Testing for Dark Matter

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Contents

Chapter 1

Introduction

1.1 Prerequisites and Basic Formulae

1.1.1 Mathematical Notation

- $\equiv$ - definition. We will use a lot of example calculations, this symbols means that this is the definition of the thing.
-

1.1.2 General Relativity

- $\eta_{\mu\nu}$ - denotes the Minkowski metric. It has the form $\text{diag}(-1, 1, 1, 1)$ in the *mostly-minus* signature, which we use.
- $g_{\mu\nu}$ - denotes the metric of a Lorentian spacetime.
- $\partial_\mu = \frac{\partial}{\partial x^\mu}$ - denotes a directional derivative. Often we will use shorthand such as V^μ
- Christoffel Connection (${\Gamma^\alpha}_{\mu\nu}$) are a system of numbers which are constructed from the derivatives of the metric,

$$ {\Gamma^\alpha}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu} \right). \tag{1.1.1} $$

  The Christoffel Connection allows us to construct a *covariant derivative*, as well as other covariant properties of manifolds.

Chapter 2

Perturbation Theory in Gravity

2.1 Linearized Gravity

We will consider gravity in the weak-field limit where the variations with time of the metric perturbations are small and we place no restrictions on particle speeds. By *weak* we mean that the metric can be expanded as a linear sum of the Minkowski metric and a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \text{ where } |h_{\mu\nu}| \ll 1, \tag{2.1.1}$$

where $\eta_{\mu\nu}$ takes the canonical form in the *mostly-plus* convention. The assumption that $h_{\mu\nu}$ is small allows us to ignore terms of $\mathcal{O}(h^2, h\partial h, h^{-1}\partial h...)$. The inverse metric is written,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2.1.2}$$

which can be shown by multiplying (2.1.1) and (2.1.2), and ignoring $\mathcal{O}(h^2)$ terms, does indeed give a Kronecker delta. Furthermore, we use the $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to raise and lower indices respectively. We can think of this whole limit as a linearized version of GR describing the theory of *symmetric tensor field*, $h_{\mu\nu}$ propagating on a flat background spacetime. This theory is manifestly Lorentz invariant,

$$h_{\mu'\nu'} = \Lambda_{\mu'}^{\ \mu} \, \Lambda_{\nu'}^{\ \nu} \, h_{\mu\nu}. \tag{2.1.3}$$

To note, we could have chosen a curved background, however, it would introduce an additional level of complexity which we should just avoid for now.

We want the equations of motion for this perturbation, $h_{\mu\nu}$. We begin with the Christoffel symbols up to linear order in $h_{\mu\nu}$,

$$\begin{aligned}\Gamma^{\rho}{}_{\mu\nu} &\equiv \frac{1}{2}g^{\rho\lambda}\left(g_{\mu\lambda,\nu}+g_{\nu\lambda,\mu}-g_{\mu\nu,\lambda}\right)\\ &= \frac{1}{2}\eta^{\rho\lambda}(\partial_\nu h_{\mu\lambda}+\partial_\mu h_{\nu\lambda}-\partial_\lambda h_{\mu\nu}).\end{aligned} \tag{2.1.4}$$

Next, we calculate the Riemann tensor, and note that the Γ^2 terms can be ignored (because they contain terms terms quadratic in $(\partial h)^2$),

$$\begin{aligned}R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho\Gamma^{\lambda}{}_{\nu\sigma}-\eta_{\mu\lambda}\partial_\sigma\Gamma^{\lambda}{}_{\nu\rho}\\ &= \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma}+\partial_\sigma\partial_\mu h_{\nu\rho}-\partial_\rho\partial_\mu h_{\nu\sigma}-\partial_\sigma\partial_\nu h_{\mu\rho}\right).\end{aligned} \tag{2.1.5}$$

This leads to the Ricci tensor, $R_{\mu\nu}=R^{\sigma}{}_{\mu\sigma\nu}$,

$$R_{\mu\nu}=\frac{1}{2}\left(\partial_\nu\partial^\sigma h_{\sigma\mu}+\partial_\mu\partial^\sigma h_{\sigma\nu}-\Box h_{\mu\nu}-\partial_\mu\partial_\nu h\right), \tag{2.1.6}$$

where the $h=\eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation metric and $\Box=-\partial_t^2+\nabla^2$, is the D'Alembertian operator on the flat background spacetime. Finally, we obtain the Ricci scalar, R,

$$\begin{aligned}R &= \frac{1}{2}\eta^{\mu\nu}\left(\partial_\nu\partial^\sigma h_{\sigma\mu}+\partial_\mu\partial^\sigma h_{\sigma\nu}-\Box h_{\mu\nu}-\partial_\mu\partial_\nu h\right)\\ &= \partial_\mu\partial_\nu h^{\mu\nu}-\Box h.\end{aligned} \tag{2.1.7}$$

We jump back to the Physics and obtain the Einstein Tensor, $G_{\mu\nu}$,

$$\begin{aligned}G_{\mu\nu} &\equiv R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\\ &= \frac{1}{2}\left(\partial_\nu\partial^\sigma h_{\sigma\mu}+\partial_\mu\partial^\sigma h_{\sigma\nu}-\Box h_{\mu\nu}-\partial_\mu\partial_\nu h\right)-\frac{1}{2}\eta_{\mu\nu}\left(\partial_\mu\partial_\nu h^{\mu\nu}-\Box h\right)\\ &= \partial^\sigma\partial_{(\mu}h_{\nu)\sigma}-\frac{1}{2}\left(\Box h_{\mu\nu}+\partial_\mu\partial_\nu h+\eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma}+\eta_{\mu\nu}\Box h\right)\end{aligned} \tag{2.1.8}$$

2.2 Dynamical Degrees of Freedom

With this linearised tensor we could immediately fix a gauge and solve the EFEs. However, we can accumulate some additional physics insights by first choosing a fixed inertial coordinate system in the Minkowski background spacetime and decomposing components of the metric perturbation according to their transformation properties under spatial rotations.

