USA IMO TST 2026 Solutions

United States of America — Team Selection Test

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67$^{\text{th}}$ IMO 2026 China

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§1 Solutions to Day 1

§1.1 Solution to IMO TST 1, by Carlos Rodriguez

Let n be a positive integer. Prove that one can paint the non-zero coefficients of the polynomial

$$f(x_1, x_2, \ldots, x_n) = \prod_{k=0}^{n} (x_1 + x_2 + \cdots + x_n - k)$$

with $2^n - 1$ colors such that the coefficients of each color have sum 0, and each color is used at least once.

Note that for any $v \in \{0,1\}^n$ we have $f(v) = 0$. Consider the polynomial $\tilde{f}$ obtained from f by replacing all non-zero exponents replaced by 1. Note that for any $v \in \{0,1\}^n$, we have $\tilde{f}(v) = f(v) = 0$. We first prove a claim that shows $\tilde{f}$ is the zero polynomial.

Claim — If g is a polynomial in n variables such that $g(v) = 0$ for all $v \in \{0,1\}^n$ and the degree of g in each variable is at most 1, then $g = 0$.

Proof. We can prove this by induction on n. For $n = 0$, this is clear.

Suppose the claim is true for $n-1$, we can prove it is true for n. Let g be a polynomial satisfying the conditions of the claim, then $g(x_1, x_2, \ldots, x_n) = x_n A(x_1, x_2, \ldots, x_{n-1}) + B(x_1, x_2, \ldots, x_{n-1})$ for some polynomials A and B. Setting $x_n = 0$, we see that B satisfies the induction hypothesis, so $B = 0$. Setting $x_n = 1$, we see that A also satisfies the induction hypothesis, so $A = 0$, as desired. □

We color each term in f with the set of indices of variables appearing in that term. That is, a term of the form $x_{i_1}^{e_1} x_{i_2}^{e_2} \ldots x_{i_k}^{e_k}$ with $e_1, e_2, \ldots, e_k \geq 1$ will receive the color $\{i_1, i_2, \ldots, i_k\}$.

Let $S = \{i_1, i_2, \ldots, i_k\}$ be one of the $2^n - 1$ nonempty subset of $\{1, 2, \ldots, n\}$. By the claim, the sum of the coefficients of terms colored S is 0. The term $x_{i_1}^{n-k+1} x_{i_2} \ldots x_{i_k}$ has positive coefficient in f because it can be formed from the product and every way to do so has coefficient 1. Hence S is used as a color at least once.

The constant term of f is zero, and each other term is colored with exactly one subset, so we have a coloring with $2^n - 1$ colors satisfying the desired condition.

Remark (author)**.** This problem has a generalization as follows: Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be a sequence of positive integers with sum A. Prove that one can paint the non-zero coefficients of the polynomial

$$f(x_1, x_2, \ldots, x_n) = \prod_{k=0}^{A} (a_1 x_1 + a_2 x_2 + \cdots + a_n x_n - k)$$

with $2^n - 1$ colors such that the coefficients of each color have sum 0 and each color is used at least once.

§1.2 Solution to IMO TST 2, by Daniel Zhu

Let p be a prime and let a and b be positive integers less than p. Show that

$$\sum_{k=1}^{b}(-1)^{\lfloor (a-1)k/p\rfloor+\lfloor ak/p\rfloor} \ge 0.$$

If $a = 1$ every term in the sum is 1 so assume $a \ge 2$. We proceed in three steps:

- Call an integer $1 \le k \le p-1$ *bad* if $\lfloor (a-1)k/p\rfloor + \lfloor ak/p\rfloor$ is odd and *good* otherwise. Since $(a-1)k/p$ and ak/p are nonintegers that differ by $k/p < 1$, badness is equivalent to the interval $I_k = ((a-1)k/p, ak/p)$ containing an integer.
- If $b+1$ is bad, then we can increase b by 1 to get a stronger inequality. Thus we may assume that $b+1$ is either good or equal to p.
- In this case, we claim that for every $1 \le k \le b$, at least one of $\{k, b+1-k\}$ is good, which will solve the problem. This is true since if $s \in I_k$ and $t \in I_{b+1-k}$ are integers, then $s+t \in I_{b+1}$, which is a contradiction in both cases.

