Testing for Dark Matter

Usama Aqeel

February 2024

Contents

1 Introduction **2**

2 Alternative Theories of Gravity **3**

2.1 Scalar-Tensor Theories 3

2.2 Brans-Dicke Theory 6

2.2.1 Using Conformal Transformations 6

Appendices **9**

Appendix A Conformal Transformations **10**

Appendix B Fifth forces in alternative theories of gravity **12**

B.1 Dynamics in conformal metrics 12

B.2 Conformal transformations on Connections 13

B.3 Considering forces in Local Inertial Frames 15

B.4 Calculating the Fifth Force 15

Chapter 1

Introduction

Something clever

Chapter 2

Alternative Theories of Gravity

2.1 Scalar-Tensor Theories

(Heavily borrowed from Carroll Chapter 4) Despite the continual success of General Relativity in describing gravitational interactions, it is always possible that the next experiment we perform could show considerable deviations and reveal effects that we cannot describe using it. In principle, there is an infinite range of modifications that could be made to GR, however, there are a few that we direct special attention to. For later comparisons and a reminder the usual Einstein-Hilbert action in $(1+3)$-dimensional Lorentzian spacetime, $\mathcal{M}$ is,

$$S_R = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}R \tag{2.1.1}$$

where, R is the Ricci scalar, G is the Gravitational Constant and $g = \det(g_{\mu\nu})$ is the metric determinant of our spacetime. This leads to the usual Einstein Field Equations in the presence of matter,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.1.2}$$

A popular set of theories is known as **scalar-tensor theories** since they contain a metric tensor, $g_{\mu\nu}$, and a scalar field φ [**?**]. The action of these theories can be written as the following sum,

$$S = S_{fR} + S_{\varphi} + S_M, \tag{2.1.3}$$

where,

$$
\begin{aligned}
S_{fR} &= \int d^4x\sqrt{-g}f(\varphi)R, && (2.1.4)\\
S_{\varphi} &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}h(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi)\right], && (2.1.5)
\end{aligned}
$$

and

$$S_M = \int d^4x\sqrt{-g}\hat{\mathcal{L}}_M(g_{\mu\nu},\psi_i). \tag{2.1.6}$$

The functions $f(\varphi)$, $h(\varphi)$ and $U(\varphi)$ define the theory we are interested in and the matter Lagrangian $\hat{\mathcal{L}}_M$ is a function of the metric and the matter fields ψ_i but not of φ. By the usual variations of the inverse metric $g^{\mu\nu} \to g^{\mu\nu} + \delta g^{\mu\nu}$, we can obtain the Field Equations of this system. Variations with respect to the inverse metric can be used to obtain the variations with respect to the metric using the following conversion,

$$
\begin{aligned}
\delta(g^{\mu\sigma}g_{\nu\sigma}) &= \delta g^{\mu\sigma}g_{\nu\sigma} + g^{\mu\sigma}\delta g_{\nu\sigma} = 0\\
\delta g_{\mu\nu} &= -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} && (2.1.7)
\end{aligned}
$$

Focusing on the gravitational contributions for the moment, from the Leibnitz product rule, we have,

$$\delta S_{fR} = \delta S_1 + \delta S_2 + \delta S_3 \tag{2.1.8}$$

where,

$$\delta S_1 = \int d^4x\sqrt{-g}f(\varphi)g^{\mu\nu}\delta R_{\mu\nu} \tag{2.1.9}$$

$$\delta S_2 = \int d^4x\sqrt{-g}f(\varphi)R_{\mu\nu}\delta g^{\mu\nu} \tag{2.1.10}$$

$$\delta S_3 = \int d^4x\sqrt{-g}f(\varphi)R\delta\sqrt{-g} \tag{2.1.11}$$

We see that δS_2 is in the form that we require, however, δS_1 and δS_3 require some extra care. Starting with δS_1, we use the Riemann tensor,

$$R^{\rho}{}_{\mu\lambda\nu} = \partial_\lambda\Gamma^{\rho}{}_{\nu\mu} + \Gamma^{\rho}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\mu} - \partial_\nu\Gamma^{\rho}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\lambda\mu} \tag{2.1.12}$$

