TSTST 2025 Solutions

United States of America — TST Selection Test

Andrew Gu, Milan Haiman, Krishna Pothapragada

67th IMO 2026 China and 15th EGMO 2026 France

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§1 Solutions to Day 1

§1.1 Solution to TSTST 1, by Milan Haiman

In a finite group of people, some pairs are friends (friendship is mutual). Each person p has a list $f_1(p), f_2(p), \dots, f_{d(p)}(p)$ of their friends, where $d(p)$ is the number of distinct friends p has. Additionally, any two people are connected by a series of friendships. Each person also has a *water balloon*. The following game is played until someone ends up with more than one water balloon: on round r, each person p throws the current water balloon they have to their friend $f_s(p)$ such that $d(p)$ divides $r - s$. Show that if the game never ends, then everyone has the same number of friends.

Given a person p, let $F(p)$ be the set of friends of p. Choose a person p with the most friends. Note that for each friend q of p, p receives a water balloon from q once out of every $d(q)$ turns. Since p always receives 1 water balloon, we must have

$$\sum_{q \in F(p)} \frac{1}{d(q)} = 1.$$

Since this sum has $d(p)$ terms, and since $d(q) \le d(p)$ for all q, we have

$$1 \ge d(p) \cdot \frac{1}{d(p)} = 1.$$

Thus we must have equality for all friends q of p. In particular, $d(q) = d(p)$. Thus all friends of any person with the most number of friends also have the most number of friends.

Again, let p be a person with the most friends. Now for any other person q, there exists a sequence of people $p = p_0, p_1, \dots, p_n = q$. Repeatedly applying the previous result gives us $d(p) = d(p_0) = d(p_1) = \cdots = d(p_n) = d(q)$. Thus any person has the maximum number of friends out of the group, which means that each person has the same number of friends. ■

§1.2 Solution to TSTST 2, by Daniel Zhu

Find all sets $S \subseteq \mathbb{Z}$ for which there exists a function $f\colon \mathbb{R} \to \mathbb{Z}$ such that

- $f(x-y) - 2f(x) + f(x+y) \geq -1$ for all $x, y \in \mathbb{R}$, and
- $S = \{f(z) \mid z \in \mathbb{R}\}$.

The answer is $\{a\}$, $\{a, a+1\}$, $\{a, a+1, a+2, \ldots\}$, and $\mathbb{Z}$, for arbitrary $a \in \mathbb{Z}$. For constructions, it is not hard to show that if $g\colon \mathbb{R} \to \mathbb{R}$ is a convex function, then $\lfloor g \rfloor$ satisfies the functional equation. Thus $f(x) = a$, $f(x) = \lfloor x \rfloor$, and $f(x) = \lfloor x^2 \rfloor + a$ work, covering the first, fourth, and third class of answers respectively. Furthermore, it is not hard to show that $f(x) = a + \mathbf{1}_{x>0}$ also works to cover the second class.

Let $P(x, y)$ denote the condition. To prove that nothing else works, the key result is to prove an "intermediate value theorem": if a and b are in the range of f, then so is every integer between a and b. Let's first see how this finishes. If we assume the intermediate value theorem, then all we need to show is that if the range of the range of f is at least 2, then the range of f is unbounded above. Indeed, if $f(x) - f(y) \geq 2$, then $P(x, y-x)$ gives us that $f(2x - y) > f(x)$, so iterating this procedure finishes.

We will now prove the intermediate value theorem. We will repeatedly use the fact that if $f(x)$ is a solution, so is $f(ax+b) + c$ for $a, b \in \mathbb{R}$ and $c \in \mathbb{Z}$.

Lemma 1.1

If $f(0) \leq -1$, then $f(2^k) \geq 2^k f(1)$ for $k \geq 0$.

Proof. $P(2^k, 2^k)$ yields that $f(2^{k+1}) \geq 2f(2^k)$. □

Lemma 1.2

If $f(-1) \leq -2$ and $f(0) = 0$, then $f(2^k) \geq 2^k - 1$ for all positive integers k.

Proof. Applying Lemma 1.1 to $f(x-1)+1$ yields that $f(2^k - 1) \geq 2^k - 1$. Then, applying Lemma 1.1 to $f(2^k - x) - f(2^k) - 1$ yields that

$$f(0) - f(2^k) - 1 \geq 2^k(f(2^k - 1) - f(2^k) - 1) \implies f(2^k) + 1 \geq \frac{2^k f(2^k - 1)}{2^k - 1} \geq 2^k. \quad \square$$

Now to prove the intermediate value theorem, scale and shift such that $f(-1) \leq -2$ and $f(0) = 0$; it suffices to show that there exists some number strictly between $f(-1)$ and $f(0)$ in the range of f (since by iteration we can then get all values). Suppose not and let $a_k = f(-1/2^k)$. If k is minimal such that $a_k \geq 0$, then $P(-1/2^k, 1/2^k)$ yields a contradiction. Thus $a_k \leq -2$ for all k. However, applying Lemma 1.2 to $f(x/2^k)$ yields that $a_k \leq -2 \implies f(1) \geq 2^k - 1$, which cannot hold for all k since $f(1)$ is constant.

