y\right)dx+g(y)$$
$$=x^2y^4+x\sin y+g(y).$$

Remember that :

$$* \quad \int x^n\,dx=\frac{x^{n+1}}{n+1}+C \ , \ n\neq -1$$

* To find $g(y)$ put $\dfrac{\partial F(x,y)}{\partial y}=N(x,y)$

$$4x^2y^3+x\cos y+g'(y)=4x^2y^3+x\cos y .$$

* Then $g'(y)=0$, and

$$g(y)=C_1$$

where C_1 is an arbitrary constant.

* The general solution is $F(x,y)=C$, then

$$\boxed{x^2y^4+x\sin y=C} .$$

Example (19):

Solve the initial-value problem (IVP)

$$(\tan y - 2)\,dx + \left(x\sec^2 y + \frac{1}{y}\right)dy = 0 \ ,\ y(0) = 1 .$$

Solution

$$(\tan y - 2)\,dx + \left(x\sec^2 y + \frac{1}{y}\right)dy = 0 .$$

* *Comparing the given equation with the equation*

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} .$$

* *Then* $M(x,y) = \tan y - 2$, $N(x,y) = x\sec^2 y + \dfrac{1}{y}$.

* *Since*

$$\boxed{\left(\frac{\partial M(x,y)}{\partial y} = \sec^2 y\right) = \left(\frac{\partial N(x,y)}{\partial x} = \sec^2 y\right)} ,$$

then the given equation is exact .

* *The solution of given equation can be put in the form* $F(x,y) = C$ *, where*

$$F(x,y) = \int M(x,y)\,dx + g(y)$$

$$= \int (\tan y - 2)\,dx + g(y)$$

$$= x\tan y - 2x + g(y) .$$

Remember that :

* $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ ,\ n \neq -1$$

* To find $g(y)$ put $\dfrac{\partial F(x,y)}{\partial y} = N(x,y)$

$$x \sec^2 y + g'(y) = x \sec^2 y + \frac{1}{y} .$$

* Then $g'(y) = \dfrac{1}{y}$, and

$$g(y) = \int \frac{1}{y} dy = \ln|y| .$$

Remember that :

* $\int \frac{1}{x} dx = \ln|x| + C$

* The general solution is $F(x,y) = C$, then

$$\boxed{x \tan y - 2x + \ln|y| = C} .$$

* To find the value of the constant C, we use the initial condition $y(0) = 1$, i.e. $y = 1$ when $x = 0$.

$$(0)\tan 1 - 2(0) + \ln|1| = C$$

Remember that :

* $\ln 1 = 0$

$$0 - 0 + 0 = C \quad \Rightarrow \quad C = 0 .$$

* Then the particular solution is

$$\boxed{x \tan y - 2x + \ln|y| = 0}$$

Example (23):

Solve the differential equation

$$\frac{1}{x}\frac{dy}{dx}-\frac{2}{x^2}y=x\,e^{x}\ .$$

Solution

* The given differential equation is

$$\frac{1}{x}\frac{dy}{dx}-\frac{2}{x^2}y=x\,e^{x}.$$

* Multiplying both sides of the given equation by x

$$\frac{dy}{dx}-\frac{2}{x}y=x^2e^{x}. \qquad (1)$$

* Comparing equation (1) with the standard form of linear equation

$$\boxed{\frac{dy}{dx}+P(x)y=Q(x)}. \qquad (2)$$

* Then $\quad P(x)=-\dfrac{2}{x} \quad , \quad Q(x)=x^2e^{x}.$

* Then the integrating factor is

$$\boxed{\mu(x)=e^{\int P(x)\,dx}}$$

$$\mu(x)=e^{\int\frac{-2}{x}dx}=e^{-2\ln|x|}=e^{\ln|x|^{-2}}=e^{\ln\left|\frac{1}{x^2}\right|}=\left|\frac{1}{x^2}\right|=\frac{1}{x^2}.$$

Remember that :

$$*\ \boxed{\int\frac{g'(x)}{g(x)}\,dx=\ln|g(x)|+C}$$

* $\ln x^r = r \ln x$

* $e^{\ln x} = x \ , \ x > 0$

* $|x|^n = |x^n| = \begin{cases} x^n & , \ n \text{ is an even integer} \\ -x^n & , \ n \text{ is an odd integer} \end{cases}$

* Then the general solution is

$$\boxed{y = \frac{1}{\mu(x)}\left[\int \mu(x)\, Q(x)\, dx + C\right]}$$

where C is an arbitrary constant .

$$y = \frac{1}{\left(\frac{1}{x^2}\right)}\left[\int \left(\frac{1}{x^2}\right) x^2 e^x\, dx + C\right]$$

$$= x^2\left[\int e^x\, dx + C\right]$$

<u>Remember that :</u>

* $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$

* Then $\boxed{y = x^2\left(e^x + C\right)}$.

<u>Example (25):</u>

Solve the initial-value problem (IVP)

$$y' + \frac{1}{x} y = 3x^2 \quad , \quad x > 0 \quad , \quad y(1) = 1 .$$

Solution

* The given differential equation is

$$y' + \frac{1}{x} y = 3x^2 . \qquad (1)$$

* *Comparing equation (1) with the standard form of linear equation*

$$\boxed{\frac{dy}{dx} + P(x) y = Q(x)} . \qquad (2)$$

* *Then* $\quad P(x) = \frac{1}{x} \quad , \quad Q(x) = 3x^2 .$

* *Then the integrating factor is*

$$\boxed{\mu(x) = e^{\int P(x)\, dx}}$$

$$\mu(x) = e^{\int \frac{1}{x} dx} = e^{\ln|x|} = |x| = x \quad , \quad x > 0 .$$

Remember that :