Following steps... The variations of the metric lead a gnarly integral,

$$\delta S_{fR} = \int \sqrt{-g} f(\varphi) \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g^{\mu\nu} + \nabla_\sigma \nabla^\sigma (g_{\mu\nu} \delta g^{\mu\nu}) - \nabla_\mu \nabla_\nu (\delta g^{\mu\nu}) \right] \tag{2.1.13}$$

For the usual Einstein-Hilbert action, $f(\varphi)$ is a constant which means the last two terms are total derivatives, which we can integrate by parts. This will furnish the action with terms involving derivatives of $f(\varphi)$,

$$\delta S_{fR} = \int d^4x \sqrt{-g} \left[G_{\mu\nu} f(\varphi) + g_{\mu\nu} \Box f(\varphi) - \nabla_\mu \nabla_\nu f(\varphi) \right] \delta g^{\mu\nu} \tag{2.1.14}$$

with the usual $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. The Field equations are then given by,

$$G_{\mu\nu} = \frac{1}{f(\varphi)} \left(\frac{1}{2} T^{(M)}_{\mu\nu} + \frac{1}{2} T^{(\varphi)}_{\mu\nu} + \nabla_\mu \nabla_\nu f - g_{\mu\nu} \Box f \right) \tag{2.1.15}$$

wherein the energy-momentum tensors are defined in the standard sense in GR,

$$T^{(I)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_I}{\delta g^{\mu\nu}}, \tag{2.1.16}$$

where the index I denotes the matter or scalar parts of the actions. In particular,

$$T^{(\varphi)}_{\mu\nu} = h(\varphi) \nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} \left[\frac{1}{2} h(\varphi) g^{\rho\sigma} \nabla_\rho \varphi \nabla_\sigma \varphi + U(\varphi) \right] \tag{2.1.17}$$

Looking closely at (2.1.15) and (2.1.2), we can identify $f(\varphi) = 1/16\pi G$, in the Einstein-Hilbert case. This identification gives us the usual GR and so is nothing new. However, this identifies a gravity-like theory, but with a strength that varies with the value of the scalar φ at different spacetime positions. This places a bound on how large the scalar field amplitude can be on cosmological scale and the scale of the Solar System (because it would be obviously observable... and we can't observe it). We can determine the equations of motion for φ,

$$h(\varphi) \Box \varphi + \frac{1}{2} h'(\varphi) g^{\mu\nu} (\nabla_\mu \varphi)(\nabla_\nu \varphi) - U'(\varphi) + f'(\varphi) R = 0 \tag{2.1.18}$$

where the prime $'$ denotes differentiation with respect to φ. If $h(\varphi) = 1$ the conventional scalar field equations are obtained,

$$\Box\varphi + f'(\varphi)R = U'(\varphi) \tag{2.1.19}$$

but with this curious Ricci scalar coupling. From the previous discussion, we can see that with little variations in $f(\varphi)$ this new term can be dropped to obtain the usual dynamics of φ. Another mechanism to lessen this effect is to choose a potential with a minimum and ensuring that φ cannot deviate from this minimum without a sufficient energy input. In other words, a very large mass for φ. Take a potential with $U(0) = U'(0) = 0$,

$$U(\varphi) \;\approx\; \frac{1}{2}m_\varphi^2\varphi^2 + ... \tag{2.1.20}$$

where we identify $m_\varphi^2 = U''(0) > 0$ and $m_\varphi^2 \gg 1$. Or we could choose f and h so that large changes in φ give rise to relatively small changes in the effective changes in G.

2.2 Brans-Dicke Theory

A famous example of a scalar-tensor theory is **Brans-Dicke theory**, and corresponds to the following choices,

$$f(\varphi) = \frac{\varphi}{16\pi}, \qquad h(\varphi) = \frac{\xi}{8\pi\varphi}, \qquad U(\varphi) = 0, \tag{2.2.1}$$

where ξ is a coupling constant. The scalar-tensor action is written as,

$$S_{BD} = \int d^4x\sqrt{-g}\left[\frac{\varphi}{16\pi}R - \frac{\xi}{16\pi}g^{\mu\nu}\frac{(\partial_\mu\varphi)(\partial_\nu\varphi)}{\varphi}\right]. \tag{2.2.2}$$

Brans-Dicke theory chooses a massless scalar. But in the limit $\xi \to \infty$ the field becomes non-dynamical and ordinary GR is recovered. Current tests in the solar system imply $\xi > 500$, of if there is such a scalar field it must couple only weakly to R.