§1.3 Solution to TSTST 3, by Carlos Rodriguez, Albert Wang, Kevin Wu, Isaac Zhu, Nathan Cho

Let a_1, a_2, r, and s be positive integers with r and s odd. The sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+2} = ra_{n+1} + sa_n$$

for all $n \geq 1$. Determine the maximum possible number of integers $1 \leq \ell \leq 2025$ such that a_ℓ divides $a_{\ell+1}$, over all possible choices of a_1, a_2, r, and s.

¶ Answer 1350.

¶ Solution We first provide the upper bound. We start by dividing out any common factors of a_1 and a_2 from the whole sequence. Note that since r and s are odd, and a_1 and a_2 cannot both be divisible by 2, the sequence $a_1, a_2, a_3, \ldots \pmod 2$ must be some cyclic shift of the sequence $1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots$. This means that exactly $\frac{2025}{3} = 675$ of the values of ℓ satisfy $a_\ell \equiv 0 \pmod 2$ and $a_{\ell+1} \equiv 1 \pmod 2$. An even number can never divide an odd number, so we have an upper bound of $2025 \times \frac{2}{3} = 1350$.

Now we provide a construction so that a_ℓ divides $a_{\ell+1}$ for 1350 values of ℓ. Let $F_1, F_2, F_3, \ldots$ denote the Fibonacci sequence with $F_1 = F_2 = 1$. Let

$$C = \frac{F_1 \cdot F_2 \cdot F_3 \cdot F_4 \cdots F_{2025}}{F_3 \cdot F_6 \cdot F_9 \cdot F_{12} \cdots F_{2025}} = F_1 \cdot F_2 \cdot F_4 \cdot F_5 \cdots F_{2024}.$$

Note that C is odd, since it is the product of all odd Fibonacci numbers up to F_{2025}. We let $a_n = C^{n-1}F_n$, which satisfies the recurrence with $r = C^2$ and $s = C$. This gives

$$\frac{a_{\ell+1}}{a_\ell} = C \cdot \frac{F_{\ell+1}}{F_\ell},$$

which is an integer whenever $3 \nmid \ell$ for $1 \leq \ell \leq 2025$.

§2 Solutions to Day 2

§2.1 Solution to TSTST 4, by Maxim Li

Let $n \geq 2$ be a positive integer. Let $a_1, a_2, \ldots, a_n$ be a sequence of positive integers such that

$$\gcd(a_1, a_2), \gcd(a_2, a_3), \ldots, \gcd(a_{n-1}, a_n)$$

is a strictly increasing sequence. Find, in terms of n, the maximum possible value of

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

over all such sequences.

We claim the maximum possible value is 2. To see that this is achievable, let the sequence $(a_i)_{i=1}^n$ be $1, 2, 4, \ldots, 2^{n-2}, 2^{n-2}$. Then $\gcd(a_i, a_{i+1}) = 2^{i-1}$, which is an increasing sequence, and it is easy to check that $\sum \frac{1}{a_i} = 2$. We now show this is the maximum.

Let $d_i = \gcd(a_i, a_{i+1})$. Since $d_{i-1} < d_i$, we have that $\frac{a_i}{d_{i-1}} > \frac{a_i}{d_i}$. But these are both integers, so we get that

$$1 \leq \frac{a_i}{d_{i-1}} - \frac{a_i}{d_i} \implies \frac{1}{a_i} \leq \frac{1}{d_{i-1}} - \frac{1}{d_i}.$$

Adding everything up, we get that

$$\sum_{i=1}^{n} \frac{1}{a_i} \leq \frac{1}{a_1} + \frac{1}{d_1} - \frac{1}{d_{n-1}} + \frac{1}{a_n}.$$

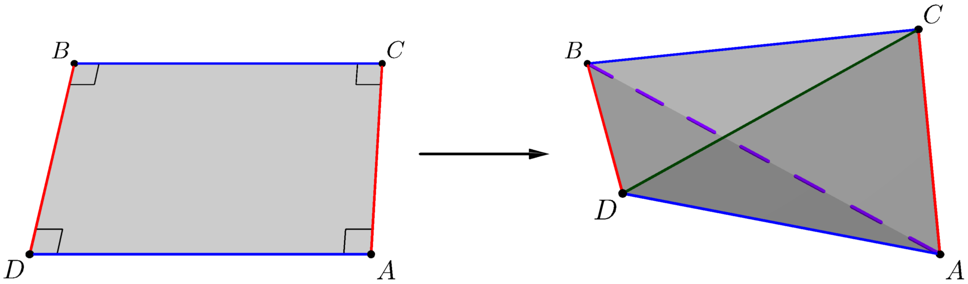
But $d_{n-1} \mid a_n$, so $d_{n-1} \leq a_n$, and $\frac{1}{d_{n-1}} \geq \frac{1}{a_n}$. Thus, this sum is at most $\frac{1}{a_1} + \frac{1}{d_1} \leq 2$. $\square$