* $\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + C$

* $e^{\ln x} = x \ , \ x > 0$

* $|x| = \begin{cases} x & , \quad x \geq 0 \\ -x & , \quad x < 0 \end{cases}$

* *Then the general solution is*

$$\boxed{y = \frac{1}{\mu(x)} \left[\int \mu(x)\, Q(x)\, dx + C \right]}$$

where C *is an arbitrary constant .*

$$y = \frac{1}{x} \left[\int (x)\, 3x^2\, dx + C \right]$$

$$= \frac{1}{x}\left[\int 3x^3\,dx + C\right].$$

Remember that :

$$* \quad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C\,, n \neq -1$$

* *Then*

$$\boxed{y = \frac{1}{x}\left(\frac{3}{4}x^4 + C\right)}.$$

* *To find the value of the constant C, we use the initial condition $y(1) = 1$, i.e. $y = 1$ when $x = 1$.*

$$1 = \frac{1}{1}\left(\frac{3}{4}(1)^4 + C\right)$$

$$1 = \frac{3}{4} + C \quad \Rightarrow \quad C = \frac{1}{4}.$$

* *Then the particular solution is*

$$\boxed{y = \frac{1}{4x}\left(3x^4 + 1\right)}.$$

Example (27):

Solve the initial-value problem (IVP)

$$x\,y' - y = -y^2 \ln x \quad , \quad x > 0 \quad , \quad y(1) = 1.$$

Solution

* *The given differential equation is*

$$x\,y' - y = -y^2 \ln x \;.$$

* *Dividing both sides by* x

$$y' - \frac{1}{x} y = -\frac{\ln x}{x} y^2. \qquad (1)$$

* *The given equation* (1) *is Bernoulli* $(n = 2)$

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \ , \quad n \neq 1}$$

* *Dividing both sides of equation* (1) *by* y^2

$$y^{-2} \frac{dy}{dx} - \frac{1}{x} y^{-1} = -\frac{\ln x}{x}. \qquad (2)$$

* *Putting* $\boxed{z = y^{1-n} = y^{-1}}$

Remember that :

$$* \ \frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} . g'(x)$$

$$\frac{dz}{dx} = -y^{-2} \frac{dy}{dx} \quad \Rightarrow \quad y^{-2} \frac{dy}{dx} = -\frac{dz}{dx}.$$

* *Substituting in equation* (2)

$$-\frac{dz}{dx} - \frac{1}{x} z = -\frac{\ln x}{x}.$$

* *Multiplying both sides by* -1

$$\frac{dz}{dx} + \frac{1}{x} z = \frac{\ln x}{x}. \qquad (3)$$

* *Equation* (3) *is linear in* z *, and comparing with the standard form of linear equation*

$$\boxed{\frac{dz}{dx}+P(x)\,z=Q(x)}\,.$$

* Then $\quad P(x)=\dfrac{1}{x} \quad , \quad Q(x)=\dfrac{\ln x}{x}\,.$

* Then the integrating factor is

$$\boxed{\mu(x)=e^{\int P(x)\,dx}}$$

$$\mu(x)=e^{\int \frac{1}{x}\,dx}=e^{\ln|x|}=|x|=x \quad , \quad x>0\,.$$

Remember that :

* $\boxed{\int \dfrac{g'(x)}{g(x)}\,dx=\ln|g(x)|+C}$

* $\boxed{e^{\ln x}=x\ ,\ x>0}$

* $\boxed{|x|=\begin{cases} x & , \quad x\ge 0 \\ -x & , \quad x<0 \end{cases}}$

* Then the general solution of (3) is

$$\boxed{z=\frac{1}{\mu(x)}\left[\int \mu(x)\,Q(x)\,dx+C\right]}$$

where C is an arbitrary constant .

$$z=\frac{1}{x}\left[\int x\,\frac{\ln x}{x}\,dx+C\right]$$

$$=\frac{1}{x}\left[\int \ln x\,dx+C\right].$$

* To evaluate the last integral we use the integration by parts

Let $\quad I=\int \ln x\,dx$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}dx \qquad v = x$$

Remember that :

* $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \ , n \neq -1$

* $\int u \, dv = uv - \int v \, du$

$$I = uv - \int v \, du$$
$$= x \ln x - \int dx$$
$$= x \ln x - x \ .$$

* Then

$$z = \frac{1}{x}\left[\left(x \ln x - x\right) + C\right]$$

$$z = \ln x - 1 + \frac{C}{x} \ .$$

* Substituting $\boxed{z = y^{-1}}$

* Then the general solution of (1) is

$$\boxed{y^{-1} = \ln x - 1 + \frac{C}{x}} \ .$$

* To find the value of the constant C, weuse the initial condition $y(1) = 1$, i.e. $y = 1$ when $x = 1$.

$$(1)^{-1} = \ln 1 - 1 + \frac{C}{1}$$

Remember that :

* $\ln 1 = 0$

$$C = 2.$$

* Then the particular solution is

$$\boxed{y^{-1} = \ln x - 1 + \frac{2}{x}}.$$

Example (30):

Solve the differential equation

$$\frac{dy}{dx} = -8xy^2 + 4x(4x+1)y - \left(8x^3 + 4x^2 - 1\right);$$

if $f(x) = x$ is a given solution.