2.2.1 Using Conformal Transformations

One way to deal with scalar-tensor theories is to perform a conformal transformation to make the connection to gravity more apparent. We choose the conformal metric,

$\tilde{g}_{\mu\nu}$,

$$\tilde{g}_{\mu\nu} = 16\pi\tilde{G}f(\varphi)g_{\mu\nu} \tag{2.2.1}$$

where $\tilde{G}$ is the Gravitational constant in the conformal frame. The connection coefficients in this new frame ($\tilde{\Gamma}^{\mu}{}_{\rho\sigma}$) are,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\sigma}\left(\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}\right) \tag{2.2.2}$$

Substituting the conformal metric (2.2.1) in terms of the usual metric give,

$$\begin{aligned}
\tilde{\Gamma}^{\rho}{}_{\mu\nu} &= \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}f(\varphi)^{-1}\big((\partial_\nu f)\delta^{\rho}_{\mu} + (\partial_\mu f)\delta^{\rho}_{\nu} - g^{\rho\sigma}g_{\mu\nu}(\partial_\sigma f)\big) && (2.2.3)\\
&= \Gamma^{\rho}{}_{\mu\nu} + C^{\rho}{}_{\mu\nu}, && (2.2.4)
\end{aligned}$$

where $C^{\rho}{}_{\mu\nu}$ is the difference between connections and so is a tensor (see Appendix B). Using the chain rule, $\partial_\mu f(\varphi) = (\partial f/\partial\varphi)\partial_\mu\varphi$, we can write,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\delta^{\rho}_{\mu} + (\partial_\mu\varphi)\delta^{\rho}_{\nu} - g^{\rho\sigma}g_{\mu\nu}(\partial_\sigma\varphi)\right] \tag{2.2.5}$$

This connection defines a new covariant derivative on a vector field, $A = A^{\mu}(x)\partial_\mu$,

$$\begin{aligned}
\tilde{\nabla}_\mu A^{\rho} &= \partial_\mu A^{\rho} + \tilde{\Gamma}^{\rho}{}_{\mu\nu}A^{\nu},\\
&= \partial_\mu A^{\rho} + \Gamma^{\rho}{}_{\mu\nu}A^{\nu} + \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(A^{\nu}\partial_\nu\varphi)\delta^{\rho}_{\mu} + (\partial_\mu\varphi)A^{\rho} - g^{\rho\sigma}A_\mu(\partial_\sigma\varphi)\right]
\end{aligned}$$

and on a covector field, $\omega = \omega_\mu(x)dx^{\mu}$,

$$\begin{aligned}
\tilde{\nabla}_\mu\omega_\nu &= \partial_\mu\omega_\nu - \tilde{\Gamma}^{\rho}{}_{\mu\nu}\omega_\rho,\\
&= \partial_\mu\omega_\nu - \Gamma^{\rho}{}_{\mu\nu}\omega_\rho - \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\omega_\mu + (\partial_\mu\varphi)\omega_\nu - g_{\mu\nu}(\partial_\sigma\varphi)\omega^{\sigma}\right]
\end{aligned}$$

which generalises to higher-order tensors. Continuing with the general calculations of the Riemann and Ricci tensors, we obtain the following action in terms of the conformal Ricci Scalar,

$$S_{fR} = \int d^4x\,\frac{\sqrt{-g}}{16\pi\tilde{G}}\left[\tilde{R} + \frac{3}{2}\tilde{g}^{\rho\sigma}f^{-2}\left(\frac{df}{d\varphi}\right)^2(\tilde{\nabla}_\rho\varphi)(\tilde{\nabla}_\sigma\varphi)\right] \tag{2.2.6}$$

where we have integrated by parts and discarded surface terms. In the conformal frame, therefore, the curvature scalar appears by itself, which looks a lot like GR. For this reason, this frame is called the **Einstein frame**, since the Einstein equations for the conformal metric $\tilde{g}_{\mu\nu}$ take on their conventional form. The original frame with metric $g_{\mu\nu}$ is called the **Jordan frame**.