§1.3 Solution to IMO TST 3, by Ryan Alweiss

Prove that for any subset S of $\mathbb{R}^2$, there exists a (not necessarily axis-aligned) rectangle of area 1 that contains either 0 or more than 2025 points in its strict interior.

¶ Solution 1 First we prove the following claim.

> **Claim** — Let C be the circle centered at the origin with area $\varepsilon < 1$ and let P be a point at distance d from the origin, where $1 \le d$. Then there is an ellipse E which contains C and P and has area at most $2d\sqrt{\varepsilon}$.

Proof. Let C have radius $r = \sqrt{\varepsilon/\pi}$ and without loss of generality, let P be on the x-axis. Now choose E to be the axis-aligned ellipse passing through P and $(0, r)$. Then the area of E is at most $d\varepsilon/r \le 2d\sqrt{\varepsilon}$. (We can enlarge by an arbitrary factor $r > 1$ to ensure strict containment.) $\square$

Now we use induction to prove the following claim, which has most of the substance of the solution.

> **Claim** — Let n be a positive integer. For any $\varepsilon > 0$, there exists an ellipse of area at most ε containing at least n points of S.

Proof. Clearly, the base case $n = 1$ is true. For the inductive step, we will prove the claim for $n + 1$ given the claim for n.

Given $\varepsilon > 0$, choose $\varepsilon_1 = \max\left(0.5, \left(\frac{\varepsilon}{6}\right)^2\right)$ so that $\varepsilon_1 < 1$ and $6\sqrt{\varepsilon_1} < \varepsilon$. By the inductive hypothesis, we can find an ellipse E_1 of area ε_1 containing at least n points of S.

Without loss of generality, let E_1 be centered at the origin and be axis aligned. Apply the affine transformation T scaling along the x-axis and the y-axis such that E_1 is sent to the circle C centered at the origin with the same area ε_1. Note that T sends axis-aligned rectangles to axis-aligned rectangles of the same area.

Therefore the preimage T^{-1} of the square $[1, 2] \times [0, 1]$ must contain some point P of S, and conversely $Q = T(P)$ lies in $[1, 2] \times [0, 1]$. Then $T(E_1) = C$ and Q satisfy the conditions of the previous claim with $d = 3$, so there exists an ellipse E of area at most $6\sqrt{\varepsilon_1} < \varepsilon$ containing C and Q.

The preimage $T^{-1}(E)$ is then an ellipse of area ε containing at least $n + 1$ points of S (namely, P and the points that were already in E_1). $\square$

To complete the argument, note that any ellipse of area ε is contained in a rectangle of area $4\varepsilon/\pi$. Thus we can find a rectangle of area 1 containing at least 2026 points of S, which is a contradiction.

¶ Solution 2 We proceed similarly to Solution 1, using induction to prove the following claim.

> **Claim** — Let n be a positive integer. For any $\varepsilon > 0$, there exists a *triangle* of area at most ε containing at least n points of S.

For simplicity in the proof, suppose that a triangle contains its boundary.

Proof. The base case of $n = 1$ is easy. For the inductive step, for any ε, suppose that ABC is a triangle with n points of S inside it by the inductive hypothesis. Let $K = [ABC]$, further suppose that $K + 2\sqrt{K} < \varepsilon$. We can now prove the claim for $n + 1$.

WLOG $\angle A \geq \angle B, \angle C$. Let D be the point on line AB closer to A than B such that $AD = \sqrt{2/K} \cdot AB$, and define E on line AC similarly. Then $ADE \sim ABC$ and $[ADE] = 2/K \cdot K = 2$. Let M, N, P be the midpoints of AD, AE, DE respectively. Let M', N' be the feet from M, N to $\overline{DE}$. Since $\angle A \geq \angle D, \angle E$, we have that rectangle $MM'N'N$ lies inside $\triangle ADE$. We have $[MM'N'N] = 2[MNP] = 2 \cdot [ADE]/4 = 1$, so $MM'N'N$ has a point Q in its interior which is also in S. Since $Q \in \triangle ADE$, we have that $\triangle ABC \subseteq \triangle BCQ$, so $\triangle BCQ$ contains $n + 1$ points of S.