Solution

* The given differential equation is

$$\frac{dy}{dx} = -8xy^2 + 4x(4x+1)y - \left(8x^3 + 4x^2 - 1\right). \qquad (1)$$

* Equation (1) is in the form of Riccati equation :

$$\boxed{\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)}.$$

* Since $f(x) = x$ is a given solution, we use the transformation

$\boxed{y = f(x) + \frac{1}{v}}$, then

$$y = x + \frac{1}{v} \qquad \text{and} \qquad \frac{dy}{dx} = 1 - \frac{1}{v^2}\frac{dv}{dx}.$$

Remember that :

* $\frac{d}{dx}\left[g(x)\right]^n = n\left[g(x)\right]^{n-1} . g'(x)$

* Substituting in equation (1)

$$1-\frac{1}{v^{2}}\frac{dv}{dx}=-8x\left(x+\frac{1}{v}\right)^{2}+4x(4x+1)\left(x+\frac{1}{v}\right)-\left(8x^{2}+4x-1\right)$$

$$=-8x\left(x^{2}+\frac{2x}{v}+\frac{1}{v^{2}}\right)+4x\left(4x^{2}+\frac{4x}{v}+x+\frac{1}{v}\right)-\left(8x^{3}+4x^{2}-1\right)$$

$$=-8x^{3}-\frac{16x^{2}}{v}-\frac{8x}{v^{2}}+16x^{3}+\frac{16x^{2}}{v}+4x^{2}+\frac{4x}{v}-8x^{3}-4x^{2}+1$$

$$=\frac{-8x}{v^{2}}+\frac{4x}{v}+1.$$

* Then
$$-\frac{1}{v^{2}}\frac{dv}{dx}=\frac{-8x}{v^{2}}+\frac{4x}{v}.$$

* Multiplying both sides by $-v^{2}$

$$\frac{dv}{dx}=8x-4xv$$

$$\frac{dv}{dx}=4x(2-v). \qquad (2)$$

* Equation (2) is linear in v, and we notice that it is separable equation in the same time.

* By separating variables, we obtain

$$\frac{dv}{2-v}=4x\,dx.$$

* *Integrating both sides*

$$\int \frac{-1}{2-v}\,dv = -4\int x\,dx$$

Remember that :

* $\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + C$

* $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \;,\; n \neq -1$

$$\ln|2-v| = -2x^2 + \ln|C|$$

where C *is an arbitrary constant .*

$$\ln|2-v| - \ln|C| = -2x^2$$

$$\ln\left|\frac{2-v}{C}\right| = -2x^2$$

Remember that :

* $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

* $e^{\ln x} = x$

$$e^{\ln\left|\frac{2-v}{C}\right|} = e^{-2x^2}$$

$$\left|\frac{2-v}{C}\right| = e^{-2x^2}$$

$$|2-v| = |C|\,e^{-2x^2}.$$

* Since $y = x + \frac{1}{v}$, then substituting $v = \frac{1}{y - x}$.

* Then the general solution of (1) is

$$\boxed{\left| 2 - \frac{1}{y - x} \right| = |C| \, e^{-2x^2}} .$$

Solutions of Homework for Chapter (4)
Higher-Order Linear Differential Equations with Applications

Example (4):

Solve the differential equation $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-\frac{dy}{dx}+2y=0$.

Solution

* The given differential equation

$$y'''-2y''-y'+2y=0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* The characteristic equation (auxialiary equation) of the given differential equation is

$$m^3-2m^2-m+2=0 \qquad (2)$$

* Factorizing , we get

$$m^2(m-2)-(m-2)=0$$

$$(m-2)\left(m^2-1\right)=0$$

$$(m-2)(m-1)(m+1)=0$$

Remember that :

* $\boxed{a^2-b^2=(a-b)(a+b)}$

* The roots of (2) are $m_1=2$, $m_2=1$, $m_3=-1$. They are distinct real roots .

* Then the general solution of (1) is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}}$$

$$\boxed{y = c_1 e^{2x} + c_2 e^{x} + c_3 e^{-x}}$$

where c_1, c_2, c_3 are arbitrary constants .

Example (6):

Solve the initial-value problem (IVP)

$$y'' - y = 0 \;,\; y(0) = 5 \;,\; y'(0) = 3 .$$

Solution

* The given differential equation

$$y'' - y = 0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* The characteristic equation (auxialiary equation) of the given differential equation is

$$m^2 - 1 = 0 \qquad (2)$$

* Factorizing , we get

$$(m-1)(m+1) = 0$$

* The roots of (2) are $m_1 = 1$, $m_2 = -1$. They are distinct real roots .

* Then the general solution of (1) is

$$\boxed{y = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x}} \qquad (3)$$

where c_1, c_2 are arbitrary constants.

* To find the values of the two constants c_1, c_2 we use the two initial conditions.

* Using the first initial condition $y(0)=5$, i.e. $y=5$ when $x=0$, from (3) we get

$$5 = c_1 e^0 + c_2 e^0$$

Remember that :

* $\boxed{e^0 = 1}$

$$5 = c_1 + c_2 \qquad (4)$$

* Differentiating equation (1), we obtain

$$y' = c_1 e^x - c_2 e^{-x} \qquad (5)$$

* Using the second initial condition $y'(0)=3$, i.e. $y'=3$ when $x=0$, from (5) we get

$$3 = c_1 e^0 - c_2 e^0$$

$$3 = c_1 - c_2 \qquad (6)$$

* To find the values of the two constants c_1, c_2 we solve the two equations (4) and (6).

* Adding (4) and (6) yields

$$8 = 2c_1 \quad \Rightarrow \quad c_1 = \frac{8}{2} = 4$$

* From (4) we obtain

$$c_2 = 5 - c_1 = 5 - 4 = 1$$

* Then the particular solution is

$$\boxed{y = 4e^{x} + e^{-x}} .$$

Example (8):

Solve the differential equation $\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} - 2y = 0$.

