Normalization Constant Exercises for PMF/PDF 1

Exercise 5 (6 points).
Let X be a discrete random variable with support $\mathbb{N} = \{1, 2, 3, \ldots\}$. Suppose X has law defined by:

$$P(X = k) = C_\theta \cdot \theta^{k+1}(1-\theta)^{k-1} \quad \text{for } k \in \mathbb{N},\ \theta \in (0, 1),$$

where $C_\theta \in \mathbb{R}$. Find C_θ.

> **Solution:**
> The normalization condition requires summing probabilities over the support:
>
> $$\sum_{k=1}^{\infty} C_\theta \theta^{k+1}(1-\theta)^{k-1} = 1.$$
>
> Factorize θ^2:
>
> $$C_\theta \theta^2 \sum_{k=1}^{\infty} [\theta(1-\theta)]^{k-1} = 1.$$
>
> Recognize the geometric series $\sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}$:
>
> $$C_\theta \theta^2 \cdot \frac{1}{1-\theta(1-\theta)} = 1.$$
>
> Simplify the denominator ($1 - \theta + \theta^2$):
>
> $$C_\theta = \frac{1-\theta+\theta^2}{\theta^2}.$$

Progressive Variations

Variation 1: Geometric Distribution with Offset Support

Exercise:
A discrete random variable X has support $\{2, 3, 4, \dots\}$ and PMF:

$$P(X = k) = C_p (1-p)^{k-1} \quad \text{for } p \in (0, 1).$$

Find C_p.

> **Solution:**
> Shift index $j = k - 1$:
>
> $$\sum_{j=1}^{\infty} C_p (1-p)^j = 1.$$
>
> Geometric series sum:
>
> $$C_p \cdot \frac{1-p}{p} = 1 \implies C_p = \frac{p}{1-p}.$$

Variation 2: Poisson Distribution via Taylor Series

Exercise:
Let X have support $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and PMF:

$$P(X = k) = C_\lambda \frac{\lambda^k}{k!} \quad \text{for } \lambda > 0.$$

Find C_λ.

> **Solution:**
> Sum over support:
>
> $$C_\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1.$$

> Recognize Taylor series for e^λ:
>
> $$C_\lambda e^\lambda = 1 \implies C_\lambda = e^{-\lambda}.$$

Variation 3: Weighted Geometric Series (Expectation Form)

Exercise:
Let X have support $\mathbb{N}$ and PMF:

$$P(X = k) = C_p \cdot k \cdot p^k \quad \text{for } p \in (0, 1).$$

Find C_p.

> **Solution:**
> Use the derivative of the geometric series:
>
> $$\sum_{k=1}^{\infty} kp^k = \frac{p}{(1-p)^2}.$$
>
> Thus:
>
> $$C_p \cdot \frac{p}{(1-p)^2} = 1 \implies C_p = \frac{(1-p)^2}{p}.$$

Variation 4: Joint Distribution with Double Sum

Exercise:
Let (X, Y) have support $\mathbb{N}_0 \times \mathbb{N}_0$ and joint PMF:

$$P(X = k, Y = j) = C \cdot \frac{2^k 3^j}{k! j!}.$$

Find C.

> **Solution:**
> Separate sums using independence:
>
> $$C\left(\sum_{k=0}^{\infty}\frac{2^k}{k!}\right)\left(\sum_{j=0}^{\infty}\frac{3^j}{j!}\right)=1.$$
>
> Recognize e^2 and e^3:
>
> $$Ce^5=1 \implies C=e^{-5}.$$

Variation 5: Gamma Distribution via Gamma Function

Exercise:
Let X be continuous with support $(0,\infty)$ and PDF:

$$f(x)=C_{\alpha,\beta}\cdot x^{\alpha-1}e^{-\beta x}\quad\text{for }\alpha,\beta>0.$$

Find $C_{\alpha,\beta}$.