§2.2 Solution to TSTST 5, by Karthik Vedula

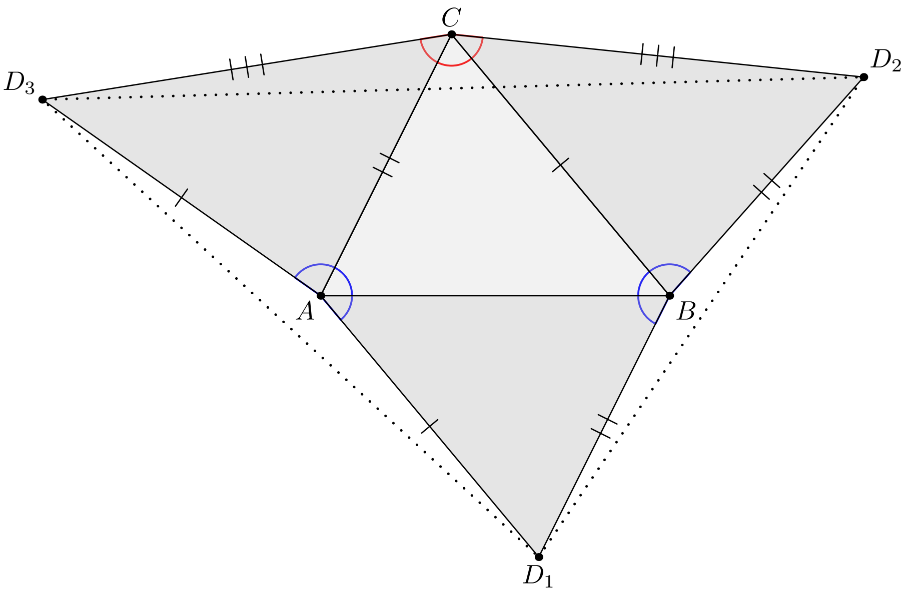
A tetrahedron $ABCD$ is said to be *angelic* if it has nonzero volume and satisfies

$$\angle BAC + \angle CAD + \angle DAB = \angle ABC + \angle CBD + \angle DBA,$$
$$\angle ACB + \angle BCD + \angle DCA = \angle ADB + \angle BDC + \angle CDA.$$

Across all angelic tetrahedrons, what is the maximum number of distinct lengths that could appear in the set $\{AB, AC, AD, BC, BD, CD\}$?



We claim the maximum cardinality is $\boxed{4}$. This is attained by taking a non-square rectangle (or parallelogram) $ACBD$ and folding it along diagonal AB, creating edges AB and CD in the process. Here, $AC = BD$ and $AD = BC$, while every other length is distinct in the general case. This tetrahedron is congruent to itself under the permutation of vertices $(A, B, C, D) \mapsto (B, A, D, C)$, and we can verify that the required angle conditions follow from this symmetry.



In the other direction, let $f(X)$ denote the sum of the angles at X, so the conditions of the problem statement can be written as $f(A) = f(B)$ and $f(C) = f(D)$. Unfold the three faces that meet at D to create a net of the tetrahedron. Along with the face $\triangle ABC$, we also create the faces $\triangle ABD_1$, $\triangle BCD_2$, and $\triangle ACD_3$. Note that

$f(A)+f(B)+f(C)+f(D)$ is the sum of all of the angles of all four faces of the tetrahedron, which is 720°. Therefore, $f(A) = f(B) = 360^\circ - f(C) = 360^\circ - f(D)$. Furthermore, we have that $\angle D_1AD_3 = f(A)$ and $\angle D_2CD_3 = 360^\circ - f(C)$. Since $AD_1 = AD_3$ and $CD_2 = CD_3$ by the definition of unfolding, this gives $\triangle D_3AD_1 \sim \triangle D_3CD_2$. Thus, due to spiral similarity, we have $\triangle D_3AC \sim \triangle D_3D_1D_2$.

Similarly, we also have $\triangle D_2BD_1 \sim \triangle D_2CD_3 \implies \triangle CBD_2 \sim \triangle D_3D_1D_2$. This means that $\triangle CBD_2 \sim \triangle D_3AC$, and since $CD_3 = CD_2$, the two triangles are actually congruent. Therefore, $AC = BD_2 = BD$ and $BC = AD_3 = AD$. Since we have two pairs of equal edge lengths, the number of distinct edge lengths is at most 4, as desired.

§2.3 Solution to TSTST 6, by Max Lu, Kevin Wu

Alice and Bob play a game on n vertices labelled $1, 2, \dots, n$. They take turns adding edges $\{i, j\}$, with Alice going first. Neither player is allowed to make a move that creates a cycle, and the game ends after $n - 1$ total turns.

Let the weight of the edge $\{i, j\}$ be $|i - j|$, and let W be the total weight of all edges at the end of the game. Alice plays to maximize W and Bob plays to minimize W. If both play optimally, what will W be?

¶ Solution Let $k = \lceil \frac{n-1}{2} \rceil$. The answer is

$$\frac{1}{2}(k+1)(2n-k-2) = (n-k-1) + (n-k) + \cdots + (n-1).$$

When $n = 1$, this is clear.

Consider now when $n \geq 2$. Note that the game consists of $n - 1$ moves, with Alice making k moves and Bob making $n - k - 1$ moves.

We first show Alice can guarantee a total of at least $\frac{1}{2}(k+1)(2n-k-2)$.

Claim — Alice can ensure her ith move is an edge of weight at least $n - i$.

