Exercises

I. Sets & Functions (Chapters 1 & 2)

1. **True or False**: For any two sets A and B, $(A \cup B)^c = A^c \cap B^c$. [Exercise 1.11]
   > **True** (De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$ holds universally.)

2. **Short Answer**: Let $\Omega = \{1, 2, 3\}$. List all elements of the power set $\mathcal{P}(\Omega)$. [Example 4.2]
   > $\mathcal{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ (Power set includes all subsets.)

3. **True or False**: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ is an injective function. [Definition 2.5]
   > **False** ($\sin(x_1) = \sin(x_2)$ does not imply $x_1 = x_2$, e.g., $\sin(0) = \sin(\pi)$.)

4. **Short Answer**: Let $f(x) = x^3 - x$. Is f a strictly monotonic function on $\mathbb{R}$? Justify your answer. [Definition 2.9, Proposition 2.5]
   > **No**, since $f'(x) = 3x^2 - 1$ changes sign (e.g., decreasing around $x = 0$, increasing when $|x| > \frac{1}{\sqrt{3}}$).

5. **True or False**: The set of rational numbers $\mathbb{Q}$ is countable. [Proposition 2.6]
   > **True** ($\mathbb{Q}$ is countable because it can be enumerated as a sequence.)

II. Sequences (Chapter 3)

1. **True or False**: Every bounded sequence of real numbers converges. [Proposition 3.3]
   > **False** (Bounded sequences need not converge; e.g., $a_n = (-1)^n$ is bounded but divergent.)

2. **Short Answer**: Consider the sequence $a_n = (-1)^n \cdot \left(1 + \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Find its limit inferior ($\liminf_{n\to\infty} a_n$) and limit superior ($\limsup_{n\to\infty} a_n$). [Definition 3.10, Example 3.12]
   > $\liminf_{n\to\infty} a_n = -1$, $\limsup_{n\to\infty} a_n = 1$ (Odd terms $\to -1$, even terms $\to 1$.)

3. **True or False**: If a sequence $(a_n)_{n\in\mathbb{N}}$ diverges to $-\infty$, then $\liminf_{n\to\infty} a_n = -\infty$ and $\limsup_{n\to\infty} a_n = -\infty$. [Exercise 3.10]
   > **True** (Divergence to $-\infty$ implies both $\liminf$ and $\limsup$ are $-\infty$.)

III. Measurable Spaces & Measures (Chapters 4 & 5)

1. **True or False**: The family $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}\}$ is a σ-field on $\Omega = \{a, b, c\}$. [Definition 4.1]
   > **False** (Missing $\{a, b, c\}$; a σ-field must include Ω itself.)

2. **Short Answer**: For a non-empty set Ω, what is the smallest possible σ-field on Ω? What is the largest possible σ-field on Ω? [Example 4.1, Example 4.2]
   > **Smallest**: $\{\emptyset, \Omega\}$. **Largest**: $\mathcal{P}(\Omega)$.

3. **True or False**: Given a measurable space $(\Omega, \mathcal{F})$, the set function $\mu(A) = 5 \cdot \#A$ (if A is finite, ∞ otherwise) for $A \in \mathcal{P}(\Omega)$ is a measure on $\mathcal{P}(\Omega)$. [Proposition 5.2, Definition 5.1]
   > **True** (It satisfies $\mu(\emptyset) = 0$ and countable additivity.)

4. **Short Answer**: Let δ_x be the Dirac measure at a point $x \in \Omega$. What is $\delta_x(A)$ if $x \notin A$? [Definition 2, Example 5.1]
   > $\delta_x(A) = 0$ (Dirac measure assigns 1 only if $x \in A$; else 0.)

5. **True or False**: For any two sets A, B in a measurable space $(\Omega, \mathcal{F})$ with measure μ, if $A \subset B$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$ is always true. [Proposition 5.3 (iii)]
   > **False** (Only true if $\mu(A) < \infty$; else $\mu(B) - \mu(A)$ is undefined.)

IV. Measurable Functions & Integration (Chapters 7, 8 & 9)

1. **True or False**: Any continuous function $f : \mathbb{R}^m \to \mathbb{R}^k$ is a Borel function. [Proposition 7.2]
   > **True** (Continuous functions are Borel-measurable since pre-images of open sets are open.)