> **Solution:**
> Substitute $u=\beta x$:
>
> $$C_{\alpha,\beta}\int_0^{\infty}\left(\frac{u}{\beta}\right)^{\alpha-1}e^{-u}\frac{du}{\beta}=1.$$
>
> Simplify and identify $\Gamma(\alpha)$:
>
> $$C_{\alpha,\beta}\cdot\frac{\Gamma(\alpha)}{\beta^\alpha}=1 \implies C_{\alpha,\beta}=\frac{\beta^\alpha}{\Gamma(\alpha)}.$$

Part 2

Original Exercise & Solution

Exercise 5 (6 points)

Let X be a discrete random variable with support $\{1, \ldots, N\}$, where $N \geq 2$ and N is even. Suppose X has PMF:

$$P(X = k) = C_N \cdot \max\{k, N - k\}, \quad \text{for } k = 1, \ldots, N,$$

where $C_N \in \mathbb{R}$. Find C_N.

> ## Solution
>
> **Normalization Condition**:
>
> $$\sum_{k=1}^{N} C_N \cdot \max\{k, N - k\} = 1$$
>
> **Split the Sum**:
>
> - For $k \leq N/2$: $\max\{k, N - k\} = N - k$
> - For $k > N/2$: $\max\{k, N - k\} = k$
>
>   Thus:
>
> $$S = \underbrace{\sum_{k=1}^{N/2} (N - k)}_{\text{Sum 1}} + \underbrace{\sum_{k=N/2+1}^{N} k}_{\text{Sum 2}}$$
>
> 1. **Evaluate Sums**:
>
> - **Sum 1**: Arithmetic series $(N - 1) + (N - 2) + \cdots + N/2$
>
> $$\frac{N/2}{2} \left(\frac{N}{2} + (N - 1) \right) = \frac{3N^2}{8} - \frac{N}{4}$$
>
> - **Sum 2**: Arithmetic series $(N/2 + 1) + \cdots + N$

> $$\frac{N/2}{2}\left(\frac{N}{2}+1+N\right)=\frac{3N^2}{8}+\frac{N}{4}$$
>
> 2. **Total Sum** S:
>
> $$S=\left(\frac{3N^2}{8}-\frac{N}{4}\right)+\left(\frac{3N^2}{8}+\frac{N}{4}\right)=\frac{3N^2}{4}$$
>
> 3. **Normalization Constant**:
>
> $$C_N\cdot\frac{3N^2}{4}=1 \implies C_N=\frac{4}{3N^2}$$

Progressive Variations

Variation 1: Odd N

Exercise:
Same as original, but $N \geq 3$ is odd.

> **Solution**:
> Split sums at $k \leq (N-1)/2$ (terms: $N-1, \dots, \frac{N+1}{2}$) and $k \geq (N+1)/2$ (same terms).
>
> $$S=2\cdot\frac{(N+1)/2}{2}\left(\frac{N+1}{2}+N\right)=\frac{(N+1)(3N+1)}{4}$$
>
> $$C_N=\frac{4}{(N+1)(3N+1)}$$

Variation 2: Sum of Squares

Exercise:
$P(X=k)=C_N\cdot k^2$ for support $\{1,\dots,N\}$.

> **Solution**:
> Using $\sum_{k=1}^{N} k^2=\frac{N(N+1)(2N+1)}{6}$:

> $$C_N = \frac{6}{N(N+1)(2N+1)}$$

Variation 3: Symmetric Absolute Function

Exercise:
$P(X=k) = C_N \cdot \left|k - \frac{N+1}{2}\right|$.

> **Solution**:
>
> - **Odd N**:
>
> $$S = \frac{N^2-1}{4}, \quad C_N = \frac{4}{N^2-1}$$
>
> - **Even N**:
>
> $$S = \frac{N^2}{4}, \quad C_N = \frac{4}{N^2}$$

Variation 4: 2D Joint Distribution

Exercise:
(X,Y) on grid $\{1,\ldots,N\}^2$ with PMF $P(X=k, Y=j) = C_N \cdot (k+j)$.

> **Solution**:
> Double summation yields:
>
> $$S = N \cdot \frac{N(N+1)}{2} + N \cdot \frac{N(N+1)}{2} = N^2(N+1)$$
>
> $$C_N = \frac{1}{N^2(N+1)}$$

Variation 5: Continuous 2D Square (Hardest)

Exercise:
(X, Y) on $[0, L]^2$ with PDF $f(x, y) = C_L \cdot \max\{x, y\}$.