2.2.1 Degrees of freedom in EM

We break coordinate invariance for convenience, just as with $\vec{E}$ and $\vec{B}$ components of the 2-form Electromagnetic field strength tensor, $F_{\mu\nu} = (\mathrm{d}A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where we define a one-form potential $A_\mu = \left(\phi, \vec{A}\right)$, where ϕ is the *electrostatic potential* and $\vec{A}$ is the *magnetic vector potential*. Combinations of derivatives of components of A_μ produce our physical fields,

$$\vec{E} \equiv -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \tag{2.2.1}$$

$$\vec{B} \equiv \vec{\nabla} \times \vec{A}. \tag{2.2.2}$$

A particle of charge q moving through an electromagnetic field with velocity $\mathbf{v}$ obeys the Lorentz force law,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right). \tag{2.2.3}$$

In a vacuum (source-free Maxwell) the dynamics of the theory are generated purely by the following action,

$$S[A_\mu] = -\frac{1}{4}\int d^4x \; F^{\mu\nu}F_{\mu\nu}, \tag{2.2.4}$$

which leads to the following equation of motion,

$$\partial_\rho F^{\rho\mu} = \Box A^\mu - \partial^\mu(\partial_\rho A^\rho) = 0. \tag{2.2.5}$$

But, we make note of the gauge freedom $A'_\mu = A_\mu + \partial_\mu \Lambda(x)$, where $\Lambda(x)$ is some scalar function. Vector potentials that differ by a gradient of a scalar are said to be equivalent, $A'_\mu \sim A_\mu$ and $\sim$ is an equivalence relation. We could go further, however, at this step it is customary to fix a particular *gauge*.

Coulomb Gauge

The **Coulomb gauge** (also known as the transverse gauge) is a popular gauge fixing condition. Coulomb gauge states,

$$\nabla \cdot \vec{A} = 0 \tag{2.2.6}$$

which forces the spatial degrees of freedom to have vanishing divergence. It's definitely an interesting choice of gauge, however, the condition itself breaks Lorentz Invariance, so we could care less about it.

Lorenz Gauge

On the other hand, the **Lorenz gauge** is a Lorentz invariant and the condition is written,

$$\partial_\mu A^\mu = 0. \tag{2.2.7}$$

Watch what this does to the gauge freedom equation,

$$\partial_\mu A'^\mu = \partial_\mu A^\mu + \Box\Lambda(x) = 0. \tag{2.2.8}$$

But also, we require that $\partial_\mu A^\mu = 0$, therefore, we have an equation of motion for the $\Lambda(x)$ degree of freedom,

$$\Box\Lambda(x) = 0. \tag{2.2.9}$$

Which is the equation of motion for a massless spin-0 field. What about (2.2.5)? We see that the second term contains a $\partial_\rho A^\rho$, which we have chosen to vanish, therefore,

$$\Box A^\mu = 0. \tag{2.2.10}$$

This is an interesting equation, because it's four spin-0 fields propagating independently. We're almost there, however, we have two too many degrees of freedom. Electromagnetic radiation carries two transverse polarisation vectors. Luckily, the gauge condition can help us here.

Solving equations in Lorenz gauge

Consider the plane wave solution,

$$A^\mu = \epsilon^\mu e^{-ik\cdot x}, \tag{2.2.11}$$

then, from $\Box A^{\mu}$, we have $k^2 = 0$, a null wavevector. Further, from the Lorenz Gauge condition, we have $\partial_{\mu} A^{\mu} \rightarrow \epsilon \cdot k = 0$, which means that the polarisation vector is normal to the propagation direction. From arguments of gauge choice, we have been able to accurately reason and describe the nature of electromagnetic radiation.

Now we choose a particular frame in which light is propagating along the z-direction. The natural choice from the $k^2 = 0$ condition for k^{μ} is,

$$k^{\mu} = (k\ 0\ 0\ k)^T. \tag{2.2.12}$$

The polarisation vector ϵ^{μ} can be ascertained as,

$$\epsilon^{\mu} = (\alpha\ e^1\ e^2\ \alpha)^T \tag{2.2.13}$$

Now, we from $\epsilon \cdot k = 0$

2.3 Back to linearized gravity

Subsequent to beating this dead horse of an analogy, we move back to the case at hand of linearized gravity. The metric perturbations $h_{\mu\nu}$ is a symmetric tensor, as opposed to the antisymmetric $F_{\mu\nu}$. We can decompose this tensor into its constituent degrees of freedom. In particular, the h_{00} component is a scalar, the h_{0i} components form a vector in $\mathbb{R}^3$ and the spatial h_{ij} components for a two index symmetric spatial tensor. The tensor is further reducible into *trace* and *traceless* parts. In mathematical form, we usually choose,

$$\begin{aligned} h_{00} &= -2\Phi && (2.3.1)\\ h_{0i} &= w_i && (2.3.2)\\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij} && (2.3.3) \end{aligned}$$

where Ψ contains information about the trace of h_{ij} and s_{ij} is traceless. The trace of h_{ij} is the contraction of h_{ij} with the metric on $\mathbb{R}^3$, δ^{ij},

$$\begin{aligned} \delta^{ij} h_{ij} &= 2\delta^{ij} s_{ij} - 2\Psi\delta^{ij}\delta_{ij} = -6\Psi \\ \Psi &= -\frac{1}{6}\delta^{ij} h_{ij} && (2.3.4) \end{aligned}$$

which leads to the form of s_{ij},

$$s_{ij} = \frac{1}{2}\left[h_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}h_{kl}\right]. \tag{2.3.5}$$

The full metric $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ is,

$$\mathrm{d}s^2 = -(1+2\Phi)\mathrm{d}t^2 + w_i(\mathrm{d}t\mathrm{d}x^i + \mathrm{d}x^i\mathrm{d}t) + [(1-2\Psi)\delta_{ij} + 2s_{ij}]\mathrm{d}x^i\mathrm{d}x^j. \tag{2.3.6}$$

As of yet, we have performed only convenient window-dressing, we still have to choose the gauge and solve equations. But not in vain. This choice shows will make apparent s_{ij} containing information about *gravitational radiation*. But we have fixed an inertial frame for convenience, to express $p^\mu = dx^\mu/d\lambda$ ($\lambda = \tau/m$ if the particle is massive). As usual,

$$p^0 = \frac{dt}{d\lambda} = E \quad \text{and} \quad p^i = Ev^i \tag{2.3.7}$$

Consider the geodesic equation,

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu{}_{\rho\sigma}p^\rho p^\sigma = 0. \tag{2.3.8}$$