Appendices

Appendix A

Conformal Transformations

A conformal transformation is a transformation between metric spaces which We choose the conformal metric, $\tilde{g}_{\mu\nu}$,

$$\tilde{g}_{\mu\nu} = 16\pi\tilde{G}f(\varphi)g_{\mu\nu} \tag{A.0.1}$$

where $\tilde{G}$ is the Gravitational constant in the conformal frame. The connection coefficients in this new frame ($\tilde{\Gamma}^{\mu}{}_{\rho\sigma}$) are,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \frac{1}{2}\tilde{g}^{\rho\sigma}\left(\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}\right) \tag{A.0.2}$$

Substituting the conformal metric (A.0.1) in terms of the usual metric give,

$$\begin{aligned}
\tilde{\Gamma}^{\rho}{}_{\mu\nu} &= \frac{1}{2}f(\varphi)^{-1}g^{\rho\sigma}\left(\partial_\nu(fg_{\mu\sigma}) + \partial_\mu(fg_{\nu\sigma}) - \partial_\sigma(fg_{\mu\nu})\right) && \text{(A.0.3)}\\
&= \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}f(\varphi)^{-1}g^{\rho\sigma}\left((\partial_\nu f)g_{\mu\sigma} + (\partial_\mu f)g_{\nu\sigma} - (\partial_\sigma f)g_{\mu\nu}\right) && \text{(A.0.4)}\\
&= \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}f(\varphi)^{-1}\big((\partial_\nu f)\delta^{\rho}_{\mu} + (\partial_\mu f)\delta^{\rho}_{\nu} - g^{\rho\sigma}g_{\mu\nu}(\partial_\sigma f)\big) && \text{(A.0.5)}
\end{aligned}$$

Using the chain rule, $\partial_\mu f(\varphi) = (\partial f/\partial\varphi)\partial_\mu\varphi$, we can write,

$$\tilde{\Gamma}^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\mu\nu} + \frac{1}{2}\frac{\partial\ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\delta^{\rho}_{\mu} + (\partial_\mu\varphi)\delta^{\rho}_{\nu} - g^{\rho\sigma}g_{\mu\nu}(\partial_\sigma\varphi)\right] \tag{A.0.6}$$

This connection defines a new covariant derivative on a vector field, $A = A^\mu(x)\partial_\mu$,

$$\begin{aligned}\tilde{\nabla}_\mu A^\rho &= \partial_\mu A^\rho + \tilde{\Gamma}^\rho{}_{\mu\nu} A^\nu, \\ &= \partial_\mu A^\rho + \Gamma^\rho{}_{\mu\nu} A^\nu + \frac{1}{2}\frac{\partial \ln[f(\varphi)]}{\partial\varphi}\left[(A^\nu \partial_\nu\varphi)\delta^\rho_\mu + (\partial_\mu\varphi)A^\rho - g^{\rho\sigma}A_\mu(\partial_\sigma\varphi)\right]\end{aligned}$$

and on a covector field, $\omega = \omega_\mu(x)dx^\mu$,

$$\begin{aligned}\tilde{\nabla}_\mu \omega_\nu &= \partial_\mu \omega_\nu - \tilde{\Gamma}^\rho{}_{\mu\nu}\omega_\rho, \\ &= \partial_\mu \omega_\nu - \Gamma^\rho{}_{\mu\nu}\omega_\rho - \frac{1}{2}\frac{\partial \ln[f(\varphi)]}{\partial\varphi}\left[(\partial_\nu\varphi)\omega_\mu + (\partial_\mu\varphi)\omega_\nu - g_{\mu\nu}(\partial_\sigma\varphi)\omega^\sigma\right]\end{aligned}$$

which generalises to higher-order tensors.

Appendix B

Fifth forces in alternative theories of gravity

B.1 Dynamics in conformal metrics

Consider a metric in the Jordan Frame,

$$g_{\mu\nu} = A(\varphi)^2 \tilde{g}_{\mu\nu} \tag{B.1.1}$$

in a scalar-tensor theory $(\tilde{g}_{\mu\nu}, \varphi)$. $A(\varphi)$ is usually give by,

$$A(\varphi) = 1 + \frac{\varphi^2}{2M^2} + \mathcal{O}\left(\frac{\varphi^4}{M^4}\right), \tag{B.1.2}$$

which is an effective field theory [**?**]. The geodesics of this theory are those of the Jordan frame metric. A massive particle will move on some affinely-parametrised geodesic,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0, \tag{B.1.3}$$

where $\Gamma^\mu{}_{\rho\sigma}$ defines a Levi-Civita connection of $g_{\mu\nu}$, where $\frac{dx^\mu}{d\tau}$ are tangent vectors of the geodesics in the Jordan Frame. It is important to note that the analogous equation for the Einstein frame metric, $\tilde{g}_{\mu\nu}$, deviates from the geodesics

$$a_E^\mu = \frac{d^2x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \neq 0. \tag{B.1.4}$$

where a_E^μ is an acceleration of a massive particle in the Einstein frame. This is because the tangent vectors are generated from the geodesics in the Jordan frame and so are not necessarily geodesics in the Einstein frame. Our goal is to calculate the Γ-coefficients of Jordan Frame in terms of $\tilde{g}_{\mu\nu}$ given the above considerations.