> **Solution**:
>
> 1. Split domain via $y = x$:
>   Lower triangle ($\max\{x, y\} = x$), upper triangle symmetric.
> 2. Compute integral:
>
> $$S = 2\int_0^L \left(\int_0^x x\,dy\right) dx = \frac{2L^3}{3}$$
>
> 3. Normalization:
>
> $$C_L = \frac{3}{2L^3}$$

Key Techniques

1. **Symmetry Splitting**: Divide sums/integrals at symmetry points (e.g., midpoint $N/2$, diagonal $y = x$).
2. **Arithmetic Series**: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum k^2 = \frac{n(n+1)(2n+1)}{6}$.
3. **Case Handling**: Separate solutions for even/odd N.
4. **Dimensionality**: Extend logic to joint PMFs/PDFs with multivariate summations/integrals.

Part 3

Variation 1 (Simplest)

Exercise (2 points)

Let X be a discrete random variable with support $\{1, 2\}$. Suppose its probability mass function is given by $P(X = k) = C \cdot k$ for $k = 1, 2$. Find the constant C.

> **Solution**
> $\sum_{k=1}^{2} P(X = k) = 1$

> $C \cdot 1 + C \cdot 2 = 1$
> $3C = 1$
> $C = \frac{1}{3}$

Variation 2 (Slightly Harder)

Exercise (4 points)
Let X be a discrete random variable with support $\{1, 2, 3\}$. The probability mass function is $P(X = k) = C_3 \cdot |k - 1.5|$ for $k = 1, 2, 3$. Find C_3.

> **Solution**
> Values: $|1 - 1.5| = 0.5$, $|2 - 1.5| = 0.5$, $|3 - 1.5| = 1.5$
> Sum: $0.5 + 0.5 + 1.5 = 2.5$
> $C_3 \cdot 2.5 = 1$
> $C_3 = \frac{2}{5}$

Variation 3 (Moderate)

Exercise (5 points)
Let X be a discrete random variable with support $\{1, 2, 3, 4\}$. The probability mass function is $P(X = k) = C_4 \cdot \max(k, 5 - k)$. Find C_4.

> **Solution**
> Values:
> $k = 1 : \max(1, 4) = 4$
> $k = 2 : \max(2, 3) = 3$
> $k = 3 : \max(3, 2) = 3$
> $k = 4 : \max(4, 1) = 4$
> Sum: $4 + 3 + 3 + 4 = 14$
> $C_4 \cdot 14 = 1$
> $C_4 = \frac{1}{14}$

Variation 4 (Challenging)

Exercise (7 points)

Let X be a discrete random variable with support $\{1, \ldots, N\}$, where $N \geq 2$ is even. The probability mass function is $P(X = k) = C_N \cdot \left|k - \frac{N}{2}\right|$. Find C_N.

> **Solution**
> Let $m = N/2$
> Sum splits:
> $\sum_{k=1}^{m}(m-k) + \sum_{k=m+1}^{N}(k-m)$
> First sum: $\sum_{j=0}^{m-1} j = \frac{(m-1)m}{2}$
> Second sum: $\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$
> Total: $\frac{(m-1)m}{2} + \frac{m(m+1)}{2} = m^2$
> Substitute $m = N/2$: Sum = $N^2/4$
> $C_N \cdot \frac{N^2}{4} = 1$
> $C_N = \frac{4}{N^2}$

Variation 5 (Hardest)

Exercise (9 points)

Let X be a discrete random variable with support $\{1, \ldots, N\}$, where $N \geq 4$ is even. The probability mass function is:

$$P(X = k) = C_N \cdot \left(\left|k - \frac{N}{2}\right| + 1\right)$$

Find C_N and compute $E[X]$.