We can write this in the following form,

$$\frac{dp^\mu}{dt} = -\Gamma^\mu{}_{\rho\sigma}\frac{p^\rho p^\sigma}{E} \tag{2.3.9}$$

The $\mu = 0$ term describes the rate of change in energy over time (power),

$$\frac{dE}{dt} = -E\left[\partial_0\Phi + 2(\partial_k\Phi)v^k - \left(\partial_{(j}w_{k)} - \frac{1}{2}\partial_0 h_{jk}\right)v^j v^k\right] \tag{2.3.10}$$

The spatial component $\mu = i$ of the geodesic equation becomes,

$$\frac{dp^i}{dt} = -E\left[\partial_i\Phi + \partial_0 w_i + 2\left(\partial_{[i}w_{j]} + \partial_0 h_{ij}\right)v^j + \left(\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right)v^j v^k\right]. \tag{2.3.11}$$

To interpret this physically, it is convenient to define the *gravitoelectric* and *gravito-*

magnetic vector fields,

$$\vec{G} = -\nabla\Phi - \frac{\partial\vec{w}}{\partial t}, \tag{2.3.12}$$
$$\vec{H} = \nabla\times\vec{w} \tag{2.3.13}$$

There is a direct parallel between these equations and (2.2.1). The net force acting on a particle is given by,

$$\frac{dp^i}{dt} = E\left(G^i + (v\times H)^i - 2(\partial_0 h_{ij})v^j + \left(\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right)v^j v^k\right) \tag{2.3.14}$$

This is analogous to the Lorentz Force Law (2.2.3) in EM. We should also examine the evolution of the linearized Einstein Field Equations. The Riemann tensor components are,

$$R_{0j0l} = \partial_j\partial_l\Phi + \partial_0\partial_{(j}w_{l)} - \frac{1}{2}\partial_0\partial_0 h_{jl} \tag{2.3.15}$$
$$R_{0jkl} = \partial_j\partial_{[k}w_{l]} - \partial_0\partial_{[k}h_{l]j} \tag{2.3.16}$$
$$R_{ijkl} = \partial_j\partial_{[k}h_{l]i} - \partial_i\partial_{[k}h_{l]j}. \tag{2.3.17}$$

The Ricci tensor components are,

$$R_{00} = \nabla^2\Phi + \partial_0\partial_k w^k + 3\partial_0^2\Psi \tag{2.3.18}$$
$$R_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_k\Psi + \partial_0\partial_l s^k_j \tag{2.3.19}$$
$$R_{ij} = -\partial_i\partial_j(\Phi-\Psi) - \partial_0\partial_{(i}w_{j)} + (\Box\Psi)\delta_{ij} + \Box s_{ij} + 2\partial_k\partial_{(i}s^k_{j)} \tag{2.3.20}$$

Finally, we can calculate Einstein tensor,

$$G_{00} = 2\nabla^2\Psi + \partial_k\partial_l s^{kl} \tag{2.3.21}$$
$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s^k_j \tag{2.3.22}$$
$$G_{ij} = (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi-\Psi) + \delta_{ij}\partial_0\partial_k w^k - \partial_0\partial_{(i}w_{j)}\ldots\text{finish this off.} \tag{2.3.23}$$

As the EFEs are give by $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, we can write the Einstein tensor in terms of matter content, the 00 equation is,

$$\nabla^2\Psi = 4\pi G T_{00} - \frac{1}{2}\partial_i\partial_j s^{ij} \tag{2.3.24}$$

which is an equation for which there are no time derivatives of Ψ, T_{00} and s_{ij} are sufficient to express Ψ up to boundary condition. As it completely depends on other variables, Ψ is not a propagating degree of freedom. Next, the $0i$ contributions,

$$\left(\delta_{jk}\nabla^2 - \partial_j\partial_k\right) w^k = -16\pi G T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s_j{}^k \tag{2.3.25}$$

where there are also no time derivatives of w^i which means the components of the three vector are also not physical degrees of freedom.

Finally, the ij component equation gives,

$$\begin{aligned}\left(\delta_{ij}\nabla^2 - \partial_i\partial_j\right)\Phi &= 8\pi G T_{ij} + (\delta_{ij}\nabla^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2)\Psi \\ &\quad -\delta_{ij}\partial_0\partial_k w^k + \partial_0\partial_{(i}w_{j)} + \Box s_{ij} - 2\partial_k\partial_{(i}s_{j)}{}^k \\ &\quad -\delta_{ij}\partial_k\partial_l s^{kl},\end{aligned} \tag{2.3.26}$$

where once again, we have no time derivative of Φ, therefore, it is not a propagating degree of freedom. Propagating tensor fields under quantisation give rise to particles of different spins. Despite the fact that some of the fields in this analysis were not physical degrees of freedom, in some alternative theories of gravity, it is possible for these **residual gauge modes** to become physical. In this case, Ψ and Φ would have spin-0, w^i would have spin-1 and s_{ij} would have spin-2. The only physical degree of freedom in regular GR is the **strain tensor**, s_{ij} transforms under a spin-2 representation. The gauge transformation on the perturbation is described as the following,

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

The infinitesimal change in $h_{\mu\nu}$ can be written in terms of the Lie derivative of the Minkowski metric,

$$h_{\mu\nu}^{(\varepsilon)} = h_{\mu\nu} + \varepsilon\mathcal{L}_\xi\eta_{\mu\nu} \tag{2.3.27}$$

We will set $\varepsilon = 1$ and think of ξ itself as small. Under this transformation, the fields of the perturbation will transform as,

$$
\begin{aligned}
\Phi &\rightarrow \Phi + \partial_0 \xi^0 && (2.3.28)\\
w_i &\rightarrow w_i + \partial_0 \xi_i - \partial_i \xi^0 && (2.3.29)\\
\Psi &\rightarrow \Psi - \frac{1}{3}\partial_i \xi^i && (2.3.30)\\
s_{ij} &\rightarrow s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k \xi^k \delta_{ij} && (2.3.31)
\end{aligned}
$$

Now we shall discuss some gauge choices.