B.2 Conformal transformations on Connections

We begin by substituting the Jordan frame metric into the standard definition of the Christoffel Symbols.

$$\Gamma^\mu{}_{\rho\sigma} = \frac{1}{2}A^{-2}\tilde{g}^{\mu\nu}\left[(A^2\tilde{g}_{\rho\nu}),_\sigma + (A^2\tilde{g}_{\nu\sigma}),_\rho - (A^2\tilde{g}_{\rho\sigma}),_\nu\right] \tag{B.2.1}$$

Expanding this out and noticing that $\tilde{\Gamma} \sim \tilde{g}^{-1}\partial\tilde{g}$, we can write,

$$\Gamma^\mu{}_{\rho\sigma} = \tilde{\Gamma}^\mu{}_{\rho\sigma} + A^{-1}\left[\delta^\mu_\rho(\partial_\sigma A) + \delta^\mu_\sigma(\partial_\rho A) - \tilde{g}_{\rho\sigma}\tilde{g}^{\mu\nu}(\partial_\nu A)\right]. \tag{B.2.2}$$

Since the second term is the difference between two connections, we can identify it as $(2,1)$ tensor, $C^\mu{}_{\rho\sigma}$. As $A(\varphi)$ is only explicitly a function of φ, the chain rule can be used to convert this expression to the following, expression for $C^\mu{}_{\rho\sigma}$,

$$\begin{aligned} C^\mu{}_{\rho\sigma} &= A^{-1}\frac{\partial A}{\partial\varphi}\left[\delta^\mu_\rho(\partial_\sigma\varphi) + \delta^\mu_\sigma(\partial_\rho\varphi) - \tilde{g}_{\rho\sigma}\tilde{g}^{\mu\nu}(\partial_\nu\varphi)\right] \\ &= \frac{\partial(\ln A(\varphi))}{\partial\varphi}\left[\delta^\mu_\rho(\partial_\sigma\varphi) + \delta^\mu_\sigma(\partial_\rho\varphi) - \tilde{g}_{\rho\sigma}\tilde{g}^{\mu\nu}(\partial_\nu\varphi)\right] \end{aligned} \tag{B.2.3}$$

This is a tensor because we can be expressed completely covariantly,

$$C^\mu{}_{\rho\sigma} = A^{-1}\frac{\partial A}{\partial\varphi}\left[\delta^\mu_\rho(\nabla_\sigma\varphi) + \delta^\mu_\sigma(\nabla_\rho\varphi) - \tilde{g}_{\rho\sigma}\tilde{g}^{\mu\nu}(\nabla_\nu\varphi)\right], \tag{B.2.4}$$

because $\nabla_\mu\varphi = \partial_\mu\varphi$.

This can be proven in terms of general coordinate transformations:

The difference between two Levi-Civita connections $C^{\mu}{}_{\rho\sigma} = \Gamma^{\mu}{}_{\rho\sigma} - \tilde{\Gamma}^{\mu}{}_{\rho\sigma}$ is a tensor under general coordinate transformations.

Proof: Consider a matrix coordinate transformation x and x',

$$M : x^{\mu} \to x^{\mu'}.$$

A general Levi-Civita connection will transform as follows,

$$\Gamma^{\mu}{}_{\rho\sigma} \to \Gamma^{\mu'}{}_{\rho'\sigma'} = \frac{1}{2} g^{\mu'\nu'} \left(g_{\rho'\nu',\sigma'} + g_{\nu'\sigma',\rho'} - g_{\rho'\sigma',\nu'} \right) \tag{B.2.5}$$