2. **Short Answer**: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. If $f : \Omega \to \mathbb{R}$ is a non-negative $\mathcal{F}$-measurable function such that $f(\omega) = 0$ for all ω except for a set $N \in \mathcal{F}$ with $\mu(N) = 0$, what can you say about $\int_\Omega f(\omega)\mu(d\omega)$? [Example 8.3]
   > $\int_\Omega f(\omega)\mu(d\omega) = 0$ (Integral over a null set is zero.)

3. **Short Answer**: Calculate the integral: $\int_{\mathbb{N}} \mathbf{1}_{\{1,2,3\}}(x)\mu(dx)$, where μ is the counting measure on $\mathcal{P}(\mathbb{N})$. [Exercise 1 (d) from Mock Exam 1 (Solutions).pdf, Section 5.1.2]
   > $= 3$ (Counting measure gives 1 for each of $\{1, 2, 3\}$, sum $= 3$.)

4. **True or False**: If $f : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function, then $|f|$ is also Borel measurable. [Exercise 7.5 (c) and (b)]
   > **True** (Absolute value preserves measurability.)

5. **Short Answer**: Let $\phi(x) = \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$. Calculate $\int_{\mathbb{R}} x\phi(x)dx$. [Proposition 9.4, Example 9.2]
   > $= 0$ (Odd function $x\phi(x)$ integrated over symmetric interval $[-1, 1]$.)

V. Probability (Chapter 10, 11 & 12)

1. **True or False**: If A and B are two events such that $P(A \cap B) = P(A)P(B)$, then A and B are independent. [Definition 11.1]
   > **True** (Definition of independence.)

2. **Short Answer**: A discrete random variable X has support $E = \{0, 1, 2\}$ and $P(X = 0) = 0.3$, $P(X = 1) = 0.5$, $P(X = 2) = 0.2$. Calculate $E[X]$ and $\mathrm{Var}(X)$. [Section 10.3, Definition 10.11]
   > - $E[X] = 0.9$, $\mathrm{Var}(X) = 0.49$:
   > - $E[X] = 0 \cdot 0.3 + 1 \cdot 0.5 + 2 \cdot 0.2 = 0.9$.
   > - $E[X^2] = 0^2 \cdot 0.3 + 1^2 \cdot 0.5 + 2^2 \cdot 0.2 = 1.3 \implies \mathrm{Var}(X) = 1.3 - 0.9^2 = 0.49$.

3. **True or False**: If a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges in probability to X (i.e., $X_n \to_P X$), then it must also converge almost surely to X (i.e., $X_n \to_{a.s.} X$). [Proposition 12.3 and Example 12.2]
   > **False** (Convergence in probability does not imply a.s. convergence; e.g., "drifting spikes" counterexample.)

4. **True or False**: If X and Y are two independent random variables, then $E[XY] = E[X]E[Y]$. [Proposition 11.2]
   > **True** (Independence implies $E[XY] = E[X]E[Y]$.)

5. **Short Answer**: Let $X \sim \mathrm{Poisson}(\lambda)$, where $\lambda > 0$. What is $E[X]$? [Example 10.11, Exercise 10.3]
   > $E[X] = \lambda$ (Mean of $\mathrm{Poisson}(\lambda)$.)

Mathematics and Probability Questions

Chapter 1: Primary tools: First part

1. Define the Cartesian product of $A_1,\ldots,A_n$ sets.

> A set of all ordered n-tuples $(a_1,\ldots,a_n)$ where $a_i \in A_i$ for each i: $A_1 \times \cdots \times A_n = \{(a_1,\ldots,a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}$.

2. State the distributive law relating union and intersection of sets.

> $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

3. Explain the concept of the empty set and provide one of its elementary properties with respect to any given set A.

> The empty set $\varnothing$ contains no elements. Elementary property: $\varnothing \subseteq A$ for any set A.

4. Given a set Ω, define the complement of a subset A in Ω, denoted A^c.

> $A^c = \{\omega \in \Omega \mid \omega \notin A\}$.

5. What is the purpose of the principle of induction in mathematics, as indicated in the text?

> To prove statements for all natural numbers by verifying a base case and an inductive step.

