Original Exercise & Solution

Exercise 4 (6 points).
Let X_1 and X_2 be two random variables that are independent with a common uniform distribution on $[0, 1]$.
Question: What is the PDF of the random vector $Y = \left(\frac{1+X_1}{2}, X_2\right)$?

> **Solution:**
>
> - **Transformation:** $Y_1 = \frac{1+X_1}{2}, Y_2 = X_2$.
> - **Inverse:** $X_1 = 2Y_1 - 1, X_2 = Y_2$.
> - **Jacobian:**
>
> $$|J| = \det \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 2.$$
>
> - **Support for Y:**
> $\frac{1}{2} \le Y_1 \le 1$ (from $0 \le 2Y_1 - 1 \le 1$) and $0 \le Y_2 \le 1$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} 2 & \text{if } (y_1, y_2) \in [\frac{1}{2}, 1] \times [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Variation 1: Simple Linear Transformation

Exercise:
Let $X_1, X_2 \sim \text{Uniform}(0, 1)$ be independent. Find the PDF of $Y = (2X_1, 3X_2)$.

> **Solution:**
>
> - **Inverse:** $X_1 = Y_1/2, X_2 = Y_2/3$.
> - **Jacobian:** $|J| = \det \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix} = \frac{1}{6}$.
> - **Support:** $Y \in [0, 2] \times [0, 3]$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} \frac{1}{6} & \text{if } (y_1, y_2) \in [0, 2] \times [0, 3], \\ 0 & \text{otherwise.} \end{cases}$$

Variation 2: Coupled Linear Transformation

Exercise:
For $X_1, X_2 \sim \text{Uniform}(0,1)$, find the PDF of $Y = (X_1 + X_2, X_1 - X_2)$.

> **Solution:**
>
> - **Inverse:** $X_1 = \frac{Y_1+Y_2}{2}, X_2 = \frac{Y_1-Y_2}{2}$.
> - **Jacobian:** $|J| = \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \frac{1}{2}$.
> - **Support:** Parallelogram bounded by $y_1 + y_2 = 0, 2$ and $y_1 - y_2 = 0, 2$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} \frac{1}{2} & \text{in the support,} \\ 0 & \text{otherwise.} \end{cases}$$

Variation 3: Non-Linear Transformation

Exercise:
For $X_1, X_2 \sim \text{Uniform}(0,1)$, find the PDF of $Y = (X_1^2, X_2)$.

> **Solution:**
>
> - **Inverse:** $X_1 = \sqrt{Y_1}, X_2 = Y_2$.
> - **Jacobian:** $|J| = \det \begin{pmatrix} \frac{1}{2\sqrt{y_1}} & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2\sqrt{y_1}}$.
> - **Support:** $Y \in [0,1] \times [0,1]$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} \frac{1}{2\sqrt{y_1}} & \text{if } (y_1, y_2) \in (0,1] \times [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Variation 4: Exponential Distribution

Exercise:
Let $X_1, X_2 \sim \text{Exp}(1)$. Find the PDF of $Y = \left(\frac{1+X_1}{2}, X_2\right)$.

> **Solution:**
>
> - **Inverse/Jacobian:** Identical to Original (but $f_X(x_1, x_2) = e^{-(x_1+x_2)}$).
> - **Support:** $Y_1 \geq \frac{1}{2}, Y_2 \geq 0$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} 2e^{1-2y_1-y_2} & \text{if } y_1 \geq \frac{1}{2}, y_2 \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Variation 5: Ratio Transformation

Exercise:
For $X_1, X_2 \sim \text{Uniform}(0, 1)$, find the PDF of $Y = (X_1/X_2, X_2)$.

> **Solution:**
>
> - **Inverse:** $X_1 = Y_1 Y_2, X_2 = Y_2$.
> - **Jacobian:** $|J| = \det \begin{pmatrix} y_2 & y_1 \\ 0 & 1 \end{pmatrix} = y_2$.
> - **Support:** $0 \leq Y_2 \leq 1, 0 \leq Y_1 \leq \frac{1}{Y_2}$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} y_2 & \text{in the support,} \\ 0 & \text{otherwise.} \end{cases}$$

Variation 6: Order Statistics

Exercise:
Let $X_1, X_2 \sim \text{Uniform}(0, 1)$ be independent. Find the PDF of $Y = (Y_1, Y_2)$ where $Y_1 = \min(X_1, X_2)$ and $Y_2 = \max(X_1, X_2)$.