Proof. On her first move, Alice chooses the edge $(1, n)$, which has weight $n - 1$. Now, consider the ith move, for $i > 1$.

Consider the induced subgraph on vertices $\{1, \dots, i\} \sqcup \{n - i + 1, \dots, n\}$. (Note that $i < n - i + 1$ for all $1 \leq i \leq k$.) Before Alice's ith move, exactly $2i - 2$ moves have been made, so since this subgraph has $2i$ vertices, it is disconnected.

Take a vertex j such that j and 1 are in different connected components. Since Alice's first move is $(1, n)$, j is also not connected to n. Now, if $j \in \{1, \dots, i\}$, Alice adds (j, n), otherwise Alice adds $(1, j)$. □

Since each of Bob's edges has weight at least 1, his edges in total have weight at least $n-k-1$. Thus, Alice can ensure a total weight of at least $(n-k-1)+(n-k)+\cdots+(n-1)$.

We now give a strategy for Bob.

Claim — Bob can ensure that for each $1 \leq i \leq k$, there are at most i edges of weight at least $n - i$, while only adding edges of weight 1.

Proof. For $i = 1$, this is clear. Consider $i > 1$. As above, consider the induced subgraph on vertices $\{1, \dots, i\} \sqcup \{n - i + 1, \dots, n\}$ before Bob's $(i - 1)$st move.

We show that there exists some edge of weight 1 in this subgraph that Bob can choose. Suppose not. Then, since none of $(1, 2), \dots, (i - 1, i)$ are valid moves, the vertices $1, \dots, i$ are connected. Likewise, the vertices $(n - i + 1, \dots, n)$ are connected.

Hence, there are at least $2(i - 1)$ edges in this subgraph. However, only $2i - 3$ moves have been played up to this point, contradiction. So, Bob can choose an edge of weight 1 in this subgraph.

When n is even, Bob makes $k - 1$ moves, so this accounts for all his moves. When n is odd, on the final move, since not all the vertices are connected, there exists some edge of weight 1 that Bob can add.

Thus, Bob can select only edges of weight 1. Now, following the above strategy, for any $1 \le i \le k$, Bob chooses at least $k-1$ edges in the induced subgraph on vertices $\{1, \dots, i\} \cup \{n-i+1, \dots, n\}$. Since the induced subgraph contains no cycles, Alice plays at most $(2i-1)-(i-1) = i$ edges in this subgraph.

Now, note that any edge of weight at least $n-i$ must have both vertices contained in this subgraph. Thus, there are at most i edges of weight at least $n-i$, as desired. (When $n > 2$, $n-i \ge n-k > 1$, and when $n = 2$, Bob plays no edges, so the bound still holds.) $\square$

Using the above strategy, the total weight of Bob's edges is $n-k-1$. Let a_i be the number of edges Alice plays of weight at least i. Then, the total weight of Alice's edges is

$$\begin{aligned}\sum_{i=1}^{n-1} a_i &\le k(n-k-1) + \sum_{i=n-k}^{n-1} a_i \\ &\le k(n-k-1) + (k + (k-1) + \cdots + 1) \\ &= (n-1) + (n-2) + \cdots + (n-k).\end{aligned}$$

Thus, Bob can ensure a total weight of at most $(n-k-1) + (n-k) + \cdots + (n-1)$.

§3 Solutions to Day 3

§3.1 Solution to TSTST 7, by Luke Robitaille

For a positive real number c, the sequence $a_1, a_2, \dots$ of real numbers is defined as follows. Let $a_1 = c$, and for $n \geq 2$, let

$$a_n = \sum_{i=1}^{n-1} (a_i)^{n-i+1}.$$

Find all positive real numbers c such that $a_i > a_{i+1}$ for all positive integers i.

¶ Solution (author) The answer is $c < \frac{\sqrt{5}-1}{2}$.

To show this is necessary, note that $a_2 = c^2$ and $a_3 = c^3 + c^4$, so if the sequence is decreasing, we have $c^2 > c^3 + c^4$, implying $c < \frac{\sqrt{5}-1}{2}$.

In the other direction, suppose c is a positive real number with $c < \frac{\sqrt{5}-1}{2}$. We will apply strong induction. We have

$$\begin{aligned} a_1 &= c \\ a_2 &= c^2 \\ a_3 &= c^3 + c^4 \end{aligned}$$

so $a_1 > a_2$ (since $c < 1$) and $a_2 > a_3$ (since $c < \frac{\sqrt{5}-1}{2}$). For the inductive step, we'll show $a_n > a_{n+1}$ assuming that $n \geq 3$ and $a_i > a_{i+1}$ for all $i < n$. We have

$$a_1^n = c^n > c^n(c + c^2) > c^n(c + c^n) = a_1^{n+1} + a_2^n.$$

Also, since $a_i > a_{i+1}$ for $i = 2, \dots, n-1$, we have

$$\sum_{i=2}^{n-1} a_i^{n-i+1} > \sum_{i=2}^{n-1} a_{i+1}^{n-i+1} = \sum_{j=3}^{n} a_j^{n-j+2},$$

where the equality comes from shifting indices to take $j = i + 1$. Thus

$$a_n = a_1^n + \sum_{i=2}^{n-1} a_i^{n-i+1} > a_1^{n+1} + a_2^n + \sum_{j=3}^{n} a_j^{n-j+2} = a_{n+1},$$

as desired. This completes the induction step, finishing the solution.