Chapter 2: Primary tools: Second part

1. What is the difference between a surjective function and an injective function?

> Surjective: Range = codomain (every element in codomain is mapped to). Injective: Distinct inputs map to distinct outputs.

2. When are two sets A and B said to have the same cardinality (#A = #B)?

> When there exists a bijection (one-to-one correspondence) between them.

3. Provide the definition of an open set $U \subset \mathbb{R}^k$.

> $\forall x \in U, \exists \varepsilon > 0$ such that the open ball $B(x,\varepsilon) \subseteq U$.

4. If $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$, is a strictly monotonic function, what can be concluded about its inverse function on f(I)?

> The inverse f^{-1} exists and is strictly monotonic on f(I).

5. State the Euclidean norm of a vector $x \in \mathbb{R}^k$ and the Euclidean distance between two points $x, y \in \mathbb{R}^k$.

> $\|x\| = \sqrt{(\sum_i x_i^2)}$, $d(x,y) = \|x - y\|$.

Chapter 3: Primary tools: Third part

1. Define a real-valued sequence.

> A function from $\mathbb{N}$ to $\mathbb{R}$, denoted $(a_n)_{n \in \mathbb{N}}$ where $a_n \in \mathbb{R}$.

2. What is a monotonic sequence?

> Non-decreasing ($a_n \leq a_{n+1}\ \forall n$) or non-increasing ($a_n \geq a_{n+1}\ \forall n$).

3. Explain the concept of a series $\sum_{i\in\mathbb{N}} a_i$ as a sequence.

> Defined via partial sums $S_n = \sum_{i=1}^n a_i$; the series converges if (S_n) converges.

4. Define an accumulation point of a sequence $(a_n)_{n\in\mathbb{N}}$.

> A point L such that $\forall\varepsilon>0$, infinitely many a_n lie in $(L-\varepsilon, L+\varepsilon)$.

5. If a sequence $(a_n)_{n\in\mathbb{N}}$ is increasing and diverges, what is its limit?

> $\lim a_n = +\infty$.

Chapter 4: Measurable spaces

1. List the three properties that define a σ-field $\mathcal{F}$ on a nonempty set Ω.

> (i) $\Omega \in \mathcal{F}$, (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, (iii) $A_n \in \mathcal{F}\ \forall n \Rightarrow \cup_n A_n \in \mathcal{F}$.

2. What is the largest possible σ-field on a nonempty set Ω, and what is it commonly denoted as?

> The power set of Ω, denoted $\mathcal{P}(\Omega)$.

3. If $\{A_i : i \in \mathbb{N}\}$ is a collection of sets belonging to a σ-field $\mathcal{F}$, what can be said about their intersection?

> $\cap_i A_i \in \mathcal{F}$ (closed under countable intersections).

4. Define the Borel σ-field $\mathcal{B}(\mathbb{R}^k)$ on $\mathbb{R}^k$.

> The σ-field generated by all open sets in $\mathbb{R}^k$.

5. Is the union of two σ-fields on Ω necessarily a σ-field on Ω?

> No, not necessarily (fails closure properties).

Chapter 5: Measure spaces

1. What two conditions must a function $\mu: \mathcal{F} \to \mathbb{R}^+$ satisfy to be considered a measure on a measurable space $(\Omega,\mathcal{F})$?

> (i) $\mu(\varnothing) = 0$, (ii) Countable additivity: $\mu(\cup_i A_i) = \sum_i \mu(A_i)$ for disjoint $A_i \in \mathcal{F}$.

2. Define a "point measure" $\mu(A) = \sum_{i\in I} \alpha_i\delta_{x_i}(A)$.

> Dirac masses: $\delta_{x_i}(A) = 1$ if $x_i \in A$, else 0. μ assigns weights α_i to points x_i.

3. In the context of a finite set Ω, how is the counting measure μ(A) defined on the power set $\mathcal{P}(\Omega)$?

> μ(A) = number of elements in A.

4. Given a measure μ on $(\Omega,\mathcal{F})$, if $A, B \in \mathcal{F}$ such that $A \subset B$, what relationship holds between $\mu(A)$ and $\mu(B)$?
> $\mu(A) \leq \mu(B)$ (monotonicity).