2.3.1 Choices of gauge

Transverse gauge (Coulomb-like gauge)

As in the Electromagnetism analogy, we choose a gauge which sets the divergence of the strain tensor to zero (analogous to the Coulomb gauge),

$$\partial_i s^{ij} = 0. \tag{2.3.1}$$

This can be imposed from (2.3.31) by choosing,

$$\nabla^2 \xi^j + \frac{1}{3}\partial_j(\partial_i \xi^i) = -2\partial_i s^{ij} \tag{2.3.2}$$

But the value of ξ^0 is still undetermined, but we can impose a condition on it by choosing,

$$\partial_i w^i = 0. \tag{2.3.3}$$

Similarly, we can use (2.3.29) to allow this,

$$\nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i \tag{2.3.4}$$

It is called the transverse gauge because, in momentum space, the *polarisations* of these degrees of freedom are orthogonal to the propagation direction. If we expand $\partial_i w^i$ in terms of Fourier modes, we have

$$\partial_i w^i = i \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}[k_i \omega^i(k)]e^{ikx} = 0 \tag{2.3.5}$$

which leads to the condition that $\vec{k}\cdot\vec{\omega}=0$, where $\vec{\omega}$ is the set of *polarisation* degrees of freedom of $\vec{w}$. Similarly, $k_i\sigma^{ij}(k)=0$, where σ^{ij} is the polarisation tensor for s_{ij}.

Under this gauge choice, the Einsten Equations become,

$$\begin{aligned}
8\pi G T_{00} &= 2\nabla^2\Psi && (2.3.6)\\
8\pi G T_{0j} &= \frac{1}{2}\left[4\partial_0\partial_j\Psi-\nabla^2 w_j\right] && (2.3.7)\\
8\pi G T_{ij} &= \left[\delta_{ij}\nabla^2-\partial_i\partial_j\right](\Phi-\Psi)-\partial_0\partial_{(i}w_{j)}+2\delta_{ij}\partial_0^2\Psi-\Box s_{ij}. && (2.3.8)
\end{aligned}$$

which are considerably simpler than the equations before.

Synchronous Gauge

Another choice of gauge is the *synchronous gauge*, it can be thought of the gravitational analogue of the temporal gauge in Electromagnetism (vanishing electrostatic potential $A^0=0$), since this removes the non-spatial components of the perturbation metric. We begin by setting $\Phi=0$, which can be achieved by setting,

$$\partial_0\xi^0=-\Phi. \tag{2.3.9}$$

Furthermore, we are free to choose ξ^i by setting the vector degree of freedom to zero, $w^i=0$, this is achieved by making the choice,

$$\partial_0\xi^i=-w^i+\partial_i\xi^0. \tag{2.3.10}$$

The metric in this synchronous gauge takes on a simple and attractive form,

$$ds^2=-dt^2+(\delta_{ij}+h_{ij})dx^i dx^j$$

The Einstein Equations in the synchronous gauge are,

$$\begin{aligned}
8\pi G T_{00} &= \partial_i\partial_j s^{ij}+2\nabla^2\Psi && (2.3.11)\\
8\pi G T_{0j} &= 2\partial_0\partial_j\Psi+\partial_0\partial_k s_j{}^k && (2.3.12)\\
8\pi G T_{ij} &= 2\partial_k\partial_{(i}s_{j)}{}^k+\delta_{ij}\partial_k\partial_l s^{kl}-\Box s_{ij}\\
&\quad+\left(2\delta_{ij}\partial_0^2+\partial_i\partial_j-\delta_{ij}\nabla^2\right)\Psi. && (2.3.13)
\end{aligned}$$

These are equations of the spatial parts of the metric perturbation h_{ij}.

2.4 Newtonian Fields and Photon Trajectories

We will extend the definition of the Newtonian limit. Relativistic particles respond to spatial components of the metric as well. We can model static gravitating sources by dust, a perfect fluid for which the pressure vanishes. Most of the matter in the universe is well approximated by dust, stars, planets, galaxies and dust. We work in the rest frame of the dust, where the energy-momentum tensor takes the form, $T_{\mu\nu} = \rho U_\mu U_\nu$.

Since the background is Minkowski space, it is straightforward to accomodate moving sources by performing a Lorentz transformation into their rest frame. We are unable to deal with multiple sources moving at large relativistic velocities. Turning to the Einstein equations in the transverse gauge and static sources (time derivatives vanishing), we have,

$$\begin{aligned} \nabla^2\Psi &= 4\pi G\rho \\ \nabla^2 w_j &= 0 \\ \nabla^2 s_{ij} - (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) &= 0 \end{aligned} \tag{2.4.1}$$

Since we are looking for both non-singular and well-behaved solutions at infinity, only the fields that are sourced at the right-hand side of (2.4.1) will be non-vanishing. For example, immediately by taking the trace of the third equation, we obtain,

$$2\nabla^2(\Phi - \Psi) = 0, \tag{2.4.2}$$

which enforces the equality between the scalar modes, $\Phi = \Psi$. If this is the case, we can see that the scalar potential obeys the Poisson equation, as is the case for the usual Newtonian limit. We can further simplify the third equation,

$$\nabla^2 s_{ij} = 0, \tag{2.4.3}$$

which implies that $s_{ij} = 0$ because there is no source. The perturbed metric in the Newtonian limit is simply,

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2). \tag{2.4.4}$$

Now we consider the path of a photon through this geometry. Essentially, we are solving the null geodesic equations for the *perturbed metric*.

Chapter 3

Alternative Theories of Gravity

3.1 Scalar-Tensor Theories

(Heavily borrowed from Carroll Chapter 4) Despite the continual success of General Relativity in describing gravitational interactions, it is always possible that the next experiment we perform could show considerable deviations and reveal effects that we cannot describe using it. In principle, there is an infinite range of modifications that could be made to GR, however, there are a few that we direct special attention to. For later comparisons and a reminder the usual Einstein-Hilbert action in $(1+3)$-dimensional Lorentzian spacetime, $\mathcal{M}$ is,

$$S_R = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R \tag{3.1.1}$$

where, R is the Ricci scalar, G is the Gravitational Constant and $g = \det(g_{\mu\nu})$ is the metric determinant of our spacetime. This leads to the usual Einstein Field Equations in the presence of matter,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.1.2}$$