The component terms of the above connection only differ in index placement, a particular term will transform as follows:

$$\begin{aligned}
\partial_{\mu} g_{\rho\sigma} \to \partial_{\mu'} g_{\rho'\sigma'} &= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\rho\nu} \right) \\
&= \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\rho}}{\partial x^{\rho'}} \right) g_{\rho\nu} + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu}}{\partial x^{\nu'}} \right) g_{\rho\nu} \\
&\quad + \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\rho}}{\partial x^{\rho'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \partial_{\mu} g_{\rho\nu}
\end{aligned}$$

The geodesic equation is now,

$$\frac{d^2 x^{\mu}}{d\tau^2} + \tilde{\Gamma}^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} + C^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = 0 \tag{B.2.6}$$

For which we identify the first two terms as a^{μ}_{E},

$$a^{\mu}_{E} = -C^{\mu}{}_{\rho\sigma} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau} = F^{\mu} \tag{B.2.7}$$

where the equality is the covariant force vector for a unit-mass particle. Identifying $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ we get the following force on this test particle is given by,

$$\begin{aligned}
F^{\mu} &= A^{-1} \frac{\partial A}{\partial \varphi} \left[\delta^{\mu}_{\rho} (\nabla_{\sigma} \varphi) + \delta^{\mu}_{\sigma} (\nabla_{\rho} \varphi) - \tilde{g}_{\rho\sigma} \tilde{g}^{\mu\nu} (\nabla_{\nu} \varphi) \right] u^{\rho} u^{\sigma} && \text{(B.2.8)} \\
&= A^{-1} \frac{\partial A}{\partial \varphi} \left[2 u^{\mu} (u^{\alpha} \nabla_{\alpha} \varphi) - \tilde{g}^{\mu\nu} (\nabla_{\nu} \varphi) u^{\alpha} u_{\alpha} \right] && \text{(B.2.9)}
\end{aligned}$$

B.3 Considering forces in Local Inertial Frames

Consider the expression for the force on the particle in a local inertial frame. This amounts to appropriate substitutions to use this relations, $\tilde{g}_{\mu\nu} \to \eta_{\mu\nu}$ and $\nabla_\lambda \to \partial_\lambda$,

$$F^\mu = -\frac{\partial(\ln A(\varphi))}{\partial\varphi}\left[2u^\mu(u^\alpha\partial_\alpha\varphi) - \eta^{\mu\nu}(\partial_\nu\varphi)u^\sigma u_\sigma\right]. \tag{B.3.1}$$

where $u_\alpha = \eta_{\alpha\beta}u^\beta$ now. Since u^μ is a time-like vector, in the mostly-plus Lorentzian signature, $u^\mu u_\mu < 0$ we can write the force equation in the inertial frame of the test particle using a suitable Lorentz transformation such that $u^\mu = (1, \vec{0})$,

$$F^\mu = \frac{\partial(\ln A(\varphi))}{\partial\varphi}\left[2u^\mu(u^\alpha\partial_\alpha\varphi) + \eta^{\mu\nu}(\partial_\nu\varphi)\right]. \tag{B.3.2}$$

Choosing now to separate these into equations, for spacelike and timelike components now gives us,

$$\mathbf{F}_5 = -\frac{\partial(\ln A)}{\partial\varphi}\vec{\nabla}\varphi = -\vec{\nabla}(\ln A) \tag{B.3.3}$$

$$P = \frac{\partial(\ln A)}{\partial\varphi}\dot{\varphi} \tag{B.3.4}$$

where P is the power generated by the test particle accelerated by $\mathbf{F}_5$, which could be of interest. This is a derivation of the fifth force, $\mathbf{F}_5$ generated in the Einstein Frame.

B.4 Calculating the Fifth Force

We see that we can calculate the non-relativistic force on the test particle, all that needs to be assigned is the form of $A(\varphi)$. In this particular theory we use,

$$A(\varphi) = 1 + \frac{\varphi^2}{2M^2} \tag{B.4.1}$$

with higher order terms ignored. We see that $\ln(A) = \ln(1 + \varphi^2/2M^2)$, where for $\frac{\varphi}{M} \ll 1$, we can expand in powers of $\frac{\varphi}{M}$

$$\ln A(\varphi) \approx \frac{\varphi^2}{2M^2} + ... \tag{B.4.2}$$

which derives a new fifth force,

$$\mathbf{F}_5 \approx -\frac{\varphi \vec{\nabla} \varphi}{M^2}. \tag{B.4.3}$$