5. List the three properties that define a semiring on Ω.
> (i) $\varnothing \in \mathcal{A}$, (ii) Closed under finite intersections, (iii) If $A,B \in \mathcal{A}$, $A \subseteq B$, then $B \setminus A$ is a finite disjoint union of sets in $\mathcal{A}$.

Chapter 6: From outer measure to measure

1. Define a countable covering of a set A by sets from a collection $\mathcal{A}$.
> A sequence $(E_n) \in \mathcal{A}$ such that $A \subseteq \cup_n E_n$.

2. What is an outer measure μ^*, and what three properties does it satisfy?
> (i) $\mu^*(\varnothing)=0$, (ii) Monotonicity: $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$, (iii) Subadditivity: $\mu^*(\cup A_n) \leq \sum \mu^*(A_n)$.

3. How is the outer measure ℓ^* defined for a set $A \in \mathcal{P}(\mathbb{R})$?
> $\ell^*(A) = \inf\{\sum_i (b_i - a_i) : A \subseteq \cup_i (a_i,b_i]\}$ (inf over countable open coverings by intervals).

4. Is every Borel set also Lebesgue measurable?
> Yes.

5. What is the relationship between $\mu^*(A \cap E) + \mu^*(A^c \cap E)$ and $\mu^*(E)$ for a μ^*-measurable set A?
> $\mu^*(A \cap E) + \mu^*(A^c \cap E) = \mu^*(E)$ $\forall E \subseteq \Omega$ (Carathéodory condition).

Chapter 7: Measurable functions

1. When is a function $f: (\Omega,\mathcal{F}) \to (\Omega^*,\mathcal{F}^*)$ considered measurable?
> $f^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{F}^*$.

2. Can any continuous function $f: \mathbb{R}^m \to \mathbb{R}^k$ be classified as a Borel function?
> Yes (preimages of open sets under continuous functions are Borel measurable).

3. What defines a "standard" simple function?
> $f = \sum_{i=1}^n \alpha_i \mathbb{1}_{\{A_i\}}$ with $\alpha_i \in \mathbb{R}$, disjoint $A_i \in \mathcal{F}$.

4. For $\mathcal{F}$-measurable functions f, g: $\Omega \to \mathbb{R}$, what can be said about the set $\{\omega \in \Omega: f(\omega) = g(\omega)\}$?
> It belongs to $\mathcal{F}$.

5. If a function $f: \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable and nonnegative, does there exist a sequence of standard simple functions $(f_n)_{\{n\in\mathbb{N}\}}$ that approximates f?

> Yes, $0 \le f_n \uparrow f$ pointwise.

Chapter 8: Integration: First part

1. What is a partition of a set Ω?
   > A disjoint collection $\{A_1,\dots,A_n\} \subseteq \mathcal{F}$ with $\cup_i A_i = \Omega$.

2. How is the integral of a simple function $f(\omega) = \sum_{i=1}^N \alpha_i A_i(\omega)$ defined over Ω with respect to a measure μ?
   > $\int f\, d\mu = \sum_i \alpha_i\, \mu(A_i)$ (assuming $\mu(A_i) < \infty$ or $\alpha_i \ge 0$).

3. What does it mean for a random variable X to be "integrable" with respect to a probability measure P?
   > $\mathbb{E}[|X|] = \int |X|\, dP < \infty$.

4. For a measure space $(\Omega,\mathcal{F},\mu)$ and a $\mathcal{F}$-measurable function $f: \Omega \to \mathbb{R}$, if $\{A_i : i \in I\} \subset \mathcal{F}$ is a disjoint collection with $I \subset \mathbb{N}$, how can the integral of f over $\cup_{i\in I} A_i$ be expressed?
   > $\int_{\cup A_i} f\, d\mu = \sum_i \int_{A_i} f\, d\mu$.

5. In a measure space $(\mathbb{N},\mathcal{P}(\mathbb{N}),\mu)$ where μ is the counting measure, how does the integral of a nonnegative $\mathcal{P}(\mathbb{N})$-measurable function $f: \mathbb{N} \to \mathbb{R}$ relate to summation?
   > $\int f\, d\mu = \sum_{n=1}^\infty f(n)$.