Let h be the height from A to $\overline{BC}$. Then $\overline{DE}$ is distance $h(1 + \sqrt{2/K})$ from $\overline{BC}$, so Q is at most this distance from $\overline{BC}$. Thus $\triangle BCQ$ has area at most $\frac{1}{2} \cdot BC \cdot h \cdot (1 + \sqrt{2/K}) = K + 2\sqrt{K} < \varepsilon$, as desired. $\square$

To finish, note that any triangle of area ε can be contained in a rectangle of area 2ε, so we can find a rectangle of area 1 containing 2026 points of S in its strict interior, as desired.

§2 Solutions to Day 2

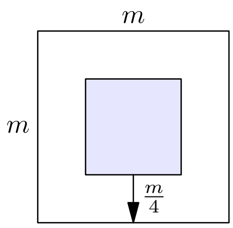
§2.1 Solution to IMO TST 4, by Ruben Carpenter

Let n be a positive integer. In the infinite lattice $\mathbb{Z}^2$, n points are colored red while the rest are colored blue. Each red point is labeled with the distance to the nearest blue point in the same row or column. Find the smallest real number α for which the sum of all labels does not exceed $100n^\alpha$, independent of n and the placement of the red points.

(Note: A *row* is the set of points with a given y-coordinate, and a *column* is the set of points with a given x-coordinate.)

¶ Answer. We claim $\boxed{\alpha = 3/2}$. We present a construction achieving the lower bound, and four proofs of the upper bound.

¶ Lower bound. We estimate the sum of labels when the red points form an $m \times m$ square. Each of the central $(m/2)^2$ points has label at least $m/4$.



This means the total sum is at least $(m/2)^2 \cdot m/4 = m^3/16$, so $\alpha \geq \frac{3}{2}$.

¶ Upper bound (author): We estimate the sum of labels in each row.

Claim — If a row has r red points, their labels sum to at most $r\sqrt{n}$.

Proof. Considering the vertical and horizontal components in turn gives the following independent estimates.

1. Each of their labels is at most r. Thus the sum is at most r^2.
2. The label for a point is at most the number of red points in its column. Thus the sum is at most n.

Each is not good enough by itself. However we can combine them: the sum will be at most $\min(n, r^2) \leq \sqrt{n \cdot r^2} = r\sqrt{n}$, establishing the claim. $\square$

Summing across all rows shows the total sum is at most $n^{3/2}$. Thus $\alpha = \frac{3}{2}$ works.

Remark (author)**.** A perhaps more instructive way of presenting the key bound is as follows. For each row we get to choose what estimate to use: fix some threshold $C > 0$, and use the first estimate when $r > C$ (of which there are at most n/C rows) and the second when $r < C$. Then

$$N \leq n \cdot \frac{n}{C} + C \cdot n.$$

> The optimal choice is $C = \sqrt{n}$, which gives $N \le O(n^{\frac{3}{2}})$.

¶ Upper bound (Roger Lidón): Let a_{ij} denote the label at any red point (i, j), and let x_{ij} be the indicator of (i, j) being red. We will prove

$$\sum_{i,j} a_{ij} \le \left(\sum_{i,j} x_{ij} \right)^{3/2}.$$

Clearly

$$a_{ij} \le \min \left\{ \sum_k x_{ik}, \sum_k x_{kj} \right\},$$

(we even drop a factor of 2). Crucially, $a_{ij} = 0$ if $x_{ij} = 0$, so we can strengthen our bound to

$$a_{ij} \le x_{ij} \min \left\{ \sum_k x_{ik}, \sum_k x_{kj} \right\}.$$

This will be enough. We claim that

$$\sum_{i,j} x_{ij} \min \left\{ \sum_k x_{ik}, \sum_k x_{kj} \right\} \le \left(\sum_{i,j} x_{ij} \right)^{3/2}.$$