Solution

* The given differential equation

$$y''' - 4y'' + 5y' - 2y = 0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* The characteristic equation (auxialiary equation) of the given differential equation is

$$m^3 - 4m^2 + 5m - 2 = 0 \qquad (2)$$

* To factorize , note that $m = 1$ is one of the root of (2) as $m = 1$ satisfies it

Remember that :

Integer zero theorem :

* If a polynomial with integer coefficients has an integer root , then this root is a factor of the constant term .

* Using long division by the linear factor $(m - 1)$

$$
\begin{array}{r|l}
 & m^2 - 3m + 2 \\
\hline
m-1 & m^3 - 4m^2 + 5m - 2 \\
 & m^3 - m^2 \\
\hline
 & -3m^2 + 5m - 2 \\
 & -3m^2 + 3m \\
\hline
 & 2m - 2 \\
 & 2m - 2 \\
\hline
 & 0
\end{array}
$$

* *Equation* (2) *takes the form*

$$(m-1)(m^2-3m+2)=0$$

$$(m-1)(m-1)(m-2)=0$$

* *The roots of* (2) *are* $(m_1=m_2=1=m)\neq(m_3=2)$. *They are two equal real roots and one distinct real root.*

* *Then the general solution is*

$$\boxed{y=(c_1+c_2x)e^{mx}+c_3e^{m_3x}}$$

$$\boxed{y=(c_1+c_2x)e^{x}+c_3e^{2x}}$$

where c_1, c_2, c_3 *are arbitrary constants.*

Example (10):

Solve the initial-value problem (IVP)

$$y''-8y'+16y=0 \ , \ y(0)=2 \ , \ y'(0)=-1 .$$

Solution

* *The given differential equation*

$$y''-8y'+16y=0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* *The characteristic equation (auxialiary equation) of the given differential equation is*

$$m^2-8m+16=0 \qquad (2)$$

* *Factorizing , we get*

$$(m-4)(m-4)=0$$

* *The roots of (2) are* $m_1=m_2=4=m$. *They are equal real roots .*

* *Then the general solution is*

$$\boxed{y=(c_1+c_2x)e^{mx}}$$

$$\boxed{y=(c_1+c_2x)e^{4x}} \qquad (3)$$

where c_1, c_2 *are arbitrary constants .*

* *To find the values of the two constants* c_1, c_2 *we use the two initial conditions .*

* *Using the first initial condition* $y(0)=2$, *i.e.* $y=2$ *when* $x=0$, *from* (3)

$$2=(c_1+0)\,e^0$$

<u>Remember that :</u>

* $\boxed{e^0=1}$

$$c_1=2 \qquad (4)$$

* Differentiating equation (3), we obtain

$$y'=(c_1+c_2\,x)\,4e^{4x}+c_2\,e^{4x}$$

$$y'=\left[4\,(c_1+c_2\,x)+c_2\right]e^{4x} \qquad (5)$$

* Using the second initial condition $y'(0)=-1$, i.e. $y'=-1$ when $x=0$, from (5)

$$-1=\left[4\,(c_1+0)+c_2\right]e^0$$

$$-1=4\,c_1+c_2$$

$$c_2=-1-4\,c_1$$

* From (4)

$$c_2=-1-4\,(2)=-9$$

* Then the particular solution is

$$\boxed{y=(2-9\,x)\,e^{4x}}\ .$$

<u>Example (12):</u>

Solve the differential equation $\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y=0$.

Solution

* The given differential equation

$$y''+y'+y=0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* *The characteristic equation (auxialiary equation) of the given differential equation is*

$$m^2 + m + 1 = 0 \qquad (2)$$

* *This quadratic equation is not factorable using integers as its descrimenant $b^2 - 4ac = -3$ is not positive perfect square , we use the quadratic formula*

Remember that :

* If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

* $\sqrt{-1} = i$

$$a = 1 \quad , \quad b = 1 \quad , \quad c = 1$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

* *The roots of (2) are $m_{1,2} = -\dfrac{1}{2} \pm i\,\dfrac{\sqrt{3}}{2} = \alpha \pm i\,\beta$. They are conjugate complex roots .*

* *Then the general solution is*

$$\boxed{y = e^{\alpha x}\left(c_1 \sin \beta x + c_2 \cos \beta x\right)}$$

$$\boxed{y = e^{-\frac{1}{2}x}\left(c_1 \sin\frac{\sqrt{3}}{2}x + c_2 \cos\frac{\sqrt{3}}{2}x\right)}$$

where c_1, c_2 are arbitrary constants .

Example (14):

Solve the differential equation $y^{(4)} + 13y'' + 36y = 0$.

Solution

* The given differential equation

$$y^{(4)} + 13y'' + 36y = 0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* The characteristic equation (auxialiary equation) of the given differential equation is

$$m^4 + 13m^2 + 36 = 0 \qquad (2)$$

* Factorizing , we get

$$\left(m^2 + 4\right)\left(m^2 + 9\right) = 0$$

Remember that :

* $\left|\begin{array}{l} a^2 - b^2 = (a-b)(a+b) \\ a^2 + b^2 = (a-ib)(a+ib) \end{array}\right.$

$$(m-2i)(m+2i)(m-3i)(m+3i) = 0$$

* The roots of (2) are

$$m_{1,2} = \pm i\,2 = \alpha_1 \pm i\,\beta_1 \ , \ m_{3,4} = \pm i\,3 = \alpha_2 \pm i\,\beta_2 .$$

They are two distinct pairs of conjugate complex roots .