Chapter 9: Integration: Second part

1. What is the effect of a measure μ on $(\Omega,\mathcal{F})$ being pushed forward by a measurable function $g: (\Omega,\mathcal{F}) \to (\Omega^*,\mathcal{F}^*)$?
   > Defines a measure $\mu \circ g^{-1}$ on $(\Omega^*,\mathcal{F}^*)$ by $\mu \circ g^{-1}(B) = \mu(g^{-1}(B))$ for $B \in \mathcal{F}^*$.

2. How is the integral of a continuous function $f: [a,b] \to \mathbb{R}$ defined in terms of its antiderivative?
   > Via the fundamental theorem: If $F' = f$, then $\int_{a}^{b} f\, dx = F(b) - F(a)$.

3. State the Substitution Rule for integrals involving continuous and differentiable functions.
   > $\int_{a}^{b} f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$.

4. If a continuous function $f: [-a,a] \to \mathbb{R}$ is odd, what is the value of $\int_{-a}^a f(x)dx$?
   > 0.

5. What is the Fubini-Tonnelli theorem, and what conditions does it require for measures?

> Allows interchange of integration order for product measures. Requires nonnegativity or integrability of the integral of absolute values.

Chapter 10: General notions in probability

1. How is a probability P defined on a measurable space $(\Omega,\mathscr{F})$?
   > A measure $P: \mathscr{F} \to [0,1]$ satisfying $P(\Omega) = 1$.
2. What distinguishes a random variable from a random vector?
   > Random variable: maps to $\mathbb{R}$. Random vector: maps to $\mathbb{R}^k$, $k \geq 1$.
3. When is a random vector X considered "discrete"?
   > Takes values in a countable subset of $\mathbb{R}^k$.
4. Define a "continuous" random vector, and what is its associated function called?
   > Has an absolutely continuous law w.r.t. Lebesgue measure; characterized by a probability density function (pdf).
5. State Chebyshev's inequality.
   > For X with mean μ and variance σ^2: $P(|X - \mu| \geq k\sigma) \leq 1/k^2$ $\forall k > 0$.

Chapter 11: Collections of random vectors

1. If $X_1,\ldots,X_n$ are n independent random vectors, what relationship holds for $P(\{X_1 \in B_1\}\cap\cdots\cap\{X_n \in B_n\})$ for measurable sets $B_1,\ldots,B_n$?
   > $P(\cap_i\{X_i \in B_i\}) = \prod_i P(X_i \in B_i)$.
2. What is the law of an independent collection of random vectors in terms of their individual laws?
   > Joint law is the product measure: $P_{(X_1,\ldots,X_n)} = P_{X_1} \otimes \quad \otimes P_{X_n}$.
3. If $X_1,\ldots,X_n$ are independent random variables such that $\mathbb{E}[|X_i|] < \infty$ for all i, what can be said about $\mathbb{E}[\sum_{i=1}^n X_i]$?
   > $\mathbb{E}[\sum X_i] = \sum \mathbb{E}[X_i]$.
4. What is the characteristic function of the sum of n independent random vectors?
   > $\phi_{\sum X_i}(t) = \prod_i \phi_{X_i}(t)$.
5. If $X = (X_1,\ldots,X_k)$ is a Gauss vector, what can be said about each of its components $X_1,\ldots,X_k$?
   > Each is a univariate Gaussian random variable.

Chapter 12: Convergence of random vectors

1. When does a sequence of random vectors (X_n) converge to X in L^p ($p \geq 1$)?

   > If lim $\mathbb{E}[|X_n - X|^p] = 0$.

2. If ∑_{n=1}^∞ P(A_n) < ∞ for a sequence of events {A_n: n∈ℕ} ⊂ $\mathscr{F}$, what does the Borel-Cantelli lemma state about P(lim sup_{n→∞}A_n)?

   > P(lim sup A_n) = 0 (infinitely many A_n occur with probability 0).

3. What is the relationship between convergence in L^1 and convergence in probability?

   > Convergence in L^1 implies convergence in probability.

4. If a sequence of random vectors (X_n) converges to X in probability, what can be said about the existence of a subsequence that converges almost surely?

   > ∃ subsequence (X_{n_k}) such that X_{n_k} → X a.s.

