USA TST Selection Test for 67th IMO and 15th EGMO

Aurora, IL

Day I 12:30pm – 5:00pm

Tuesday, June 24, 2025

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

You may keep the problems, but you cannot discuss them publicly until they are posted by staff online.

Problem 1. In a finite group of people, some pairs are friends (friendship is mutual). Each person p has a list $f_1(p), f_2(p), \dots, f_{d(p)}(p)$ of their friends, where $d(p)$ is the number of distinct friends p has. Additionally, any two people are connected by a series of friendships. Each person also has a *water balloon*. The following game is played until someone ends up with more than one water balloon: on round r, each person p throws the current water balloon they have to their friend $f_s(p)$ such that $d(p)$ divides $r - s$. Show that if the game never ends, then everyone has the same number of friends.

Problem 2. Find all sets $S \subseteq \mathbb{Z}$ for which there exists a function $f\colon \mathbb{R} \to \mathbb{Z}$ such that

- $f(x-y) - 2f(x) + f(x+y) \geq -1$ for all $x, y \in \mathbb{R}$, and
- $S = \{f(z) \mid z \in \mathbb{R}\}$.

Problem 3. Let a_1, a_2, r, and s be positive integers with r and s odd. The sequence $a_1, a_2, a_3, \dots$ is defined by

$$a_{n+2} = ra_{n+1} + sa_n$$

for all $n \geq 1$. Determine the maximum possible number of integers $1 \leq \ell \leq 2025$ such that a_ℓ divides $a_{\ell+1}$, over all possible choices of a_1, a_2, r, and s.

USA TST Selection Test for 67[th] IMO and 15[th] EGMO

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Day II 12:30pm – 5:00pm

Thursday, June 26, 2025

Time limit: 4.5 hours. If you need to add page headers after the time limit, you must do so under proctor supervision. Proctors may not answer clarification questions.

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Problem 4. Let $n \geq 2$ be a positive integer. Let a_1, a_2, $\ldots$, a_n be a sequence of positive integers such that

$$\gcd(a_1, a_2),\ \gcd(a_2, a_3),\ \ldots,\ \gcd(a_{n-1}, a_n)$$

is a strictly increasing sequence. Find, in terms of n, the maximum possible value of

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

over all such sequences.

Problem 5. A tetrahedron $ABCD$ is said to be *angelic* if it has nonzero volume and satisfies

$$\angle BAC + \angle CAD + \angle DAB = \angle ABC + \angle CBD + \angle DBA,$$
$$\angle ACB + \angle BCD + \angle DCA = \angle ADB + \angle BDC + \angle CDA.$$

Across all angelic tetrahedrons, what is the maximum number of distinct lengths that could appear in the set $\{AB, AC, AD, BC, BD, CD\}$?

Problem 6. Alice and Bob play a game on n vertices labelled $1, 2, \ldots, n$. They take turns adding edges $\{i, j\}$, with Alice going first. Neither player is allowed to make a move that creates a cycle, and the game ends after $n - 1$ total turns.

Let the weight of the edge $\{i, j\}$ be $|i - j|$, and let W be the total weight of all edges at the end of the game. Alice plays to maximize W and Bob plays to minimize W. If both play optimally, what will W be?

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Day III 12:30pm – 5:00pm

Saturday, June 28, 2025

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Problem 7. For a positive real number c, the sequence $a_1, a_2, \dots$ of real numbers is defined as follows. Let $a_1 = c$, and for $n \geq 2$, let

$$a_n = \sum_{i=1}^{n-1} (a_i)^{n-i+1}.$$

Find all positive real numbers c such that $a_i > a_{i+1}$ for all positive integers i.

Problem 8. Find all polynomials f with integer coefficients such that for all positive integers n,

$$n \text{ divides } \underbrace{f(f(\dots(f(0))\dots)}_{n+1\ f\text{'s}} - 1.$$

Problem 9. Let acute triangle ABC have orthocenter H. Let B_1, C_1, B_2, and C_2 be collinear points which lie on lines AB, AC, BH, and CH, respectively. Let ω_B and ω_C be the circumcircles of triangles BB_1B_2 and CC_1C_2, respectively. Prove that the radical axis of ω_B and ω_C intersects the line through their centers on the nine-point circle of triangle ABC.