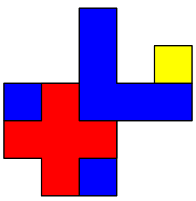
Indeed, by the min–GM inequality and Cauchy–Schwarz

$$\begin{aligned}
\sum_{i,j} x_{ij} \min \left\{ \sum_k x_{ik}, \sum_k x_{kj} \right\} &\le \sum_{i,j} x_{ij} \sqrt{\left(\sum_k x_{ik} \right) \left(\sum_k x_{kj} \right)} \\
&\le \left(\sum_{i,j} x_{ij}^2 \right)^{\frac{1}{2}} \left(\sum_{i,j} \left(\sum_k x_{ik} \right) \left(\sum_k x_{kj} \right) \right)^{\frac{1}{2}} \\
&= \left(\sum_{i,j} x_{ij} \right)^{3/2}
\end{aligned}$$

as we needed to obtain.

¶ Upper bound (Andrew Gu) This solution is most naturally phrased with the following reformulation of the problem.

> Turbo builds *crosses* by joining a $1 \times (2k+1)$ and a $(2k+1) \times 1$ piece of wood at their centers, for some integer value of k. He has many crosses and wants to place them all in an infinite square grid, aligning them with the gridlines. They can overlap, but no two crosses may cover exactly the same set of squares (he doesn't want to waste crosses).



Above, Turbo placed $N = 3$ crosses, covering 13 squares. Find the largest α such that, by placing N crosses according to his rules, he must cover at least N^α squares.

> **Remark.** This was the original formulation of the problem. However we have chosen the current problem statement because it is more direct.

Each cross can be mapped to a tuple (x, y, z) where (x, y) is the center of the cross and z is the length of the arms. We will show that if there are n crosses, then at least $n^{2/3}$ squares are covered.

By IMO 1992/5, the projection of the tuples (x, y, z) onto one of the three planes has at least $n^{2/3}$ distinct points. If the projection onto the xy-plane has $n^{2/3}$ points then we are done. Otherwise, suppose the projection onto the xz-plane has $n^{2/3}$ points, by symmetry. We will construct an injection from these points to covered points in the xy-plane.

Partition the points by x, and suppose some x value has k possible z values in the projection. Iterate through the values in increasing order of z. For each one, we can pick a corresponding y value that hasn't been chosen before by the Pigeonhole principle, which gives us an injection.

¶ Upper bound (Milan Haiman) Let n_t denote the number of points with label at least t. We will show that $\sum_t n_t \le n^{3/2}$, noting that the LHS is the sum of the labels. Let x_t be the number of distinct x-coordinates of the points with label at least t and define y_t similarly. We have that $n_t + 2tx_t \le n$ and $n_t + 2ty_t \le n$ by counting red points based on their x-coordinates and y-coordinates, respectively. Also, note that $x_t y_t \ge n_t$. Combining these inequalities with AM-GM gives

$$n_t + 2t\sqrt{n_t} \le n.$$

This rearranges to $n_t \le n + 2t^2 - 2t\sqrt{n + t^2}$, so we obtain

$$\sum_t n_t \le \int_0^\infty n + 2t^2 - 2t\sqrt{n + t^2}dt = \frac{2}{3}n^{3/2}.$$

Once we have $n_t + 2t\sqrt{n_t} \le n$ there are other ways to finish.