> **Solution**
> Let $m = N/2$
> **Part 1: Find C_N**
> Sum: $\sum_{k=1}^{m}(m-k+1) + \sum_{k=m+1}^{N}(k-m+1)$
> First sum: $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$
> Second sum: $\sum_{j=1}^{m}(j+1) = \frac{m(m+1)}{2} + m$
> Total: $\frac{m(m+1)}{2} + \frac{m(m+1)}{2} + m = m^2 + 2m$
> Substitute $m = N/2$: Sum = $\frac{N^2}{4} + N$

> $C_N \cdot \left(\frac{N^2}{4} + N\right) = 1$
> $C_N = \frac{4}{N(N+4)}$
>
> **Part 2: Compute $E[X]$**
> $E[X] = C_N \left[\sum_{k=1}^{m} k(m-k+1) + \sum_{k=m+1}^{N} k(k-m+1)\right]$
> First sum: $(m+1)\sum k - \sum k^2 = \frac{m(m+1)^2}{2} - \frac{m(m+1)(2m+1)}{6} = \frac{m(m+1)(m+2)}{6}$
> Second sum: $\sum (j+m)(j+1) = \frac{m(5m^2+15m+4)}{6}$ (after substitution $j = k-m$)
> Total inner sum: $\frac{m(m+1)(m+2)}{6} + \frac{m(5m^2+15m+4)}{6} = m(m^2+3m+1)$
> $E[X] = \frac{4}{N(N+4)} \cdot \frac{N}{2}\left(\frac{N^2}{4} + \frac{3N}{2} + 1\right) = \frac{N^2+6N+4}{2(N+4)}$

Normalization Constant Exercises for PMF/PDF 1

Exercise 5 (6 points).
Let X be a discrete random variable with support $\mathbb{N} = \{1, 2, 3, \dots\}$. Suppose X has law defined by:

$$P(X = k) = C_\theta \cdot \theta^{k+1}(1-\theta)^{k-1} \quad \text{for } k \in \mathbb{N},\ \theta \in (0,1),$$

where $C_\theta \in \mathbb{R}$. Find C_θ.

> **Solution:**
> The normalization condition requires summing probabilities over the support:
>
> $$\sum_{k=1}^{\infty} C_\theta \theta^{k+1}(1-\theta)^{k-1} = 1.$$
>
> Factorize θ^2:
>
> $$C_\theta \theta^2 \sum_{k=1}^{\infty} [\theta(1-\theta)]^{k-1} = 1.$$
>
> Recognize the geometric series $\sum_{k=1}^{\infty} r^{k-1} = \frac{1}{1-r}$:

> $$C_\theta \theta^2 \cdot \frac{1}{1-\theta(1-\theta)} = 1.$$
>
> Simplify the denominator ($1-\theta+\theta^2$):
>
> $$C_\theta = \frac{1-\theta+\theta^2}{\theta^2}.$$

Progressive Variations

Variation 1: Geometric Distribution with Offset Support

Exercise:
A discrete random variable X has support $\{2, 3, 4, \dots\}$ and PMF:

$$P(X = k) = C_p (1-p)^{k-1} \quad \text{for } p \in (0, 1).$$

Find C_p.

> **Solution:**
> Shift index $j = k - 1$:
>
> $$\sum_{j=1}^{\infty} C_p (1-p)^j = 1.$$
>
> Geometric series sum:
>
> $$C_p \cdot \frac{1-p}{p} = 1 \implies C_p = \frac{p}{1-p}.$$

Variation 2: Poisson Distribution via Taylor Series

Exercise:
Let X have support $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and PMF:

$$P(X = k) = C_\lambda \frac{\lambda^k}{k!} \quad \text{for } \lambda > 0.$$

Find C_λ.

> **Solution:**
> Sum over support:
>
> $$C_\lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = 1.$$
>
> Recognize Taylor series for e^λ:
>
> $$C_\lambda e^\lambda = 1 \implies C_\lambda = e^{-\lambda}.$$

Variation 3: Weighted Geometric Series (Expectation Form)

Exercise:
Let X have support $\mathbb{N}$ and PMF:

$$P(X = k) = C_p \cdot k \cdot p^k \quad \text{for } p \in (0, 1).$$

Find C_p.

> **Solution:**
> Use the derivative of the geometric series:
>
> $$\sum_{k=1}^{\infty} kp^k = \frac{p}{(1-p)^2}.$$
>
> Thus:
>
> $$C_p \cdot \frac{p}{(1-p)^2} = 1 \implies C_p = \frac{(1-p)^2}{p}.$$

Variation 4: Joint Distribution with Double Sum

Exercise:

Let (X, Y) have support $\mathbb{N}_0 \times \mathbb{N}_0$ and joint PMF:

$$P(X = k, Y = j) = C \cdot \frac{2^k 3^j}{k! j!}.$$

Find C.