* Then the general solution is

$$\boxed{\begin{aligned} y &= e^{\alpha_1 x}\left(c_1 \sin \beta_1 x + c_2 \cos \beta_1 x\right) \\ &\quad + e^{\alpha_2 x}\left(c_3 \sin \beta_2 x + c_4 \cos \beta_2 x\right) \end{aligned}}$$

$$\begin{aligned} y &= e^{0x}\left(c_1 \sin 2x + c_2 \cos 2x\right) \\ &\quad + e^{0x}\left(c_3 \sin 3x + c_4 \cos 3x\right) \end{aligned}$$

Remember that :

* $\boxed{e^0 = 1}$

$$\boxed{y = c_1 \sin 2x + c_2 \cos 2x + c_3 \sin 3x + c_4 \cos 3x}$$

where c_1, c_2, c_3, c_4 are arbitrary constants .

Example (16):

Solve the boundary-value problem (BVP)

$$y'' + 2y' + 2y = 0 \quad , \quad y(0) = 1 \quad , \quad y'\left(\frac{\pi}{2}\right) = 0 .$$

Solution

* The given differential equation

$$y'' + 2y' + 2y = 0 \qquad (1)$$

is homogeneous linear differential equation with constant coefficients .

* The characteristic equation (auxialiary equation) of the given differential equation is

$$m^2 + 2m + 2 = 0 \qquad (2)$$

* *This quadratic equation is not factorable using integers as its descrimenant $b^2 - 4ac = -4$ is not positive perfect square , we use the quadratic formula*

Remember that :

* $\left| \begin{array}{l} \text{If } ax^2 + bx + c = 0 , \\ \text{then } x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array} \right.$

* $\left| \sqrt{-1} = i \right.$

$$a = 1 \quad , \quad b = 2 \quad , \quad c = 2$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm i\,2}{2} = -1 \pm i$$

* *The roots of* (2) *are* $m_{1,2} = -1 \pm i = \alpha \pm i\,\beta .$ *They are conjugate complex roots .*

* *Then the general solution is*

$$\boxed{y = e^{\alpha x}\left(c_1 \sin \beta x + c_2 \cos \beta x\right)}$$

$$\boxed{y = e^{-x}\left(c_1 \sin x + c_2 \cos x\right)} \qquad (3)$$

where c_1 , c_2 *are arbitrary constants .*

* *To find the values of the two constants* c_1 , c_2 *we use the two initial conditions .*

* *Using the first condition* $y(0)=1$, *i.e.* $y=1$ *when* $x=0$, *from* (3)

$$1=e^{0}\left(c_1\sin 0+c_2\cos 0\right)$$

Remember that :

* $e^{0}=1\ ,\ \sin 0=0\ ,\ \cos 0=1$

$$1=1\left(c_1\,0+c_2\,1\right)$$

$$c_2=1 \qquad (4)$$

* *Differentiating equation* (3), *we obtain*

$$y'=e^{-x}\left(c_1\cos x-c_2\sin x\right)-e^{-x}\left(c_1\sin x+c_2\cos x\right) \qquad (3)$$

* *Using the second condition* $y'\left(\frac{\pi}{2}\right)=0$, *i.e.* $y'=0$ *when* $x=\frac{\pi}{2}$, *from* (3)

$$0=e^{-\frac{\pi}{2}}\left(c_1\cos\frac{\pi}{2}-c_2\sin\frac{\pi}{2}\right)-e^{-\frac{\pi}{2}}\left(c_1\sin\frac{\pi}{2}+c_2\cos\frac{\pi}{2}\right)$$

Remember that :

* $\sin\frac{\pi}{2}=1\ ,\ \cos\frac{\pi}{2}=0$

$$0=e^{-\frac{\pi}{2}}\left(-c_1-c_2\right) \quad\Rightarrow\quad 0=c_1+c_2 \quad\Rightarrow\quad c_1=-c_2$$

Remember that :

* $e^{x}>0\ ,\ x\in\mathbb{R}$

* *From* (4)

$$c_1=-1$$

* Then the particular solution is

$$\boxed{y = e^{-x}\left(-\sin x + \cos x\right)}\ .$$

Example (18):

Use the undetermined coefficients method to solve the nonhomogeneous differential equation $\dfrac{d^2 y}{dx^2} - y = x^2$.

Solution

* The given differential equation

$$y'' - y = x^2 \qquad (1)$$

is nonhomogeneous linear differential equation with constant coefficients .

Step 1 :

* To solve equation (1) we obtain y_c the complementary function .

* The associated homogeneous differential equation of (1) is

$$y'' - y = 0 \qquad (2)$$

* The characteristic equation of the homogeneous differential equation is

$$m^2 - 1 = 0 \qquad (3)$$

* Factorizing , we get

$$(m-1)(m+1) = 0$$

* The roots of (3) are $m_1 = 1$, $m_2 = -1$. They are distinct real roots .

* The complementary function is

$$\boxed{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}}$$

$$\boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$$

where c_1, c_2 *are arbitrary constants .*

Step 2 :

* *We obtain* y_p *the particular solution .*

* *Since* $F(x) = x^2$ *a polynomial of degree* $n = 2$ *, then* $\boxed{y_p = x^k Q(x)}$ *where* $Q(x)$ *is a polynomial of the same degree as* $F(x)$ *, and since zero is not included in the roots of characteristic equation* (3) *, then* $k = 0$ *, and*

$$y_p = A x^2 + B x + C$$

$$y'_p = 2 A x + B$$

$$y''_p = 2 A$$

* *Substituting in the given nonhomogeneos differential equation* (1) *, we obtain*

$$2A - \left(A x^2 + B x + C\right) = x^2 \qquad (4)$$

* *Equating coefficients in both sides of identity* (4)

coefficient of $x^2 : \; -A = 1 \quad \Rightarrow \quad A = -1$

coefficient of $x : \; -B = 0 \quad \Rightarrow \quad B = 0$

constant term $: \; 2A - C = 0 \quad \Rightarrow \quad C = 2A = 2(-1) = -2$

$$\boxed{y_p = -x^2 - 2} \, .$$

Step 3 :

* Then the general solution of the given nonhomogeneous equation (1) is

$$\boxed{y = y_c + y_p}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - x^2 - 2}\,.$$

Example (20):

Use the undetermined coefficients method to solve the nonhomogeneous differential equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} - 3y = 4e^{x}.$$

Solution

* The given differential equation

$$y'' + 2y' - 3y = 4e^{x} \qquad (1)$$

is nonhomogeneous linear differential equation with constant coefficients .