5. State the Law of Large Numbers for a sequence of i.i.d. random variables.

   > If X_n i.i.d. with $\mathbb{E}[|X_1|] < \infty$, then $(1/n)\sum_{i=1}^{n} X_i \to \mathbb{E}[X_1]$ almost surely and in L^1.

part 2

Mathematics and Probability Questions

Chapter 1: Primary tools: First part

1. **Define what it means for two sets A and B to be equal.**

   > Two sets are equal ($A = B$) if $A \subset B$ and $B \subset A$.

2. **Given three sets A, B, and C, state the associative law for union.**

   > $(A \cup B) \cup C = A \cup (B \cup C)$.

3. **Let A and B be two sets. Show that $A \cap B \subset A$.**

   > For any $x \in A \cap B$, by definition $x \in A$ and $x \in B$, thus $x \in A$.

4. **Given a set Ω, what is the complement of the empty set $\emptyset^c$ and the complement of Ω itself Ω^c?**

   > $\emptyset^c = \Omega$ and $\Omega^c = \emptyset$.

5. **State the definition of an upper bound for a set $A \subset \mathbb{R}$.**
   > $b \in \mathbb{R}$ is an upper bound for A if $\forall a \in A$, $a \leq b$.

Chapter 2: Primary tools: Second part

1. **Let $f : A \to B$ be a function. Define its image $f(C)$ for a subset $C \subset A$.**
   > $f(C) = \{f(x) \in B : x \in C\}$.

2. **Provide an example of a function $f : \mathbb{R} \to [0, \infty)$ that is surjective but not injective.**
   > $f(x) = x^2$.

3. **Given an open set $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{R}^k$, is their intersection $U_1 \cap U_2$ necessarily open?**
   > Yes - finite intersection of open sets is open.

4. **Define what it means for a set $V \subset \mathbb{R}^k$ to be closed.**
   > Its complement V^c is open.

5. **When are two sets A and B said to have cardinality $\#A \leq \#B$?**
   > If there exists an injective mapping from A to B.

Chapter 3: Primary tools: Third part

1. **What condition must a sequence $(a_n)_{n \in \mathbb{N}}$ satisfy to be considered convergent?**
   > $\exists L \in \mathbb{R}$ such that $\forall \epsilon > 0$, $\exists N$ with $|a_n - L| < \epsilon$ for $n \geq N$.

2. **If a sequence $(a_n)_{n \in \mathbb{N}}$ is convergent, what can be concluded about its boundedness?**
   > All convergent sequences are bounded.

3. **Let (a_n) and (b_n) be two convergent sequences with limits a and b. What is $\lim(a_n b_n)$?**
   > ab.

4. **Define divergence to ∞ for a sequence $(a_n)_{n \in \mathbb{N}}$.**
   > $\forall M > 0$, $\exists N$ such that $a_n > M$ for $n \geq N$.

5. **State the Bolzano-Weierstrass theorem.**
   > Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Chapter 4: Measurable spaces

1. **What is the smallest σ-field on a nonempty set Ω?**
   > $\{\emptyset, \Omega\}$.

2. **Show that if $\{A_i\} \subset \mathcal{F}$, then $\cap_i A_i \in \mathcal{F}$.**

> By De Morgan's: $\cap_i A_i = (\cup_i A_i^c)^c \in \mathcal{F}$.

3. **Can $G = \{A \subset \Omega : A \text{ finite or } A^c \text{ finite}\}$ be a σ-field if Ω infinite?**

   > No, because infinite unions of finite sets may not be in G.

4. **Define $\mathcal{B}(\mathbb{R})$.**

   > The σ-field generated by $\{(a, b] : a < b\}$.

5. **Define $\sigma(f)$ where $f : X \to Y$.**

   > $\sigma(f) = \{f^{-1}(B) : B \in \mathcal{B}\}$.

Chapter 5: Measure spaces

1. **Define the Dirac measure δ_x.**

   > $\delta_x(A) = 1$ if $x \in A$, else 0.

2. **If $A \subset B$ and $\mu(A) < \infty$, what is $\mu(B \setminus A)$?**

   > $\mu(B) - \mu(A)$.

3. **State the relationship between $\mu(A \cup B)$ and $\mu(A) + \mu(B)$.**

   > $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

4. **For nested sets A_i, what is $\mu(\cup_i A_i)$?**

   > $\lim_n \mu(A_n)$ (continuity from below).