> **Solution:**
>
> Separate sums using independence:
>
> $$C \left(\sum_{k=0}^{\infty} \frac{2^k}{k!} \right) \left(\sum_{j=0}^{\infty} \frac{3^j}{j!} \right) = 1.$$
>
> Recognize e^2 and e^3:
>
> $$Ce^5 = 1 \implies C = e^{-5}.$$

Variation 5: Gamma Distribution via Gamma Function

Exercise:

Let X be continuous with support $(0, \infty)$ and PDF:

$$f(x) = C_{\alpha,\beta} \cdot x^{\alpha-1} e^{-\beta x} \quad \text{for } \alpha, \beta > 0.$$

Find $C_{\alpha,\beta}$.

> **Solution:**
>
> Substitute $u = \beta x$:
>
> $$C_{\alpha,\beta} \int_0^{\infty} \left(\frac{u}{\beta} \right)^{\alpha-1} e^{-u} \frac{du}{\beta} = 1.$$
>
> Simplify and identify $\Gamma(\alpha)$:

> $$C_{\alpha,\beta} \cdot \frac{\Gamma(\alpha)}{\beta^\alpha} = 1 \implies C_{\alpha,\beta} = \frac{\beta^\alpha}{\Gamma(\alpha)}.$$

Part 2

Original Exercise & Solution

Exercise 5 (6 points)
Let X be a discrete random variable with support $\{1, \ldots, N\}$, where $N \geq 2$ and N is even. Suppose X has PMF:

$$P(X = k) = C_N \cdot \max\{k, N-k\}, \quad \text{for } k = 1, \ldots, N,$$

where $C_N \in \mathbb{R}$. Find C_N.

> ### Solution
>
> **Normalization Condition**:
>
> $$\sum_{k=1}^{N} C_N \cdot \max\{k, N-k\} = 1$$
>
> **Split the Sum**:
>
> - For $k \leq N/2$: $\max\{k, N-k\} = N - k$
> - For $k > N/2$: $\max\{k, N-k\} = k$
>   Thus:
>
> $$S = \underbrace{\sum_{k=1}^{N/2}(N-k)}_{\text{Sum 1}} + \underbrace{\sum_{k=N/2+1}^{N} k}_{\text{Sum 2}}$$
>
> 1. **Evaluate Sums**:
>
> - **Sum 1**: Arithmetic series $(N-1) + (N-2) + \cdots + N/2$

$$\frac{N/2}{2}\left(\frac{N}{2}+(N-1)\right)=\frac{3N^2}{8}-\frac{N}{4}$$

- **Sum 2**: Arithmetic series $(N/2+1)+\cdots+N$

$$\frac{N/2}{2}\left(\frac{N}{2}+1+N\right)=\frac{3N^2}{8}+\frac{N}{4}$$

2. **Total Sum** S:

$$S=\left(\frac{3N^2}{8}-\frac{N}{4}\right)+\left(\frac{3N^2}{8}+\frac{N}{4}\right)=\frac{3N^2}{4}$$

3. **Normalization Constant**:

$$C_N\cdot\frac{3N^2}{4}=1\implies C_N=\frac{4}{3N^2}$$

Progressive Variations

Variation 1: Odd N

Exercise:
Same as original, but $N\geq 3$ is odd.

> **Solution**:
> Split sums at $k\leq(N-1)/2$ (terms: $N-1,\ldots,\frac{N+1}{2}$) and $k\geq(N+1)/2$ (same terms).
>
> $$S=2\cdot\frac{(N+1)/2}{2}\left(\frac{N+1}{2}+N\right)=\frac{(N+1)(3N+1)}{4}$$
>
> $$C_N=\frac{4}{(N+1)(3N+1)}$$

Variation 2: Sum of Squares

Exercise:
$P(X=k)=C_N\cdot k^2$ for support $\{1,\ldots,N\}$.