> **Remark** (author)**.** The behavior of the sequence for other values of c is interesting. In particular, computer experiments seem to indicate that, for $c = 0.655736876792$, the sequence starting from a_2 increases to a value barely less than $\frac{1}{2}$ and then decreases, going to 0 as $n \to \infty$, while for $c = 0.655736876793$, the sequence starting from a_2 is increasing and goes to infinity. We conjecture that some form of these patterns hold in general, and that there should exist some value of c with $0.655736876792 < c < 0.655736876793$ so that the sequence starting from a_2 is increasing and $\lim_{n\to\infty} a_n = \frac{1}{2}$.

¶ Solution (Pitchayut) We get $c < \frac{\sqrt{5}-1}{2}$ in the same way as the first solution. To show that any such c gives a strictly decreasing sequence, we use induction on n. The base cases of $n = 1, 2$ are clear. Let $t = \frac{\sqrt{5}-1}{2}$, then $t^2 + t = 1$. For the inductive step, assume $n \geq 3$ and $a_1 > a_2 > \cdots > a_{n-1}$. Since $t > a_1$, t is greater than $a_1, a_2, \ldots, a_{n-1}$. Furthermore, $t^2 > a_1^2 = a_2 \geq a_{n-1}$. Thus

$$\begin{aligned} a_n &= a_1^n + a_2^{n-1} + \cdots + a_{n-1}^2 \\ &< t(a_1^{n-1} + a_2^{n-2} + \cdots + a_{n-2}^2) + t^2 a_{n-1} \\ &= t a_{n-1} + t^2 a_{n-1} \\ &= a_{n-1}, \end{aligned}$$

as desired.

§3.2 Solution to TSTST 8, by Pitchayut Saengrungkongka

Find all polynomials f with integer coefficients such that for all positive integers n,

$$n \text{ divides } \underbrace{f(f(\dots(f(0))\dots)}_{n+1\ f\text{'s}} - 1.$$

There are three families.

- $f(x) = x + 1$.
- $f(x) = x(x-1)g(x) + 1$ for any polynomial $g(x)$ (i.e., any $f(x)$ such that $f(0) = f(1) = 1$).
- $f(x) = x(x-1)(x+1)g(x) + (2x^2 - 1)$ for any polynomial $g(x)$ (i.e., any $f(x)$ such that $f(0) = -1$, $f(-1) = f(1) = 1$).

These all clearly work, so we focus on proving that these are all solutions.

Claim — For any prime p, either

(a) $f(1) \equiv 1 \pmod p$ or

(b) the directed graph of f in $\mathbb{F}_p$ forms a single cycle of size p.

Proof. Work modulo p. Consider the sequence

$$0, f(0), f(f(0)), \dots,$$

which must be eventually periodic. Clearly 1 must be in the periodic part by taking n to be a large multiple of p. Now, note that since the non-periodic part must have size less than p, we have

$$\left.\begin{aligned} f^{p+1}(0) &= 1 \\ f^{2p+1}(0) &= 1 \end{aligned}\right\} \implies f^p(1) = 1,$$

so the period must divide p and hence must be either 1 or p. If it is 1, then $f(1) = 1$. Otherwise, (b) holds. $\square$

Next, we note that if $f(x) - x$ is non-constant, then by Schur's theorem on $f(x) - x$, f has a fixed point modulo infinitely many primes p, so (b) fails for infinitely many primes p. This means that (a) holds for infinitely many primes p, so $f(1) = 1$. Therefore, either $f(x) - x$ is constant or $f(1) = 1$.

In the case that $f(x) - x$ a constant c, $f^{n+1}(0) - 1 = (n+1)c - 1$, so $n \mid c - 1$ for all n and $f(x) = x + 1$. Henceforth, assume $f(1) = 1$.

Since $f(0) \mid f^k(0)$ for all k, plugging in $n = |f(0)|$ gives

$$f(0) \mid f^{|f(0)|+1}(0) - 1 \implies f(0) \mid 1 \implies f(0) = \pm 1.$$

If $f(0) = 1$, then we are done. Otherwise, assume $f(0) = -1$. Then, $f(-1)$ is odd because $f(1) = 1$, so plugging in $n = |f(-1)| = 2k - 1$ gives

$$f(-1) \mid f^{2k}(0) - 1.$$

We have that $f^2(0) = 0$ modulo $f(-1) = f^2(0)$, so $f(-1) \mid 1$. This gives $f(-1) = \pm 1$. If $f(-1) = -1$, then we have $f^n(0) = -1$ for all $n \geq 3$, which makes the divisibility condition fail. Hence, $f(-1) = 1$, and we get the third solution set.

Remark (author)**.** An easier variant is to ask for all polynomials f with integer coefficients such that $n \mid f^n(0)$ for all positive integers n.