5. **Is $\{(a, b]\} \cup \{\emptyset\}$ a semiring?**

   > Yes, because intersection of two left-open intervals satisfies semiring properties.

Chapter 6: From outer measure to measure

1. **State Lebesgue measure λ on $(a, b]$.**

   > $\lambda((a, b]) = b - a$.

2. **What is $\ell^*(\{a\})$?**

   > 0 (singletons are null sets).

3. **Is outer measure μ^* finitely additive?**

   > No, only countably subadditive.

4. **Are there non-Lebesgue measurable sets?**

   > Yes (Vitali sets exist under AC).

5. **When is measure unique?**

> If two measures agree on a π-system generating the σ-field.

Chapter 7: Measurable functions

1. **Define Borel function $f : \mathbb{R}^m \to \mathbb{R}^k$.**
   > Preimages of Borel sets are Borel.
2. **Sufficient condition for measurability?**
   > $f^{-1}((-\infty, a]) \in \mathcal{F}$ for all $a \in \mathbb{R}$.
3. **Measurability of composition $g(f_1, ..., f_k)$?**
   > It remains measurable.
4. **Define f^+ and f^-.**
   > $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$.
5. **Can any measurable f be approximated by simple functions?**
   > Yes, via $f_n = \sum_{k=-n2^n}^{n2^n} k2^{-n} 1_{\{k2^{-n} \leq f < (k+1)2^{-n}\}}$.

Chapter 8: Integration: First part

1. **Monotonicity of integral:**
   > $\int f d\mu \leq \int g d\mu$.
2. **Monotone Convergence Theorem:**
   > If $f_n \uparrow f$, then $\int f_n d\mu \uparrow \int f d\mu$.
3. **Linearity of integral:**
   > $\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$.
4. **Integral over null set:**
   > 0 (since μ-a.e. equal to 0).
5. **Integral with respect to sum measure:**
   > $\int f d(\sum_i \mu_i) = \sum_i \int f d\mu_i$.

Chapter 9: Integration: Second part

1. **Integral with density ϕ:**
   > $\int_A f(\omega)\phi(\omega)\mu(d\omega)$.
2. **Lebesgue integrability:**
   > When both $\int_E f^+ d\lambda$ and $\int_E f^- d\lambda$ are finite.

3. **Integration by parts:**
   > $\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g.$

4. **Even function integral:**
   > $\int_{-a}^a f = 2\int_0^a f.$

5. **Product σ-field:**
   > $\sigma(\{A \times B : A \in \mathcal{X}, B \in \mathcal{Y}\}).$

Chapter 10: General probability

1. **Law of random vector X:**
   > $P_X(B) = P(X \in B).$

2. **Discrete law:**
   > $P_X = \sum_{x \in E} P(X = x)\delta_x.$

3. **Classical discrete distributions:**
   > Bernoulli and Poisson.

4. **Expectation of continuous X:**
   > $E[X] = \int_{-\infty}^{\infty} x\phi(x)dx.$

5. **Properties of variance:**
   > i. $\mathrm{Var}(X) \geq 0$
   > ii. $\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$
   > iii. $\mathrm{Var}(X) = E[X^2] - E[X]^2$

Chapter 11: Random vectors

1. **Independent vectors:**
   > $P_X = \otimes_{i=1}^n P_{X_i}$ (product measure).

2. **Covariance of independent variables:**
   > $0.$

3. **PDF of sum:**
   > $\phi_{X_1+X_2} = \phi_1 * \phi_2$ (convolution).

4. **Gauss vector:**
   > X where $\langle v, X\rangle$ is Gaussian $\forall v$.

5. **Characteristic function:**
   > $\Phi_X(v) = \exp(i\langle \mu, v\rangle - \frac{1}{2}v^T \Sigma v)$.

Chapter 12: Convergence

1. **Almost sure convergence:**
   > $P(\lim_n X_n = X) = 1$.

2. **limsup of events:**
   > $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ ("infinitely often").

3. **a.s. implies in probability?**
   > Yes.

4. **Does convergence in probability imply a.s. of subsequence?**
   > Yes (via Bolzano-Weierstrass argument).

5. **L^1 convergence condition:**
   > When dominated by integrable Y ($|X_n| \leq Y$).