> **Solution**:
> Using $\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$:
>
> $$C_N = \frac{6}{N(N+1)(2N+1)}$$

Variation 3: Symmetric Absolute Function

Exercise:
$P(X = k) = C_N \cdot \left| k - \frac{N+1}{2} \right|$.

> **Solution**:
>
> - **Odd N**:
>
> $$S = \frac{N^2 - 1}{4}, \quad C_N = \frac{4}{N^2 - 1}$$
>
> - **Even N**:
>
> $$S = \frac{N^2}{4}, \quad C_N = \frac{4}{N^2}$$

Variation 4: 2D Joint Distribution

Exercise:
(X, Y) on grid $\{1, \ldots, N\}^2$ with PMF $P(X = k, Y = j) = C_N \cdot (k + j)$.

> **Solution**:
> Double summation yields:
>
> $$S = N \cdot \frac{N(N+1)}{2} + N \cdot \frac{N(N+1)}{2} = N^2(N+1)$$
>
> $$C_N = \frac{1}{N^2(N+1)}$$

Variation 5: Continuous 2D Square (Hardest)

Exercise:
(X, Y) on $[0, L]^2$ with PDF $f(x, y) = C_L \cdot \max\{x, y\}$.

> **Solution**:
>
> 1. Split domain via $y = x$:
>   Lower triangle ($\max\{x, y\} = x$), upper triangle symmetric.
> 2. Compute integral:
>
> $$S = 2 \int_0^L \left(\int_0^x x \, dy \right) dx = \frac{2L^3}{3}$$
>
> 3. Normalization:
>
> $$C_L = \frac{3}{2L^3}$$

Key Techniques

1. **Symmetry Splitting**: Divide sums/integrals at symmetry points (e.g., midpoint $N/2$, diagonal $y = x$).
2. **Arithmetic Series**: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum k^2 = \frac{n(n+1)(2N+1)}{6}$.
3. **Case Handling**: Separate solutions for even/odd N.
4. **Dimensionality**: Extend logic to joint PMFs/PDFs with multivariate summations/integrals.

Part 3

Variation 1 (Simplest)

Exercise (2 points)
Let X be a discrete random variable with support $\{1, 2\}$. Suppose its probability mass function is given by $P(X = k) = C \cdot k$ for $k = 1, 2$. Find the constant C.

> **Solution**
> $\sum_{k=1}^{2} P(X = k) = 1$

> $C \cdot 1 + C \cdot 2 = 1$
> $3C = 1$
> $C = \frac{1}{3}$

Variation 2 (Slightly Harder)

Exercise (4 points)
Let X be a discrete random variable with support $\{1, 2, 3\}$. The probability mass function is $P(X = k) = C_3 \cdot |k - 1.5|$ for $k = 1, 2, 3$. Find C_3.

> **Solution**
> Values: $|1 - 1.5| = 0.5$, $|2 - 1.5| = 0.5$, $|3 - 1.5| = 1.5$
> Sum: $0.5 + 0.5 + 1.5 = 2.5$
> $C_3 \cdot 2.5 = 1$
> $C_3 = \frac{2}{5}$

Variation 3 (Moderate)

Exercise (5 points)
Let X be a discrete random variable with support $\{1, 2, 3, 4\}$. The probability mass function is $P(X = k) = C_4 \cdot \max(k, 5 - k)$. Find C_4.

> **Solution**
> Values:
> $k = 1 : \max(1, 4) = 4$
> $k = 2 : \max(2, 3) = 3$
> $k = 3 : \max(3, 2) = 3$
> $k = 4 : \max(4, 1) = 4$
> Sum: $4 + 3 + 3 + 4 = 14$
> $C_4 \cdot 14 = 1$
> $C_4 = \frac{1}{14}$

Variation 4 (Challenging)

Exercise (7 points)

Let X be a discrete random variable with support $\{1, \ldots, N\}$, where $N \geq 2$ is even. The probability mass function is $P(X = k) = C_N \cdot \left|k - \frac{N}{2}\right|$. Find C_N.