§3.3 Solution to TSTST 9, by Ruben Carpenter

Let acute triangle ABC have orthocenter H. Let B_1, C_1, B_2, and C_2 be collinear points which lie on lines AB, AC, BH, and CH, respectively. Let ω_B and ω_C be the circumcircles of triangles BB_1B_2 and CC_1C_2, respectively. Prove that the radical axis of ω_B and ω_C intersects the line through their centers on the nine-point circle of triangle ABC.

¶ Solution (author) The first important step is to introduce N, the circumcenter of HB_2C_2.

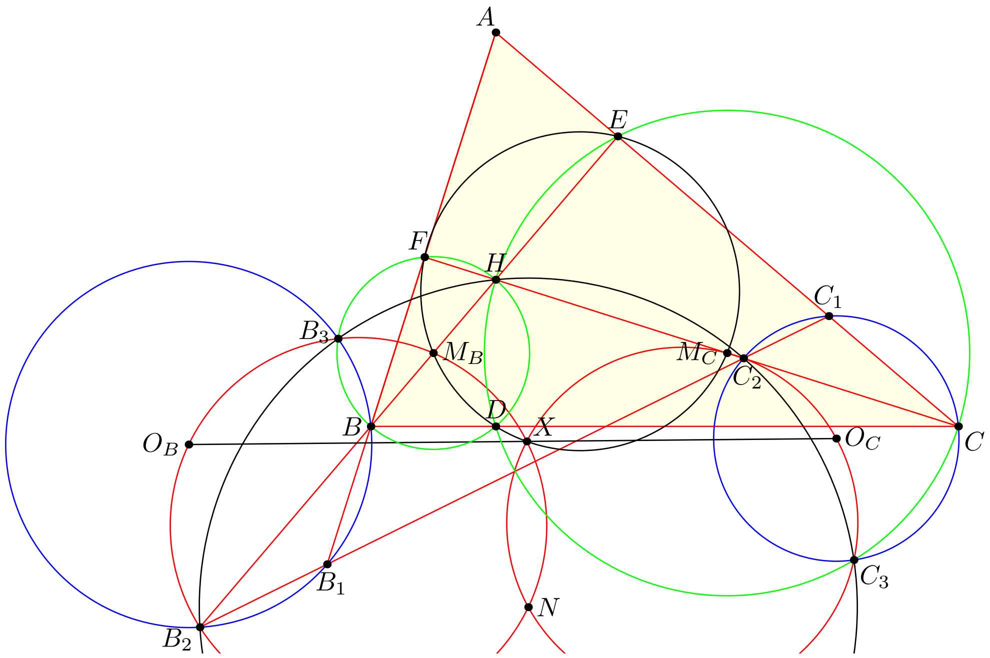
Claim — Lines NB_2 and NC_2 are tangent to (BB_1B_2) and (CC_1C_2), respectively.

Proof. This follows from chasing

$$\measuredangle NB_2B = \measuredangle NB_2H = 90^\circ - \measuredangle HC_2B_2 = \measuredangle B_2B_1B.$$

An analogous proof works for the other side. □

Since $NB_2 = NC_2$, it follows that N lies on the radical axis of (BB_1B_2) and (CC_1C_2). Thus we want to show X, the foot of N onto O_BO_C, lies on the nine-point circle.



Let D, E, F be the feet of the altitudes in $\triangle ABC$, and let M_B, M_C be the midpoints of BH, CH. Let B_3, C_3 be the second intersections of (B_2C_2H) with (BB_1B_2), (CC_1C_2).

Claim — Hexagons $O_BB_2B_3M_BXN$ and $O_CC_2C_3M_CXN$ are cyclic.

Proof. Clearly B_2, B_3 and X lie on the circle of diameter O_BN. To show M_B lies in this circle, we first note that BB_3HFD is cyclic. Indeed, this follows from chasing

$$\measuredangle BB_3H = \measuredangle B_2B_3H - \measuredangle B_2B_3B = \measuredangle B_2C_2H - \measuredangle B_2B_1B = \measuredangle B_1FC_2 = \measuredangle BFH.$$

This circle has center M_B, so O_BM_B and M_BN are the perpendicular bisectors of BB_3, B_3H. Since $BB_3 \perp B_3H$, $O_BM_B \perp M_BN$, as we wanted to show. □

Now we are ready to finish. We know $\measuredangle M_BDM_C = \measuredangle M_CHM_B = \measuredangle BAC$. We will now show that $\measuredangle M_BXM_C$ gives the same value. On one hand

$$\measuredangle M_BXO_B = \measuredangle M_BB_2O_B = \measuredangle BB_2O_B = 90^\circ - \measuredangle B_2B_1B = 90^\circ - \measuredangle C_1B_1A.$$

Similarly $\measuredangle O_CXM_C = 90^\circ - \measuredangle AC_1B_1$, so

$$\measuredangle M_BXM_C = (90^\circ - \measuredangle C_1B_1A) + (90^\circ - \measuredangle AC_1B_1) = \measuredangle BAC,$$

which establishes the result.

> **Remark** (author)**.** Since the problem is symmetric under $A \leftrightarrow H$, a similar solution can be found by considering the circumcenter of $\triangle AB_1C_1$ (instead of N). Moreover, introducing both circumcenters adds more structure to the diagram, which can make it easier to finish.
> The most important step in this solution is to introduce the point N. This can be motivated by noticing that the circles (BB_1B_2) and (HB_2C_2) are orthogonal.

