Cheat Sheet: Set Functions Failing Countable Additivity

Set functions that violate countable additivity (σ-additivity) despite satisfying other measure properties.

Table of Examples

Set Function	Defined On	Definition	Counterexample	Failure Reason
Point-Lebesgue Product	$(\mathbb{R}, B(\mathbb{R}))$	`μ(A) = λ(A)·δ₀(A)` where λ = Lebesgue measure, $\delta_0(A) = 1\_\{0\}(A)$	$A_1 = \{0\}$, $A_2 = (1,2)$ $\mu(A_1) = 0$, $\mu(A_2) = 0$ $\mu(A_1 \cup A_2) = \mu([0,2)) = 1 \neq 0$	Dirac mass dominates Lebesgue null sets when $0 \in A$
Finite/Infinite Dichotomy	$(\mathbb{N}, \mathcal{P}(\mathbb{N}))$	`μ(A) = {0 if A finite, ∞ if A infinite}`	$A_n = \{n\}$ (disjoint) $\mu(A_n) = 0\ \forall n$ $\mu(\cup A_n) = \mu(\mathbb{N}) = \infty \neq 0$	Finitely additive but not σ-additive
Binary Jump	$([0,1], B([0,1]))$	`μ(A) = {0 if λ(A)=0, 1 if λ(A)>0}`	$A_n = (1/(n+1), 1/n]$ $\mu(A_n) = 1\ \forall n$ $\mu(\cup A_n) = \mu((0,1]) = 1 \neq \Sigma 1 = \infty$	Measures only presence of positive Lebesgue measure
Asymptotic Density	$(\mathbb{N}, \mathcal{P}(\mathbb{N}))$	`μ(A) = lim_{n→∞}	A∩{1,...,n}	/n` (if exists)

Additional Examples

| **Logarithmic Density** | $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ | `μ(A) = lim_{n→∞} 1/log(n) Σ_{k∈A∩[1,n]} 1/k` | $A_n = \{n\}$
$\mu(A_n) = 0\ \forall n$
$\mu(\mathbb{N}) = \infty$ | Harmonic series behavior breaks additivity |
| **Ultrafilter Measure** | $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ | `μ(A) = {1 if A ∈ 𝒰, 0 otherwise}`
($\mathcal{U}$ = free ultrafilter) | $A_n = \{2^n\}$

$\mu(A_n) = 0 \ \forall n$
$\mu(\cup A_n) = \mu(\mathbb{N}) = 1$ | Non-measurability under ultrafilter |

Failure Patterns

1. **Point Mass Interference**: Dirac measures can override Lebesgue null sets
2. **Finite vs. Infinite**: Infinite unions of finite/null sets may have non-zero measure
3. **Discontinuous Measures**: Threshold-based measures ignore set granularity
4. **Asymptotic Limits**: Limits may not commute with infinite sums
5. **Non-σ Structures**: Ultrafilters and similar constructs violate countable additivity