Step 1 :

* To solve equation (1) we obtain y_c the complementary function .

* The associated homogeneous differential equation of (1) is

$$y'' + 2y' - 3y = 0 \qquad (2)$$

* The characteristic equation of the homogeneous differential equation (2) is

$$m^2 + 2m - 3 = 0 \qquad (3)$$

* Factorizing , we get

$$(m-1)(m+3)=0$$

* The roots of (3) are $m_1=1$, $m_2=-3$. They are distinct real roots .

* The complementary function is

$$\boxed{y_c=c_1 e^{m_1 x}+c_2 e^{m_2 x}}$$

$$\boxed{y_c=c_1 e^{x}+c_2 e^{-3x}}$$

where c_1 , c_2 are arbitrary constants .

Step 2 :

* We obtain y_p the particular solution .

* Since $F(x)=4e^{1x}$ exponential function , then $\boxed{y_p=A x^k e^{\alpha x}}$, and since $\alpha=1$ is included in the roots of characteristic equation (3) , then $k=1$, and

$$y_p=A x e^{x}$$

$$y'_p=A\left[x e^{x}+e^{x}\right]=A(x+1)e^{x}$$

$$y''_p=A\left[(x+1)e^{x}+e^{x}\right]=A(x+2)e^{x}$$

Remember that :

* $\boxed{\dfrac{d}{dx}e^{g(x)}=e^{g(x)}.g'(x)}$

* Substituting in the given nonhomogeneos differential equation (1) , we obtain

$$A(x+2)e^{x}+2A(x+1)e^{x}-3Axe^{x}=4e^{x} \qquad \left(\div e^{x}\right)$$

Remember that :

* $\boxed{e^{x} > 0 \ , \ x \in \mathbb{R}}$

$$4A = 4 \quad \Rightarrow \quad A = \frac{4}{4} = 1$$

$$\boxed{y_p = x e^{x}} \ .$$

Step 3 :

* *Then the general solution of the given nonhomogeneous equation (1) is*

$$\boxed{y = y_c + y_p}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{-3x} + x e^{x}} \ .$$

Example (22):

Use the undetermined coefficients method to solve the nonhomogeneous differential equation

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 2\cos 3x \ .$$

Solution

* *The given differential equation*

$$y'' + 2y' + y = 2\cos 3x \qquad (1)$$

is nonhomogeneous linear differential equation with constant coefficients .

Step 1 :

* *To solve equation (1) we obtain y_c the complementary function .*

* The associated homogeneous differential equation of (1) is

$$y''+2y'+y=0 \qquad (2)$$

* The characteristic equation of the homogeneous differential equation is

$$m^2+2m+1=0 \qquad (3)$$

* Factorizing , we get

$$(m+1)(m+1)=0$$

* The roots of (3) are $m_1=-1$, $m_2=-1$. They are equal real roots .

* The complementary function is

$$\boxed{y=(c_1+c_2x)e^{mx}}$$

$$\boxed{y_c=(c_1+c_2x)e^{-x}}$$

where c_1, c_2 are arbitrary constants .

Step 2 :

* We obtain y_p the particular solution .

* Since $F(x)=2\cos 3x$, then $\boxed{y_p=x^k(A\sin\beta x+B\cos\beta x)}$, and since $\pm i3$ is not included in the roots of characteristic equation (3), then $k=0$, and

$$y_p=A\sin 3x+B\cos 3x$$

$$y'_p=3A\cos 3x-3B\sin 3x$$

$$y''_p=-9A\sin 3x-9B\cos 3x$$

Remember that :

$$* \quad \frac{d}{dx}\sin(g(x)) = \cos(g(x)) . g'(x)$$

$$* \quad \frac{d}{dx}\cos(g(x)) = -\sin(g(x)) . g'(x)$$

* *Substituting in the given nonhomogeneos differential equation* (1), *we obtain*

$$(-9A\sin 3x - 9B\cos 3x) + 2(3A\cos 3x - 3B\sin 3x)$$
$$+ (A\sin 3x + B\cos 3x) = \cos 3x$$

$$(-8A - 6B)\sin 3x + (6A - 8B)\cos 3x = 0\sin 3x + 2\cos 3x \qquad (4)$$

* *Equating coefficients in both sides of identity* (4)

coefficient of $\sin 2x$: $-8A - 6B = 0 \Rightarrow 4A + 3B = 0 \qquad (5)$

coefficient of $\cos 2x$: $6A - 8B = 2 \Rightarrow 3A - 4B = 1 \qquad (6)$

* *Multiplying equation* (5) *by* 4 *and equation* (6) *by* 3

$$16A + 12B = 0 \qquad (7)$$

$$9A - 12B = 3 \qquad (8)$$

* *Adding* (7) *and* (8)

$$25A = 3 \Rightarrow A = \frac{3}{25} \qquad (9)$$

* *From* (5) *and* (9)

$$3B = -4A \Rightarrow B = -\frac{4}{3}A = -\frac{4}{3}\left(\frac{3}{25}\right) = -\frac{4}{25}$$

$$\boxed{y_p = \frac{3}{25}\sin 3x - \frac{4}{25}\cos 3x} .$$

Step 3 :

* Then the general solution of the given nonhomogeneous equation (1) is

$$\boxed{y = y_c + y_p}$$

$$\boxed{y = (c_1 + c_2 x)e^{-x} + \frac{3}{25}\sin 3x - \frac{4}{25}\cos 3x} .$$

Example (25):

Use the undetermined coefficients method to solve the nonhomogeneous differential equation

$$y'' - 3y' + 2y = e^x \sin x \ .$$

Solution

* The given differential equation

$$y'' - 3y' + 2y = e^x \sin x \qquad (1)$$

is nonhomogeneous linear differential equation with constant coefficients .