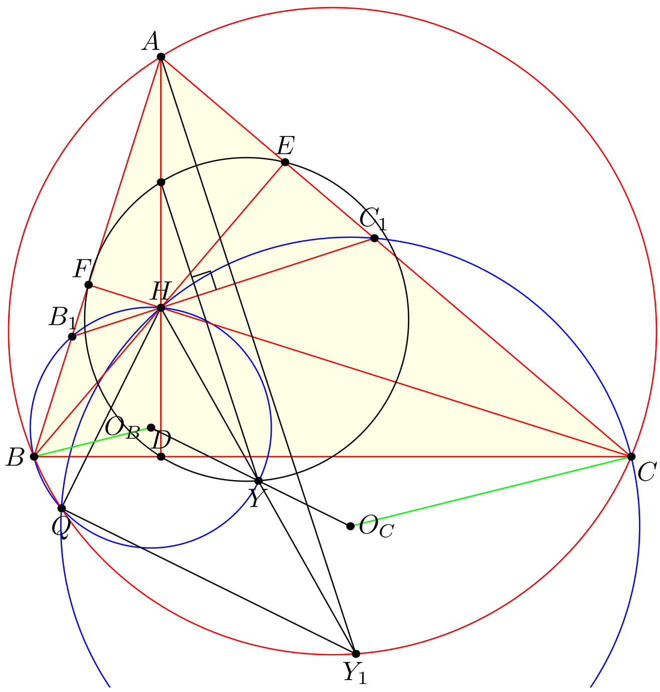
¶ Moving Points Solution (Krishna) Let E, F be the feet of the B, C altitudes, as in the first solution.

> **Claim —** As ℓ varies while staying parallel to a fixed line, $\overline{O_BO_C}$ passes through a fixed point Y on the nine-point circle.

Proof. Move ℓ linearly. Then $\triangle BB_1B_2$ is dilating at B linearly, so O_B moves linearly. Similarly O_C moves linearly. Furthermore, we have that

$$\measuredangle \overline{BO_B} = 90 + \measuredangle BA + \measuredangle BH - \measuredangle \ell = \measuredangle BA + \measuredangle BC - \measuredangle \ell,$$

which by symmetry means $\overline{BO_B} \parallel \overline{CO_C}$. Thus we need to find a point Y on the nine-point circle such that $O_B - O_C - Y$ in 2 cases.



Suppose ℓ passes through H. In that case, let $Q = (BB_1H) \cap (CC_1H)$. We get that Q lies on (ABC) by chasing

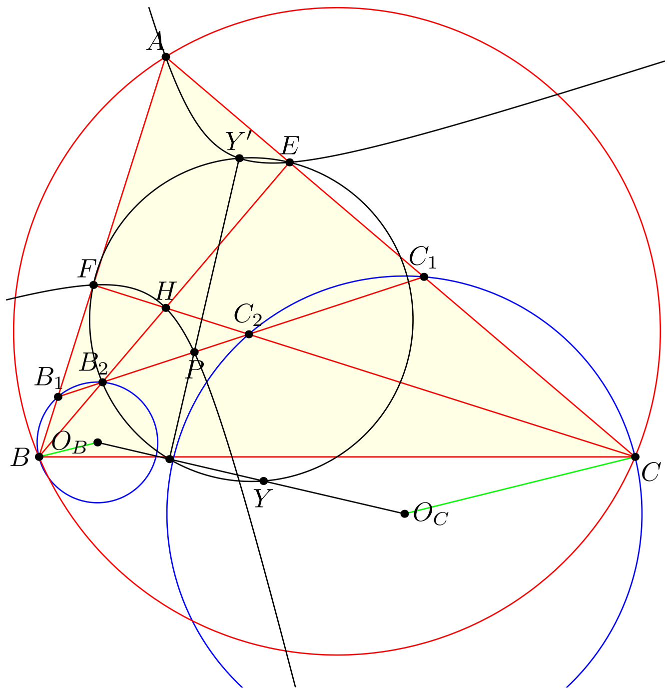
$$\measuredangle BQC = \measuredangle BQH + \measuredangle HQC = \measuredangle BB_1H + \measuredangle HC_1C = \measuredangle ABB_1 + \measuredangle BC_1A = \measuredangle BAC.$$

The perpendicular bisector of $\overline{HQ}$ is $\overline{O_BO_C}$, and the midpoint of HQ lies on the nine-point circle. Let the perpendicular from Q to $\overline{HQ}$ intersect (ABC) at Y_1. Then

$$\measuredangle BAY_1 = \measuredangle BQY_1 = 90 + \measuredangle BQH = 90 + \measuredangle AB_1H,$$

so $\overline{AY_1} \perp \ell$. If Y is the midpoint of HY_1 , then Y lies on $\overline{O_BO_C}$ and the nine-point circle of (ABC). The line between Y and the midpoint of AH is perpendicular to ℓ. Hence when ℓ passes through A, by symmetry we also get $O_B - O_C - Y$, as desired (this angle chase is basically done as part of 2019 TSTST 5). $\square$

Let P_∞ be the point at infinity of ℓ, and let $P = r \cap \ell$. Since $PB_1 \cdot PB_2 = PC_1 \cdot PC_2$ (with signed lengths), we have that $(P, P_\infty), (B_1, B_2), (C_1, C_2)$ are 3 pairs under the same involution on ℓ. Then by DIT on the 4 points A, H, E, F and the line ℓ, if $\mathcal{C}$ is the conic through A, H, E, F, P_∞, then P also lies on $\mathcal{C}$. Now we do moving points.