A popular set of theories is known as **scalar-tensor theories** since they contain a metric tensor, $g_{\mu\nu}$, and a scalar field φ [**?**]. The action of these theories can be written as the following sum,

$$S = S_{fR} + S_\varphi + S_M, \tag{3.1.3}$$

where,

$$S_{fR} = \int d^4x\sqrt{-g}f(\varphi)R, \tag{3.1.4}$$

$$S_\varphi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\mathfrak{h}(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi)\right], \tag{3.1.5}$$

and

$$S_M = \int d^4x\sqrt{-g}\hat{\mathcal{L}}_M(g_{\mu\nu}, \psi_i). \tag{3.1.6}$$

The functions $f(\varphi)$, $\mathfrak{h}(\varphi)$ and $U(\varphi)$ define the theory we are interested in and the matter Lagrangian $\hat{\mathcal{L}}_M$ is a function of the metric and the matter fields ψ_i but not of φ. By the usual variations of the inverse metric $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, we can obtain the Field Equations of this system. Variations with respect to the inverse metric can be used to obtain the variations with respect to the metric using the following conversion,

$$\begin{aligned} \delta(g^{\mu\sigma}g_{\nu\sigma}) &= \delta g^{\mu\sigma}g_{\nu\sigma} + g^{\mu\sigma}\delta g_{\nu\sigma} = 0 \\ \delta g_{\mu\nu} &= -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} \end{aligned} \tag{3.1.7}$$

Focusing on the gravitational contributions for the moment, from the Leibnitz product rule, we have,

$$\delta S_{fR} = \delta S_1 + \delta S_2 + \delta S_3 \tag{3.1.8}$$

where,

$$\delta S_1 = \int d^4x\sqrt{-g}f(\varphi)g^{\mu\nu}\delta R_{\mu\nu} \tag{3.1.9}$$

$$\delta S_2 = \int d^4x\sqrt{-g}f(\varphi)R_{\mu\nu}\delta g^{\mu\nu} \tag{3.1.10}$$

$$\delta S_3 = \int d^4x\sqrt{-g}f(\varphi)R\delta\sqrt{-g} \tag{3.1.11}$$

We see that δS_2 is in the form that we require, however, δS_1 and δS_3 require some extra care. Starting with δS_1, we use the Riemann tensor,

$$R^\rho{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^\rho{}_{\nu\mu} + \Gamma^\rho{}_{\lambda\sigma}\Gamma^\sigma{}_{\nu\mu} - \partial_\nu\Gamma^\rho{}_{\lambda\mu} - \Gamma^\rho{}_{\nu\sigma}\Gamma^\sigma{}_{\lambda\mu} \tag{3.1.12}$$

Following steps... The variations of the metric lead to a gnarly integral,

$$\delta S_{fR} = \int d^4x\sqrt{-g}f(\varphi)\left[\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \nabla_\sigma\nabla^\sigma(g_{\mu\nu}\delta g^{\mu\nu}) - \nabla_\mu\nabla_\nu(\delta g^{\mu\nu})\right] \tag{3.1.13}$$

For the usual Einstein-Hilbert action, $f(\varphi)$ is a constant which means the last two terms are total derivatives, which we can integrate by parts. This will furnish the action with terms involving derivatives of $f(\varphi)$,

$$\delta S_{fR} = \int d^4x\sqrt{-g}\left[G_{\mu\nu}f(\varphi) + g_{\mu\nu}\Box f(\varphi) - \nabla_\mu\nabla_\nu f(\varphi)\right]\delta g^{\mu\nu} \tag{3.1.14}$$

with the usual $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. The Field equations are then given by,

$$G_{\mu\nu} = \frac{1}{f(\varphi)}\left(\frac{1}{2}T^{(M)}_{\mu\nu} + \frac{1}{2}T^{(\varphi)}_{\mu\nu} + \nabla_\mu\nabla_\nu f - g_{\mu\nu}\Box f\right) \tag{3.1.15}$$

wherein the energy-momentum tensors are defined in the standard sense in GR,

$$T^{(I)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_I}{\delta g^{\mu\nu}}, \tag{3.1.16}$$

where the index I denotes the matter or scalar parts of the actions. In particular,

$$T^{(\varphi)}_{\mu\nu} = \mathfrak{h}(\varphi)\nabla_\mu\varphi\nabla_\nu\varphi - g_{\mu\nu}\left[\frac{1}{2}\mathfrak{h}(\varphi)g^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi + U(\varphi)\right] \tag{3.1.17}$$

Looking closely at (3.1.15) and (3.1.2), we can identify $f(\varphi) = 1/16\pi G$, in the Einstein-Hilbert case. This identification gives us the usual GR and so is nothing new. However, this identifies a gravity-like theory, but with a strength that varies with the value of the scalar φ at different spacetime positions. This places a bound on how large the scalar field amplitude can be on cosmological scale and the scale of the Solar System (because it would be obviously observable... and we can't observe it). We can determine the equations of motion for φ,

$$\mathfrak{h}(\varphi)\Box\varphi + \frac{1}{2}\mathfrak{h}'(\varphi)g^{\mu\nu}(\nabla_\mu\varphi)(\nabla_\nu\varphi) - U'(\varphi) + f'(\varphi)R = 0 \tag{3.1.18}$$

where the prime $'$ denotes differentiation with respect to φ. If $\mathfrak{h}(\varphi) = 1$ the conventional scalar field equations are obtained,

$$\Box\varphi + f'(\varphi)R = U'(\varphi) \tag{3.1.19}$$

but with this curious Ricci scalar coupling. From the previous discussion, we can see that with little variations in $f(\varphi)$ this new term can be dropped to obtain the usual dynamics of φ. Another mechanism to lessen this effect is to choose a potential with a minimum and ensuring that φ cannot deviate from this minimum without a sufficient energy input. In other words, a very large mass for φ. Take a potential with $U(0) = U'(0) = 0$,

$$U(\varphi) \;\approx\; \frac{1}{2}m_\varphi^2\varphi^2 + ... \tag{3.1.20}$$

where we identify $m_\varphi^2 = U''(0) > 0$ and $m_\varphi^2 \gg 1$. Or we could choose f and $\mathfrak{h}$ so that large changes in φ give rise to relatively small changes in the effective changes in G.