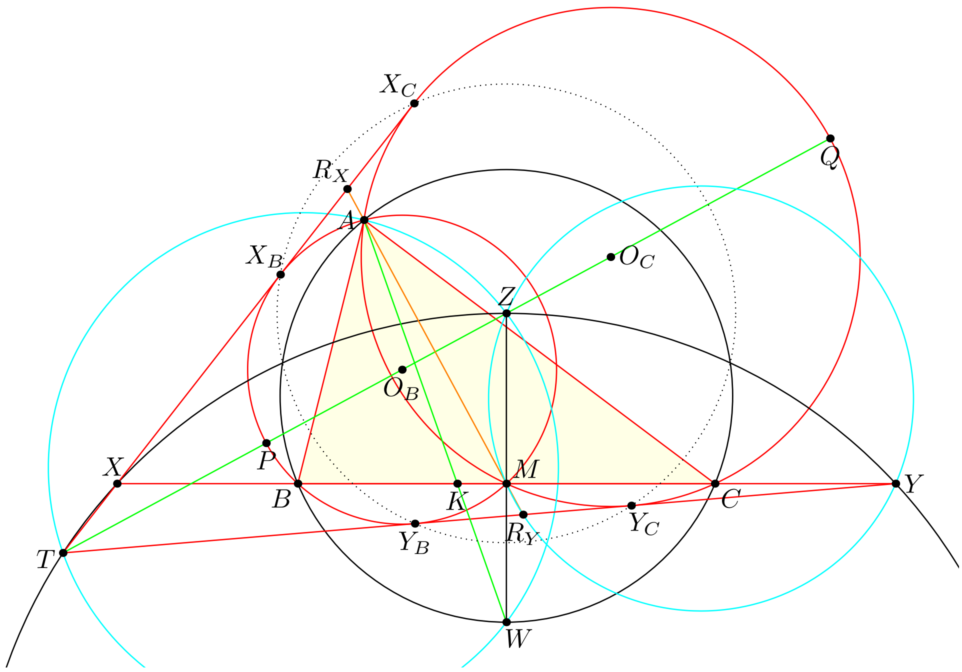
§2.2 Solution to IMO TST 5, by Ruben Carpenter

Let ABC be an acute scalene triangle with circumcircle Γ, and let M the midpoint of BC. Let ω be the circumcircle of triangle formed by BC and the two common external tangents of the circumcircles ABM and ACM. Prove that the internal bisector of $\angle BAC$ and the perpendicular bisector of AM intersect on the radical axis of ω and Γ.

¶ Solution. Let ℓ_X, ℓ_Y be the two common tangents, and introduce the following points:

- $X = \ell_X \cap BC$, $Y = \ell_Y \cap BC$, $T = \ell_X \cap \ell_Y$.
- X_B, X_C, Y_B, Y_C are the tangency points of ℓ_X, ℓ_Y with (ABM), (ACM).
- O_B, O_C, Z are the circumcenters of (ABM), (ACM), and trapezoid $X_BY_BY_CX_C$.

Since $O_BX_B \perp X_BX_C \perp O_CX_C$, the midpoint O_BO_C is equidistant to X_B and X_C. By using the same logic on Y_B and Y_C, it follows that Z is the midpoint of O_BO_C.



Claim — M is the midpoint of XY.

Angle Chasing. Let $R_X = \ell_X \cap AM$ and $R_Y = \ell_Y \cap AM$. By Power of a Point from these points to (ABM) and (ACM), they are the midpoints of X_BX_C and Y_BY_C respectively. It follows that R_XMZX and R_YMZY are cyclic, so

$$\angle ZYM = \angle ZR_YM = \angle ZR_YR_X = \angle R_YR_XZ = \angle MR_XZ = \angle ZXM,$$

as desired. $\square$

Involutions. Let U and V be the intersections of $(X_BY_BY_CX_C)$ and BC, let N_B be $X_BY_B \cap BC$, and let N_C be $X_CY_C \cap BC$. By Desargues' Involution Theorem on $X_BY_BY_CX_C$ and line BC, there is an involution on BC swapping (N_B, N_C), (U, V) and (X, Y). We claim that this is the reflection about M:

- Z is on the perpendicular bisector of BC, so U, V are symmetric over M.
- AM bisects Y_BY_C and X_BX_C, so N_B and N_C are symmetric about M.

Because the first two are pairs under reflection about M, the third one must be so too. □

ISL 2005 G6. Invert at M. We get the following problem:

> Let ABC be a triangle and let M be the midpoint of BC. Let ω_X and ω_Y be circles through M tangent to AB and AC. Let $X = \omega_X \cap BC$ and $Y = \omega_Y \cap BC$. Prove that $MX = MY$.

By taking homotheties at A sending ω_X and ω_Y to the incircle of ABC, we recover ISL 2005 G6. □

Hence Z is the intersection of the angle bisector of $\angle YTX$ and the perpendicular bisector of XY, so it is the midpoint of one of the arcs XY of ω.