- Let P_∞ be a sufficiently general fixed point at infinity (proving the problem for sufficiently general P_∞ suffices by continuity.
- Define P as a moving point with degree 2 varying on the conic through A, H, E, F, P_∞.
- As we let P vary and we let $\ell = \overline{PP_\infty}$, we get that $\overline{PP_\infty}$ has degree 1 so B_1 has degree 1. The perpendicular bisector of BB_1 then has degree 1, and $\overline{BO_B}$ is fixed so O_B has degree 1.
- Y has degree 0, and the antipode Y' of Y on the nine-point circle has degree 0. It suffices to show that $\overline{Y'P} \perp \overline{O_BY}$.
- $\overline{Y'P}$ has degree ≤ 2 and $\overline{O_BY}$ has degree 1, so we get a degree 3 statement that needs to be checked at 4 cases.

We can do $P = A, H, E, F$. $P = H$ and $P = A$ are true by previous work. By symmetry, to solve $P = E$ and $P = F$ we only need to look at $P = E$. In this case, we want to show that $E - O_B - O_C$.

$$\measuredangle EO_B = \measuredangle BA + \measuredangle \ell - \measuredangle BH = 90 + \measuredangle CH + \measuredangle \ell - 90 - \measuredangle AC = 90 - 90 + \measuredangle EO_C = \measuredangle EO_C,$$

as desired.

¶ Solution (Vivian Loh) Let $S_B = (BH) \cap (BB_1B_2)$, and $S_C = (CH) \cap (CC_1C_2)$. Then S_B and S_C are the Miquel points of complete quadrilaterals $\{B, B_1, B_2, H, C_2, CH \cap AB\}$ and $\{C, C_1, C_2, H, B_2, BH \cap AC\}$ respectively, so S_B and S_C both lie on (HB_2C_2). Furthermore, by radical center, $P = B_2S_B \cap C_2S_C$ lies on the radical axis of ω_B, ω_C. Also note that if H' is the antipode of H with respect to (HB_2C_2), then S_B and S_C are the projections of H onto BH' and CH', respectively. Note that the 6 points H, B_2, C_2, H', S_B, S_C all lie on (HH').

Claim 3.1 — If O_B and O_C are the centers of ω_B and ω_C respectively, then O_B is the intersection of tangents to (HH') at B_2 and S_B, while O_C is the intersection of tangents to (HH') at C_2 and S_C.

Proof. Angle chasing; $\angle B_2O_BS_B = 2\angle B_2BS_B = 2(90^\circ - \angle BHS_B) = 180^\circ - 2\angle BHS_B$. It works for O_C similarly. $\square$

Claim 3.2 — O_BO_C is the polar of P with respect to (HH').

Proof. By the definition of poles and polars, B_2S_B is the polar of O_B and C_2S_C is the polar of O_C, and by La Hire's, we know that a point lies on the polar of O_B and the polar of O_C if and only if O_B and O_C both lie on its polar, so O_BO_C is the polar of $B_2S_B \cap C_2S_C = P$. $\square$

Since P lies on the radical axis of ω_B and ω_C, the desired point in the problem (the intersection of the radical axis and O_BO_C) is the foot from P to O_BO_C, which is the inverse P^* of P about (HH'). It suffices to show that this point lies on the nine-point circle of $\triangle ABC$.

Let M_B and M_C be the midpoints of HB and HC. Then the nine-point circle of $\triangle ABC$ passes through M_B, M_C, and the foot from H to BC, so P^* lies on this nine-point circle if and only if $\angle M_BP^*M_C = \angle BHC$.

Claim 3.3 — $M_B^* \in B_2S_B$, $M_C^* \in C_2S_C$.

Proof. This is simply angle chasing. We will show that $M_C^* \in C_2S_C$. Let O be the center of (HH'). Then $\angle OM_CH = \angle OHM_C^*$, which equals $\angle OS_CM_C^*$ since $OM_C \perp HS_C$. However, $\angle OS_CC_2 = \angle HS_CC_2 - \angle HS_CO = \angle(HC_2, S_CH') = \angle OM_CH$, so $M_C^* \in C_2S_C$, and M_B^* follows similarly. $\square$

The last part is just angle chasing. We know from before that it is sufficient to show $\angle M_BP^*M_C = \angle BHC$. We have: $\angle M_BP^*M_C = \angle OP^*M_B + \angle OP^*M_C = \angle OM_B^*P + \angle OM_C^*P = 360^\circ - \angle M_B^*OM_C^* - \angle M_B^*PM_C^* = 360^\circ - \angle S_CPS_B - \angle S_CH'S_B = \angle H'S_BP + \angle H'S_CP = \angle B_2HC_2 = \angle BHC$, as desired.