3.2 Brans-Dicke Theory

A famous example of a scalar-tensor theory is **Brans-Dicke theory**, and corresponds to the following choices,

$$f(\varphi) = \frac{\varphi}{16\pi}, \qquad \mathfrak{h}(\varphi) = \frac{\lambda}{8\pi\varphi}, \qquad U(\varphi) = 0, \tag{3.2.1}$$

where ξ is a coupling constant. The scalar-tensor action is written as,

$$S_{\mathrm{BD}} = \int d^4x \frac{\sqrt{-g}}{16\pi}\left[R\varphi - \lambda g^{\mu\nu}\frac{(\partial_\mu\varphi)(\partial_\nu\varphi)}{\varphi}\right]. \tag{3.2.2}$$

Brans-Dicke theory chooses a massless scalar. But in the limit $\lambda \to \infty$ the field becomes non-dynamical and ordinary GR is recovered. Current tests in the solar system imply $\lambda > 500$, of if there is such a scalar field it must couple only weakly to R.

3.2.1 Using Conformal Transformations

One way to deal with scalar-tensor theories is to perform a conformal transformation to make the connection to gravity more apparent. We choose the conformal metric,

$\tilde{g}_{\mu\nu}$,

$$\tilde{g}_{\mu\nu} = 16\pi\tilde{G}f(\varphi)g_{\mu\nu} \tag{3.2.1}$$

where $\tilde{G}$ is the Gravitational constant in the conformal frame. The connection coefficients in this new frame ($\tilde{\Gamma}^{\mu}{}_{\rho\sigma}$) are,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\sigma}\left(\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}\right) \tag{3.2.2}$$

Substituting the conformal metric (3.2.1) in terms of the usual metric give,

$$\begin{aligned}
\tilde{\Gamma}^{\rho}{}_{\mu\nu} &= \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}f(\varphi)^{-1}\big((\partial_\nu f)\delta^\rho_\mu + (\partial_\mu f)\delta^\rho_\nu - g^{\rho\sigma}g_{\mu\nu}(\partial_\sigma f)\big) && (3.2.3)\\
&= \Gamma^{\rho}{}_{\mu\nu} + C^{\rho}{}_{\mu\nu}, && (3.2.4)
\end{aligned}$$

where $C^{\rho}{}_{\mu\nu}$ is the difference between connections and so is a tensor (see Appendix C). Using the chain rule, $\partial_\mu f(\varphi) = (\partial f/\partial\varphi)\partial_\mu\varphi$, we can write,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\delta^\rho_\mu + (\partial_\mu\varphi)\delta^\rho_\nu - g^{\rho\sigma}g_{\mu\nu}(\partial_\sigma\varphi)\right] \tag{3.2.5}$$

This connection defines a new covariant derivative on a vector field, $V = V^\mu(x)\partial_\mu$,

$$\begin{aligned}
\tilde{\nabla}_\mu V^\rho &= \partial_\mu V^\rho + \tilde{\Gamma}^{\rho}{}_{\mu\nu}V^\nu,\\
&= \partial_\mu V^\rho + \Gamma^{\rho}{}_{\mu\nu}V^\nu + \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(V^\nu\partial_\nu\varphi)\delta^\rho_\mu + (\partial_\mu\varphi)V^\rho - g^{\rho\sigma}(\partial_\sigma\varphi)V_\mu\right]
\end{aligned}$$

and on a covector field, $\omega = \omega_\mu(x)dx^\mu$,

$$\begin{aligned}
\tilde{\nabla}_\mu\omega_\nu &= \partial_\mu\omega_\nu - \tilde{\Gamma}^{\rho}{}_{\mu\nu}\omega_\rho,\\
&= \partial_\mu\omega_\nu - \Gamma^{\rho}{}_{\mu\nu}\omega_\rho - \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\omega_\mu + (\partial_\mu\varphi)\omega_\nu - g_{\mu\nu}(\partial_\sigma\varphi)\omega^\sigma\right]
\end{aligned}$$

which generalises to higher-order tensors in the usual manner. Continuing with the general calculations of the Riemann and Ricci tensors, we obtain the following action in terms of the conformal Ricci Scalar,

$$S_{fR} = \int d^4x\,\frac{\sqrt{-g}}{16\pi\tilde{G}}\left[\tilde{R} + \frac{3}{2}\tilde{g}^{\rho\sigma}f^{-2}\left(\frac{df}{d\varphi}\right)^2(\tilde{\nabla}_\rho\varphi)(\tilde{\nabla}_\sigma\varphi)\right] \tag{3.2.6}$$

where we have integrated by parts and discarded surface terms. In the conformal frame, therefore, the curvature scalar appears by itself, which looks a lot like GR. For this reason, this frame is called the **Einstein frame**, since the Einstein equations for the conformal metric $\tilde{g}_{\mu\nu}$ take on their conventional form. The original frame with metric $g_{\mu\nu}$ is called the **Jordan frame**.

If we make the choice,

$$f(\varphi) = e^{\varphi/\sqrt{3}}, \quad \mathfrak{h}(\varphi) = U(\varphi) = 0 \tag{3.2.7}$$

which is a specific choice of $f(\varphi)$ and turns off the S_φ contribution from the pure scalar. With these choices, in the Einstein frame, we see a conventional kinetic term for the scalar field,

$$S = \int d^4x \sqrt{-\tilde{g}} (16\pi\tilde{G})^{-1} \left[\tilde{R} - \frac{1}{2} \tilde{\nabla}_\rho \varphi \tilde{\nabla}^\rho \varphi \right] + S_M. \tag{3.2.8}$$

Even without an explicit kinetic term in the Jordan frame, we see a kinetic term generated in the Einstein frame.

Appendices

Appendix A

Representation Theory $SU(2)$ for spin-2 systems

A.1 Invariant G-modules

Consider a vector space V as an invariant left G-module which transforms under the fundamental representation of $G \cong SU(2)$. V^* will denoted the conjugate of V. Through tensor products of these vector spaces, we can construct higher dimensional vector spaces which will transform with higher dimensional representations of $SU(2)$. We can find the dimensions of these representations using *Young tableu*.

A.2 Young tableu

The *defining representation* will be denoted by a box: $\square$. The dimension of the V in the defining representation of $SU(2)$ is 2. Taking the tensor product of V with itself, produces higher dimensional Young tableu. Suppose the vector space, $W = V \otimes V$, has two invariant subspaces.