> **Solution**
> Let $m = N/2$
> Sum splits:
> $\sum_{k=1}^{m}(m-k) + \sum_{k=m+1}^{N}(k-m)$
> First sum: $\sum_{j=0}^{m-1} j = \frac{(m-1)m}{2}$
> Second sum: $\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$
> Total: $\frac{(m-1)m}{2} + \frac{m(m+1)}{2} = m^2$
> Substitute $m = N/2$: Sum = $N^2/4$
> $C_N \cdot \frac{N^2}{4} = 1$
> $C_N = \frac{4}{N^2}$

Variation 5 (Hardest)

Exercise (9 points)

Let X be a discrete random variable with support $\{1, \ldots, N\}$, where $N \geq 4$ is even. The probability mass function is:

$$P(X = k) = C_N \cdot \left(\left|k - \frac{N}{2}\right| + 1\right)$$

Find C_N and compute $E[X]$.

> **Solution**
> Let $m = N/2$
> **Part 1: Find C_N**
> Sum: $\sum_{k=1}^{m}(m-k+1) + \sum_{k=m+1}^{N}(k-m+1)$
> First sum: $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$
> Second sum: $\sum_{j=1}^{m}(j+1) = \frac{m(m+1)}{2} + m$
> Total: $\frac{m(m+1)}{2} + \frac{m(m+1)}{2} + m = m^2 + 2m$
> Substitute $m = N/2$: Sum = $\frac{N^2}{4} + N$

> $C_N \cdot \left(\frac{N^2}{4} + N\right) = 1$
> $C_N = \frac{4}{N(N+4)}$
>
> **Part 2: Compute $E[X]$**
> $E[X] = C_N \left[\sum_{k=1}^{m} k(m-k+1) + \sum_{k=m+1}^{N} k(k-m+1)\right]$
> First sum: $(m+1)\sum k - \sum k^2 = \frac{m(m+1)^2}{2} - \frac{m(m+1)(2m+1)}{6} = \frac{m(m+1)(m+2)}{6}$
> Second sum: $\sum(j+m)(j+1) = \frac{m(5m^2+15m+4)}{6}$ (after substitution $j = k - m$)
> Total inner sum: $\frac{m(m+1)(m+2)}{6} + \frac{m(5m^2+15m+4)}{6} = m(m^2+3m+1)$
> $E[X] = \frac{4}{N(N+4)} \cdot \frac{N}{2}\left(\frac{N^2}{4} + \frac{3N}{2} + 1\right) = \frac{N^2+6N+4}{2(N+4)}$

Variation 6 (Min Function on Small Support)

Exercise (4 points)

Let X be a discrete random variable with support $\{1, 2, 3, 4\}$. Its probability mass function is given by $P(X = k) = C \cdot \min(k, 3)$ for $k \in \{1, 2, 3, 4\}$. Find the constant C.

> **Solution**
> The normalization condition is $\sum_{k=1}^{4} P(X = k) = 1$.
> We evaluate $P(X = k)$ for each k in the support:
> $P(X = 1) = C \cdot \min(1, 3) = C \cdot 1 = C$
> $P(X = 2) = C \cdot \min(2, 3) = C \cdot 2 = 2C$
> $P(X = 3) = C \cdot \min(3, 3) = C \cdot 3 = 3C$
> $P(X = 4) = C \cdot \min(4, 3) = C \cdot 3 = 3C$
> Summing these probabilities:
> $C + 2C + 3C + 3C = 1$
> $9C = 1$
> $C = \frac{1}{9}$

Variation 7 (Simple Linear PDF)

Exercise (3 points)

Let X be a continuous random variable with support on the interval $[0, 2]$. Its probability density function is $f(x) = C \cdot x$ for $x \in [0, 2]$. Find the constant C.

> **Solution**
> For a PDF, the normalization condition is $\int_{-\infty}^{\infty} f(x)dx = 1$.
> Given the support is $[0, 2]$, this becomes $\int_0^2 C \cdot x\,dx = 1$.
> $C \int_0^2 x\,dx = 1$
> $C \left[\frac{x^2}{2}\right]_0^2 = 1$
> $C \left(\frac{2^2}{2} - \frac{0^2}{2}\right) = 1$
> $C \left(\frac{4}{2} - 0\right) = 1$
> $2C = 1$
> $C = \frac{1}{2}$

Variation 8 (Symmetric Quadratic PMF)

Exercise (4 points)

Let X be a discrete random variable with support $\{-2, -1, 0, 1, 2\}$. The probability mass function is $P(X = k) = C \cdot (k^2 + 1)$ for $k \in \{-2, -1, 0, 1, 2\}$. Find C.