Step 1 :

* To solve equation (1) we obtain y_c the complementary function .

* The associated homogeneous differential equation of (1) is

$$y'' - 3y' + 2y = 0 \qquad (2)$$

* The characteristic equation of the homogeneous differential equation (2) is

$$m^2 - 3m + 2 = 0 \qquad (3)$$

* Factorizing , we get

$$(m-1)(m-2)=0$$

* The roots of (3) are $m_1=1$, $m_2=2$. They are distinct real roots .

* The complementary function is

$$\boxed{y_c=c_1e^{m_1x}+c_2e^{m_2x}}$$

$$\boxed{y_c=c_1e^{x}+c_2e^{2x}}$$

where c_1, c_2 are arbitrary constants .

Step 2 :

* We obtain y_p the particular solution .

* Since $F(x)=e^x\sin x$, and since $1\pm i$ is not included in the roots of characteristic equation (3) , then $k=0$, and

$$y_p=e^x(A\sin x+B\cos x)=Ae^x\sin x+Be^x\cos x$$

$$y_p'=A\left(e^x\cos x+e^x\sin x\right)+B\left(-e^x\sin x+e^x\cos x\right)$$

$$=(A-B)e^x\sin x+(A+B)e^x\cos x$$

$$y_p''=(A-B)\left(e^x\cos x+e^x\sin x\right)$$

$$+(A+B)\left(-e^x\sin x+e^x\sin x\right)$$

$$=-2Ae^x\sin x+2Ae^x\cos x$$

* Substituting in the given nonhomogeneos differential equation (1) , we obtain

$$\left[-2Ae^{x}\sin x+2Ae^{x}\cos x\right]$$

$$-3\left[(A-B)e^{x}\sin x+(A+B)e^{x}\cos x\right] \qquad (2)$$

$$+2\left[Ae^{x}\sin x+Be^{x}\cos x\right]=e^{x}\sin x$$

* *Dividing both sides of equation* (2) *by* e^{x}

$$(-A+B)\sin x+(-A-B)\cos x=\sin x+0\cos x \qquad (3)$$

* *Equating coefficients in both sides of identity* (3)

coefficient of $\sin x$: $-A+B=1$ $\qquad (4)$

coefficient of $\cos x$: $-A-B=0$ $\qquad (5)$

* *Adding* (4) *and* (5)

$$-2A=1 \Rightarrow A=-\frac{1}{2} \qquad (6)$$

* *From* (5) *and* (6)

$$B=-A \Rightarrow B=\frac{1}{2}$$

$$\Rightarrow y_{p_2}=\sin 2x+\cos 2x$$

* *Then* $$\boxed{y_p=\frac{1}{2}e^{x}(-\sin x+\cos x)}.$$

Step 3 :

* *Then the general solution of the nonhomogeneous equation is*

$$\boxed{y=y_c+y_p}$$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{x} \left(-\sin x + \cos x \right)} .$$

Example (27):

Use the variation of parameters method to solve the nonhomogeneous equation $y'' + 9y = 3\sec 3x$.

Solution

* *The given nonhomogeneous differential equation*

$$y'' + 9y = 3\sec 3x \;. \qquad (1)$$

Step 1 :

* *To solve equation (1) we obtain y_c the complementary function .*

* *The associated homogeneous differential equation is*

$$y'' + 9y = 0 .$$

* *The characteristic equation of the homogeneous differential equation is*

$$m^2 + 9 = 0 .$$

* *Factorizing*

$$(m - 3i)(m + 3i) = 0 .$$

Remember that :

* $\left| \begin{aligned} a^2 - b^2 &= (a - b)(a + b) \\ a^2 + b^2 &= (a - ib)(a + ib) \end{aligned} \right.$

* *The roots are $m_1 = 3i$, $m_2 = -3i$, two conjugate complex roots , $m_{1,2} = \alpha \pm i\beta = 0 \pm 3i$.*

* *The complementary function*

$$\boxed{y_c(x) = e^{\alpha x}\left(c_1 \sin \beta x + c_2 \cos \beta x\right)}$$

$$y_c(x) = e^{0x}\left(c_1 \sin 3x + c_2 \cos 3x\right)$$

$$\boxed{y_c(x) = c_1 \sin 3x + c_2 \cos 3x} \qquad (2)$$

where c_1, c_2 are arbitrary constants .

Step 2 :

* **Since** $\quad y_1(x) = \sin 3x \quad , \quad y_2(x) = \cos 3x$

* **Find the Wronskian**

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix}$$

$$= -3\sin^2 x - 3\cos^2 x$$

$$= -3\left(\sin^2 x + \cos^2 x\right) = -3 .$$

Remember that :