Appendix B

Conformal Transformations

A conformal transformation is a transformation between metric spaces which we choose an arbitrary symmetric conformal metric, $\tilde{g}_{\mu\nu}$,

$$\tilde{g}_{\mu\nu} = 16\pi \tilde{G} f(\varphi) g_{\mu\nu} \tag{B.0.1}$$

where $\tilde{G}$ is the Gravitational constant in the conformal frame. The connection coefficients in this new frame ($\tilde{\Gamma}^{\mu}{}_{\rho\sigma}$) are,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\sigma}\left(\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}\right) \tag{B.0.2}$$

Substituting the conformal metric (B.0.1) in terms of the usual metric give,

$$\begin{aligned}
\tilde{\Gamma}^{\rho}{}_{\mu\nu} &= \frac{1}{2} f(\varphi)^{-1} g^{\rho\sigma}\left(\partial_\nu(f g_{\mu\sigma}) + \partial_\mu(f g_{\nu\sigma}) - \partial_\sigma(f g_{\mu\nu})\right) && \text{(B.0.3)}\\
&= \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2} f(\varphi)^{-1} g^{\rho\sigma}\left((\partial_\nu f) g_{\mu\sigma} + (\partial_\mu f) g_{\nu\sigma} - (\partial_\sigma f) g_{\mu\nu}\right) && \text{(B.0.4)}\\
&= \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2} f(\varphi)^{-1}\big((\partial_\nu f)\delta^\rho_\mu + (\partial_\mu f)\delta^\rho_\nu - g^{\rho\sigma} g_{\mu\nu}(\partial_\sigma f)\big) && \text{(B.0.5)}
\end{aligned}$$

Using the chain rule, $\partial_\mu f(\varphi) = (\partial f/\partial\varphi)\partial_\mu\varphi$, we can write,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}\frac{\partial \ln[f(\varphi)]}{\partial\varphi}\big[(\partial_\nu\varphi)\delta^\rho_\mu + (\partial_\mu\varphi)\delta^\rho_\nu - g^{\rho\sigma} g_{\mu\nu}(\partial_\sigma\varphi)\big] \tag{B.0.6}$$

This connection defines a new covariant derivative on a vector field, $A = A^\mu(x)\partial_\mu$,

$$\begin{aligned}\tilde{\nabla}_\mu A^\rho &= \partial_\mu A^\rho + \tilde{\Gamma}^\rho{}_{\mu\nu} A^\nu, \\ &= \partial_\mu A^\rho + \Gamma^\rho{}_{\mu\nu} A^\nu + \frac{1}{2}\frac{\partial \ln[f(\varphi)]}{\partial\varphi}\left[(A^\nu \partial_\nu\varphi)\delta^\rho_\mu + (\partial_\mu\varphi)A^\rho - g^{\rho\sigma}A_\mu(\partial_\sigma\varphi)\right]\end{aligned}$$

and on a covector field, $\omega = \omega_\mu(x)dx^\mu$,

$$\begin{aligned}\tilde{\nabla}_\mu \omega_\nu &= \partial_\mu \omega_\nu - \tilde{\Gamma}^\rho{}_{\mu\nu}\omega_\rho, \\ &= \partial_\mu \omega_\nu - \Gamma^\rho{}_{\mu\nu}\omega_\rho - \frac{1}{2}\frac{\partial \ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\omega_\mu + (\partial_\mu\varphi)\omega_\nu - g_{\mu\nu}(\partial_\sigma\varphi)\omega^\sigma\right]\end{aligned}$$

which generalises to higher-order tensors.

Appendix C

Fifth forces in alternative theories of gravity

C.1 Dynamics in conformal metrics

Consider a metric in the Jordan Frame,

$$g_{\mu\nu} = A(\varphi)^2 \tilde{g}_{\mu\nu} \tag{C.1.1}$$

in a scalar-tensor theory $(\tilde{g}_{\mu\nu}, \varphi)$. $A(\varphi)$ is usually give by,

$$A(\varphi) = 1 + \frac{\varphi^2}{2M^2} + \mathcal{O}\left(\frac{\varphi^4}{M^4}\right), \tag{C.1.2}$$

which is an effective field theory [**?**]. The geodesics of this theory are those of the Jordan frame metric. A massive particle will move on some affinely-parametrised geodesic,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0, \tag{C.1.3}$$

where $\Gamma^\mu{}_{\rho\sigma}$ defines a Levi-Civita connection of $g_{\mu\nu}$, where $\frac{dx^\mu}{d\tau}$ are tangent vectors of the geodesics in the Jordan Frame. It is important to note that the analogous equation for the Einstein frame metric, $\tilde{g}_{\mu\nu}$, deviates from the geodesics

$$a^\mu_E = \frac{d^2x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \neq 0. \tag{C.1.4}$$

where a_E^μ is an acceleration of a massive particle in the Einstein frame. This is because the tangent vectors are generated from the geodesics in the Jordan frame and so are not necessarily geodesics in the Einstein frame. Our goal is to calculate the Γ-coefficients of Jordan Frame in terms of $\tilde{g}_{\mu\nu}$ given the above considerations.

C.2 Conformal transformations on Connections

We begin by substituting the Jordan frame metric into the standard definition of the Christoffel Symbols.

$$\Gamma^\mu{}_{\rho\sigma} = \frac{1}{2} A^{-2} \tilde{g}^{\mu\nu} \left[(A^2 \tilde{g}_{\rho\nu})_{,\sigma} + (A^2 \tilde{g}_{\nu\sigma})_{,\rho} - (A^2 \tilde{g}_{\rho\sigma})_{,\nu} \right] \tag{C.2.1}$$

Expanding this out and noticing that $\tilde{\Gamma} \sim \tilde{g}^{-1} \partial \tilde{g}$, we can write,

$$\Gamma^\mu{}_{\rho\sigma} = \tilde{\Gamma}^\mu{}_{\rho\sigma} + A^{-1} \left[\delta^\mu_\rho (\partial_\sigma A) + \delta^\mu_\sigma (\partial_\rho A) - \tilde{g}_{\rho\sigma} \tilde{g}^{\mu\nu} (\partial_\nu A) \right]. \tag{C.2.2}$$

