Mathematical Foundations Cheat Sheet

1. Sets and Set Operations

Basic Definitions

- **Set**: A collection of distinct elements (e.g., `A = {1, 2, 3}`)
- **Element**:
  - $\omega \in A$ (element is in set)
  - $\omega \notin A$ (element is not in set)

Important Sets

Symbol	Name	Description
$\mathbb{N}$	Natural Numbers	Positive integers (1, 2, 3, ...)
$\mathbb{Z}$	Integers	(..., -2, -1, 0, 1, 2, ...)
$\mathbb{Q}$	Rational Numbers	Fractions p/q where $p,q \in \mathbb{Z}$, $q \neq 0$
$\mathbb{R}$	Real Numbers	Includes $\sqrt{2}$, π, e, etc.
$\mathbb{C}$	Complex Numbers	Numbers of form a + bi, $i^2 = -1$

Set Operations

- **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- **Intersection**: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- **Difference**: $A \setminus B = \{x \in A \mid x \notin B\}$
- **Complement**: If $A \subseteq \Omega$, then $A^c = \Omega \setminus A$
- **Subset**: $A \subseteq B$ if all elements of A are in B

Special Cases

- **Empty Set ($\varnothing$)**: The set containing no elements
  - Property: $\varnothing \subseteq A$ for any set A

- **Disjoint Sets**: $A \cap B = \varnothing$ (no common elements)

Key Properties

- **Density of $\mathbb{Q}$ in $\mathbb{R}$**: Between any two real numbers, there exists a rational number
- **Completeness of $\mathbb{R}$**: Every bounded subset has both a supremum (least upper bound) and infimum (greatest lower bound) in $\mathbb{R}$

2. Functions

Basic Definition

A **function** $f: A \to B$ is a rule that assigns each element $a \in A$ to exactly one element $f(a) \in B$.

Important Concepts

- **Image**: $f(E) = \{f(a) \mid a \in E\}$ for $E \subseteq A$
- **Preimage**: $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$ for $D \subseteq B$

Function Types

Type	Condition	Example
Injective	$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ (one-to-one)	$f(x) = 2x$
Surjective	$f(A) = B$ (onto)	$f: \mathbb{R} \to [0,\infty)$, $f(x)=x^2$
Bijective	Both injective and surjective (invertible)	$f: \mathbb{R} \to \mathbb{R}$, $f(x)=x^3$

Monotonicity

- **Increasing**: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
- **Strictly Increasing**: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Continuity

- **Topological Definition**: $f: \mathbb{R}^m \to \mathbb{R}^k$ is continuous if $f^{-1}(U)$ is open for every open $U \subseteq \mathbb{R}^k$
- **Sequence Criterion**: f is continuous at x iff for all sequences $x_n \to x$, we have $f(x_n) \to f(x)$

Important Properties

- Preimage respects:
  - Unions: $f^{-1}(\cup_j D_j) = \cup_j f^{-1}(D_j)$
  - Intersections: $f^{-1}(\cap_j D_j) = \cap_j f^{-1}(D_j)$
- Bijective functions have unique inverses

3. Sequences

Basic Definition

A **real sequence** $(a_n)_{n\in\mathbb{N}}$ is a function from $\mathbb{N}$ to $\mathbb{R}$.

Convergence

We say $\lim a_n = a$ if:
$\forall\varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - a| < \varepsilon$

Divergence

- $a_n \rightarrow +\infty$ if $\forall M > 0, \exists N, \forall n \geq N, a_n > M$
- $a_n \rightarrow -\infty$ if $\forall M < 0, \exists N, \forall n \geq N, a_n < M$

Key Properties

- **Boundedness**: Sequence is bounded if $\exists M > 0$ st $|a_n| \leq M$ $\forall n$
- **Monotonic Sequences**:
  - Increasing: $a_{n+1} \geq a_n$
  - Decreasing: $a_{n+1} \leq a_n$
  - **Theorem**: Monotonic + Bounded $\Rightarrow$ Convergent

Subsequences

- A subsequence (a_{kn}) is obtained by selecting terms with increasing indices
- **Bolzano-Weierstrass Theorem**: Every bounded sequence has a convergent