> **Solution**
> The normalization condition is $\sum_{k=-2}^{2} P(X = k) = 1$.
> We evaluate $P(X = k)$ for each k:
> $P(X = -2) = C \cdot ((-2)^2 + 1) = C \cdot (4 + 1) = 5C$
> $P(X = -1) = C \cdot ((-1)^2 + 1) = C \cdot (1 + 1) = 2C$
> $P(X = 0) = C \cdot (0^2 + 1) = C \cdot (0 + 1) = C$
> $P(X = 1) = C \cdot (1^2 + 1) = C \cdot (1 + 1) = 2C$
> $P(X = 2) = C \cdot (2^2 + 1) = C \cdot (4 + 1) = 5C$
> Summing these probabilities:
> $5C + 2C + C + 2C + 5C = 1$
> $15C = 1$
> $C = \frac{1}{15}$

Variation 9 (Finite Geometric Series PMF)

Exercise (6 points)

Let X be a discrete random variable with support $\{0, 1, \ldots, N - 1\}$ for some integer $N \geq 1$. The probability mass function is $P(X = k) = C \cdot \left(\frac{1}{2}\right)^k$. Find C.

Solution

The normalization condition is $\sum_{k=0}^{N-1} P(X=k) = 1$.
So, $\sum_{k=0}^{N-1} C \cdot \left(\frac{1}{2}\right)^k = 1$.
$C \sum_{k=0}^{N-1} \left(\frac{1}{2}\right)^k = 1$.
The sum is a finite geometric series $\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$. Here, $n = N$ and $r = 1/2$.
$\sum_{k=0}^{N-1} \left(\frac{1}{2}\right)^k = \frac{1-(1/2)^N}{1-1/2} = \frac{1-2^{-N}}{1/2} = 2(1-2^{-N})$.
So, $C \cdot 2(1-2^{-N}) = 1$.
$C = \frac{1}{2(1-2^{-N})} = \frac{1}{2-2^{1-N}}$.
This can also be written as $C = \frac{2^{N-1}}{2^N-1}$.

Variation 10 (Continuous PDF with Max Function)

Exercise (7 points)

Let X be a continuous random variable with support on the interval $[0,1]$. Its probability density function is $f(x) = C \cdot \max(x, 1-x)$ for $x \in [0,1]$. Find C.

Solution

The normalization condition is $\int_0^1 f(x)\,dx = 1$.
We need to understand the function $\max(x, 1-x)$ on $[0,1]$.
The term $1-x$ is greater than or equal to x when $1-x \geq x \implies 1 \geq 2x \implies x \leq 1/2$.
The term x is greater than $1-x$ when $x > 1-x \implies 2x > 1 \implies x > 1/2$.
So, $\max(x, 1-x) = \begin{cases} 1-x & \text{if } 0 \leq x \leq 1/2 \\ x & \text{if } 1/2 < x \leq 1 \end{cases}$.
We split the integral based on this:
$\int_0^1 C \cdot \max(x, 1-x)\,dx = C\left(\int_0^{1/2}(1-x)\,dx + \int_{1/2}^1 x\,dx\right) = 1$.
Calculate the first integral:
$\int_0^{1/2}(1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^{1/2} = \left(\frac{1}{2} - \frac{(1/2)^2}{2}\right) - 0 = \frac{1}{2} - \frac{1/8}{=}\frac{4-1}{8} = \frac{3}{8}$.
Calculate the second integral:
$\int_{1/2}^1 x\,dx = \left[\frac{x^2}{2}\right]_{1/2}^1 = \frac{1^2}{2} - \frac{(1/2)^2}{2} = \frac{1}{2} - \frac{1/8}{=}\frac{3}{8}$.
So, $C\left(\frac{3}{8} + \frac{3}{8}\right) = 1$.
$C\left(\frac{6}{8}\right) = 1$.
$C\left(\frac{3}{4}\right) = 1$.
$C = \frac{4}{3}$.