Since the second term is the difference between two connections, we can identify it as $(2,1)$ tensor, $C^\mu{}_{\rho\sigma}$. As $A(\varphi)$ is only explicitly a function of φ, the chain rule can be used to convert this expression to the following expression for $C^\mu{}_{\rho\sigma}$,

$$\begin{aligned} C^\mu{}_{\rho\sigma} &= A^{-1} \frac{\partial A}{\partial \varphi} \left[\delta^\mu_\rho (\partial_\sigma \varphi) + \delta^\mu_\sigma (\partial_\rho \varphi) - \tilde{g}_{\rho\sigma} \tilde{g}^{\mu\nu} (\partial_\nu \varphi) \right] \\ &= \frac{\partial (\ln A(\varphi))}{\partial \varphi} \left[\delta^\mu_\rho (\partial_\sigma \varphi) + \delta^\mu_\sigma (\partial_\rho \varphi) - \tilde{g}_{\rho\sigma} \tilde{g}^{\mu\nu} (\partial_\nu \varphi) \right] \end{aligned} \tag{C.2.3}$$

This is a tensor because we can be expressed completely covariantly,

$$C^\mu{}_{\rho\sigma} = A^{-1} \frac{\partial A}{\partial \varphi} \left[\delta^\mu_\rho (\nabla_\sigma \varphi) + \delta^\mu_\sigma (\nabla_\rho \varphi) - \tilde{g}_{\rho\sigma} \tilde{g}^{\mu\nu} (\nabla_\nu \varphi) \right], \tag{C.2.4}$$

because $\nabla_\mu \varphi \equiv \partial_\mu \varphi$.

This can be proven in terms of general coordinate transformations:

The difference between two Levi-Civita connections $C^{\mu}{}_{\rho\sigma} = \Gamma^{\mu}{}_{\rho\sigma} - \tilde{\Gamma}^{\mu}{}_{\rho\sigma}$ is a tensor under general coordinate transformations.

Proof: Consider a matrix coordinate transformation x and x',

$$M : x^{\mu} \to x^{\mu'}.$$

A general Levi-Civita connection will transform as follows,

$$\Gamma^{\mu}{}_{\rho\sigma} \to \Gamma^{\mu'}{}_{\rho'\sigma'} = \frac{1}{2} g^{\mu'\nu'} \left(g_{\rho'\nu',\sigma'} + g_{\nu'\sigma',\rho'} - g_{\rho'\sigma',\nu'} \right) \tag{C.2.5}$$

The component terms of the above connection only differ in index placement, a particular term will transform as follows:

$$\begin{aligned} \partial_{\mu} g_{\rho\sigma} \to \partial_{\mu'} g_{\rho'\sigma'} &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\rho\nu} \right) \\ &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\rho}}{\partial x^{\rho'}} \right) g_{\rho\nu} + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu}}{\partial x^{\nu'}} \right) g_{\rho\nu} \\ &\quad + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \partial_{\mu} g_{\rho\nu} \end{aligned}$$

The geodesic equation is now,

$$\frac{d^2 x^{\mu}}{d\tau^2} + \tilde{\Gamma}^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} + C^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = 0 \tag{C.2.6}$$

For which we identify the first two terms as a^{μ}_{E},

$$a^{\mu}_{E} = -C^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = F^{\mu} \tag{C.2.7}$$

where the equality is the covariant force vector for a unit-mass particle. Identifying $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ we get the following force on this test particle is given by,

$$F^{\mu} = A^{-1} \frac{\partial A}{\partial \varphi} \left[\delta^{\mu}_{\rho} (\nabla_{\sigma} \varphi) + \delta^{\mu}_{\sigma} (\nabla_{\rho} \varphi) - \tilde{g}_{\rho\sigma} \tilde{g}^{\mu\nu} (\nabla_{\nu} \varphi) \right] u^{\rho} u^{\sigma} \tag{C.2.8}$$

$$= A^{-1} \frac{\partial A}{\partial \varphi} \left[2u^{\mu} (u^{\alpha} \nabla_{\alpha} \varphi) - \tilde{g}^{\mu\nu} (\nabla_{\nu} \varphi) u^{\alpha} u_{\alpha} \right] \tag{C.2.9}$$

C.3 Considering forces in Local Inertial Frames

Consider the expression for the force on the particle in a local inertial frame. This amounts to appropriate substitutions to use this relations, $\lim_{\mathrm{LIF}} \tilde{g}_{\mu\nu} \to \eta_{\mu\nu}$ and $\nabla_\lambda \to \partial_\lambda$,

$$F^\mu = -\frac{\partial(\ln A(\varphi))}{\partial\varphi}\left[2u^\mu(u^\alpha\partial_\alpha\varphi) - \eta^{\mu\nu}(\partial_\nu\varphi)u^\sigma u_\sigma\right]. \tag{C.3.1}$$

where $u_\alpha = \eta_{\alpha\beta}u^\beta$ now. Since u^μ is a time-like vector, in the mostly-plus Lorentzian signature, $u^\mu u_\mu < 0$ we can write the force equation in the inertial frame of the test particle using a suitable Lorentz transformation such that $u^\mu = (1, \vec{0})$,

$$F^\mu = \frac{\partial(\ln A(\varphi))}{\partial\varphi}\left[2u^\mu(u^\alpha\partial_\alpha\varphi) + \eta^{\mu\nu}(\partial_\nu\varphi)\right]. \tag{C.3.2}$$

Choosing now to separate these into equations, for spacelike and timelike components $(\Pi, \vec{F}_5)$ now gives us,

$$\vec{F}_5 = -\frac{\partial(\ln A)}{\partial\varphi}\vec{\nabla}\varphi = -\vec{\nabla}\left(\ln A\right) \tag{C.3.3}$$

$$\Pi = \frac{\partial(\ln A)}{\partial\varphi}\dot{\varphi} \tag{C.3.4}$$

where Π is the power generated by the test particle accelerated by $\vec{F}_5$, which could be of interest. This is a derivation of the fifth force, $\vec{F}_5$ generated in the Einstein Frame.

C.4 Calculating the Fifth Force

We see that we can calculate the non-relativistic force on the test particle, all that needs to be assigned is the form of $A(\varphi)$. In this particular theory we use,

$$A(\varphi) = 1 + \frac{\varphi^2}{2M^2} \tag{C.4.1}$$

with higher order terms ignored. We see that $\ln(A) = \ln(1 + \varphi^2/2M^2)$, where for $\frac{\varphi}{M} \ll 1$, we can expand in powers of $\frac{\varphi}{M}$

$$\ln A(\varphi) \approx \frac{\varphi^2}{2M^2} + ... \tag{C.4.2}$$

which derives a new fifth force,

$$\vec{F}_5 \approx -\frac{\varphi \vec{\nabla} \varphi}{M^2}. \tag{C.4.3}$$