* $\boxed{\sin^2 x + \cos^2 x = 1}$

* **Find**

$$v_1(x) = \int \frac{-y_2 F(x)}{W(y_1, y_2)} dx = \int \frac{-\cos 3x \;\; 3 \operatorname{ses} 3x}{-3} dx$$

$$= \int \cos 3x \left(\frac{1}{\cos 3x}\right) dx = \int 1 \, dx = x \; .$$

* **Find**

$$v_2(x) = \int \frac{y_1 F(x)}{W(y_1, y_2)} dx = \int \frac{\sin 3x \;\; 3\sec 3x}{-3} dx$$

$$= \frac{1}{3} \int \frac{-3 \sin 3x}{\cos 3x} dx$$

$$= \frac{1}{3} \ln |\cos 3x| \; .$$

Remember that :

$$* \quad \boxed{\int \frac{g'(x)}{g(x)} dx = \ln |g(x)| + C}$$

* *Then the particular solution of nonhomogeneous equation will be*

$$y_p(x) = v_1(x) y_1(x) + v_2(x) y_2(x)$$

$$y_p(x) = x \sin 3x + \frac{1}{3} \ln |\cos 3x| \cos 3x$$

$$\boxed{y_p = x \sin 3x + \frac{1}{3} \ln |\cos 3x| \cos 3x} \; . \qquad (3)$$

* *From* (2) *and* (3) *, then the general solution of the nonhomogeneous equation is*

$$\boxed{y(x) = y_c(x) + y_p(x)}$$

$$\boxed{y(x) = c_1 \sin 3x + c_2 \cos 3x + \frac{1}{3} \ln |\cos 3x| \cos 3x} \; .$$

Example (29):

Use the variation of parameters method to solve the nonhomogeneous equation $y'' - y' - 12y = 2 \sinh^2 x$.

Solution

* *The given nonhomogeneous differential equation*

$$y'' - y' - 12y = 2 \sinh^2 x \; . \qquad (1)$$

Step 1 :

* To solve equation (1) we obtain y_c the complementary function .

* The associated homogeneous differential equation is

$$y''-y'-12y=0.$$

* The characteristic equation of the homogeneous differential equation is

$$m^2-m-12=0.$$

* Factorizing

$$(m+3)(m-4)=0.$$

* The roots are $m_1=-3$, $m_2=4$, two distinct real roots .

* The complementary function

$$\boxed{y_c(x)=c_1e^{m_1x}+c_2e^{m_2x}}$$

$$\boxed{y_c(x)=c_1e^{-3x}+c_2e^{4x}} \qquad (2)$$

where c_1, c_2 are arbitrary constants .

<u>Step 2 :</u>

* Since $y_1(x)=e^{-3x}$, $y_2(x)=e^{4x}$.

* Find the Wronskian

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix}$$

$$=(e^{-3x})(4e^{4x})-(-3e^{-2x})(e^{4x})$$

$$=4e^{x}+3e^{x}=7e^{x}.$$

<u>Remember that :</u>

* $e^a e^b=e^{a+b}$

*** Find**

$$v_1(x)=\int \frac{-y_2\,F(x)}{W\,(y_1,y_2)}\,dx=\int \frac{-\left(e^{4x}\right)\,2\sinh^2 x}{7e^x}\,dx$$

$$=-\frac{2}{7}\int e^{3x}\sinh^2 x\,dx$$

Remember that :

$$* \quad \sinh x=\frac{1}{2}\left(e^x-e^{-x}\right)$$

$$=-\frac{2}{7}\int e^{3x}\left[\frac{1}{2}\left(e^x-e^{-x}\right)\right]^2 dx$$

$$=-\frac{2}{7}\int e^{3x}\left[\frac{1}{4}\left(e^{2x}+e^{-2x}-2\right)\right]dx$$

$$=-\frac{1}{14}\int\left(e^{5x}+e^x-2e^{3x}\right)dx$$

Remember that :

$$* \quad \int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+C$$

$$=-\frac{1}{14}\left(\frac{1}{5}e^{5x}-\frac{2}{3}e^{3x}+e^x\right).$$

*** Find**

$$v_2(x)=\int \frac{y_1\,F(x)}{W\,(y_1,y_2)}\,dx=\int \frac{\left(e^{-3x}\right)\,2\sinh^2 x}{7e^x}\,dx$$

$$=\frac{2}{7}\int e^{-2x}\sinh^2 x\,dx$$

$$=\frac{2}{7}\int e^{-2x}\left[\frac{1}{2}\left(e^{x}-e^{-x}\right)^{2}\right]dx$$

$$=\frac{2}{7}\int e^{-2x}\left[\frac{1}{4}\left(e^{2x}+e^{-2x}-2\right)\right]dx$$

$$=\frac{1}{14}\int\left(1-2e^{-2x}+e^{-4x}\right)dx$$

$$=\frac{1}{14}\left(x+e^{-2x}-\frac{1}{4}e^{-4x}\right).$$

* ***Then the particular solution of nonhomogeneous equation will be***

$$y_p(x)=v_1(x)y_1(x)+v_2(x)y_2(x)$$

$$y_p(x)=-\frac{1}{14}\left(\frac{1}{3}e^{5x}-\frac{2}{3}e^{3x}+e^{x}\right)\left(e^{-3x}\right)$$

$$+\frac{1}{14}\left(x+e^{-2x}-\frac{1}{4}e^{-4x}\right)\left(e^{4x}\right)$$

$$=-\frac{1}{14}\left(\frac{1}{3}e^{2x}-\frac{2}{3}+e^{-2x}\right)+\frac{1}{14}\left(xe^{4x}+e^{2x}-\frac{1}{4}\right)$$

$$\boxed{y_p=\frac{1}{14}\left(xe^{4x}+\frac{2}{3}e^{2x}-\frac{1}{4}-e^{-2x}\right)}. \qquad (3)$$

* ***From (2) and (3), then the general solution of the nonhomogeneous equation is***

$$\boxed{y(x)=y_c(x)+y_p(x)}$$

$$\boxed{y(x)=c_1e^{-3x}+c_1e^{4x}+\frac{1}{14}\left(xe^{4x}+\frac{2}{3}e^{2x}-\frac{1}{4}-e^{-2x}\right)}.$$