Claim — $\angle ZTA = \frac{\angle MBA - \angle ACM}{2}$.

Proof. This is a standard fact about exsimillicenters. Let TZ intersect (ABM) and (ACM) at P and Q as shown in the diagram. Then an inversion at T sending X_B to Y_B swaps (ABM) and (ACM), so it swaps P, Q and fixes A. Hence $\triangle TAP \sim \triangle TQA$, so

$$\angle ZTA = \angle PTA = \angle QPA - \angle TAP = \frac{\angle MPA}{2} - \frac{\angle AQM}{2} = \frac{\angle MBA - \angle ACM}{2}. \quad \square$$

Let $K = AW \cap BC$. Note that

$$\angle ZMA = 90^\circ - \angle AKB = \angle BAK + \angle KBA - 90^\circ$$
$$= \frac{\angle BAC}{2} + \angle CBA - \frac{\angle BAC + \angle ACB + \angle CBA}{2} = \frac{\angle CBA - \angle ACB}{2} = \angle ZTA.$$

It follows that $AZWT$ is cyclic. Applying the radical axis theorem on $(AZWT)$,(ABC), and ω finishes the problem.

§2.3 Solution to IMO TST 6, by Pitchayut Saengrungkongka

A positive integer is called *chaotic* if it can be expressed as $a^3 + b^3 + abc$ for positive integers $a \geq b \geq c$. Show that there is no infinite increasing arithmetic progression consisting of only chaotic positive integers.

We start with a lemma.

Lemma

For a prime p, the number of triples $(a, b, c) \in \mathbb{F}_p^3$ such that $a^3 + b^3 + abc \equiv 0 \pmod p$ is $p^2 - p + 1$.

Proof. For all $a, b \in \mathbb{F}_p$ with $a, b \not\equiv 0 \bmod p$, there is exactly one c such that $a^3+b^3+abc \equiv 0 \bmod p$. This gives $(p-1)^2$ triples.

If $a \equiv 0 \bmod p$ and $a^3 + b^3 + abc \equiv 0 \bmod p$, then $b \equiv 0 \bmod p$, and vice-versa. If $a \equiv b \equiv 0 \bmod p$, then any c gives $a^3 + b^3 + abc \equiv 0 \bmod p$. Hence there are $(p-1)^2 + p$ triples in total, as desired. □

Now assume for the sake of contradiction that an arithmetic progression with only chaotic numbers exists. Let D be the common difference of this arithmetic progression. We know that $\prod_p \frac{p^2-p+1}{p^2} \leq \prod_p \left(1 - \frac{1}{p}\right)^{0.001} = 0$. Hence we can find primes $p_1 < \cdots < p_k$ such that p_i does not divide D for all i and

$$\prod_{i=1}^{k} \frac{p_i^2 - p_i + 1}{p_i^2} < \frac{1}{2D}.$$

Let $P = p_1 \dots p_k$. For any N, we now count the number of chaotic integers less than $(PN)^3$ that are divisible by P.

- On one hand, there are at least $(PN)^3/(PD) - 1$ chaotic integers, because of the numbers divisible by P in the arithmetic progression.
- On the other hand, we note that for any a, b, c with $a \geq b \geq c$ and $a^3 + b^3 + abc \leq (PN)^3$, we have $a, b, c \leq PN$. By the Chinese remainder theorem, the number of triples (a, b, c) such that $a, b, c \leq PN$ and $P \mid a^3 + b^3 + abc$ is

  $$(PN)^3 \prod_{i=1}^{k} \frac{p_i^2 - p_i + 1}{p_i^3} < \frac{P^2 N^3}{2D},$$

  using the condition we imposed on the p_i's.

The two bullet points give a contradiction when N is large.

> **Remark** (author)**.** A more natural variant is to replace $a^3 + b^3 + abc$ by $a^3 + b^3 + c^3$, which will remove the asymmetry in the problem. However, this is unsuitable for a math contest because the solution has the same idea as above, but using the Sato-Tate law on the elliptic curve $x^3 + y^3 + z^3 = 0$.