> **Solution:**

- **Note:** This transformation is not one-to-one. For any pair (y_1, y_2) in the support of Y, there are two points in the support of X, namely (y_1, y_2) and (y_2, y_1), that map to it. We must sum their contributions.
- **Inverse Mappings:**
  i. $X_1 = Y_1, X_2 = Y_2$
  ii. $X_1 = Y_2, X_2 = Y_1$
- **Jacobians:**
  i. $|J_1| = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$
  ii. $|J_2| = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |-1| = 1$
- **Support for Y:** Since $0 \le X_1, X_2 \le 1$, the support for Y is the triangle defined by $0 \le Y_1 \le Y_2 \le 1$.
- **PDF:**
  The PDF is given by $f_Y(y_1, y_2) = f_X(y_1, y_2)|J_1| + f_X(y_2, y_1)|J_2|$. Since $f_X(x_1, x_2) = 1$ on its support.

$$f_Y(y_1, y_2) = \begin{cases} 1 \cdot 1 + 1 \cdot 1 = 2 & \text{if } 0 \le y_1 \le y_2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Variation 7: Polar Coordinates Transformation

Exercise:
Let $X_1, X_2 \sim N(0,1)$ be independent. Find the PDF of (R, Θ) where $X_1 = R\cos\Theta$ and $X_2 = R\sin\Theta$.

Solution:

- **Original PDF:** $f_X(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}$.
- **Inverse:** The transformation is given, so the inverse maps from (x_1, x_2) to (r, θ). We need the Jacobian of the forward transformation $(r, \theta) \to (x_1, x_2)$.
- **Jacobian:**

$$|J| = \det \begin{pmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

- **Support for (R, Θ):** $x_1, x_2 \in (-\infty, \infty)$ implies $R \ge 0$ and $\Theta \in [0, 2\pi)$.
- **PDF:** Note that $x_1^2 + x_2^2 = r^2$.

$$f_{R,\Theta}(r,\theta) = f_X(r\cos\theta, r\sin\theta)|J| = \frac{1}{2\pi}e^{-r^2/2} \cdot r.$$

$$f_{R,\Theta}(r,\theta) = \begin{cases} \frac{r}{2\pi}e^{-r^2/2} & \text{if } r \geq 0, \theta \in [0, 2\pi), \\ 0 & \text{otherwise.} \end{cases}$$

Variation 8: Sum and One Variable (Exponential)

Exercise:
Let $X_1, X_2 \sim \mathrm{Exp}(1)$ be independent. Find the PDF of $Y = (X_1, X_1 + X_2)$.

Solution:

- **Original PDF:** $f_X(x_1, x_2) = e^{-(x_1+x_2)}$ for $x_1, x_2 > 0$.
- **Inverse:** $X_1 = Y_1, X_2 = Y_2 - Y_1$.
- **Jacobian:**

$$|J| = \det \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = 1.$$

- **Support for Y:** $X_1 > 0 \Rightarrow Y_1 > 0$. $X_2 > 0 \Rightarrow Y_2 - Y_1 > 0 \Rightarrow Y_2 > Y_1$. So the support is $0 < y_1 < y_2$.
- **PDF:**

$$f_Y(y_1, y_2) = f_X(y_1, y_2 - y_1)|J| = e^{-(y_1+(y_2-y_1))} \cdot 1 = e^{-y_2}.$$

$$f_Y(y_1, y_2) = \begin{cases} e^{-y_2} & \text{if } 0 < y_1 < y_2, \\ 0 & \text{otherwise.} \end{cases}$$

Variation 9: Logarithmic Transformation

Exercise:
Let $X_1, X_2 \sim \mathrm{Uniform}(0, 1)$ be independent. Find the PDF of $Y = (-\ln X_1, -\ln X_2)$.

Solution:

- **Inverse:** $X_1 = e^{-Y_1}, X_2 = e^{-Y_2}$.
- **Jacobian:**

$$|J| = \det \begin{pmatrix} -e^{-y_1} & 0 \\ 0 & -e^{-y_2} \end{pmatrix} = e^{-y_1}e^{-y_2} = e^{-(y_1+y_2)}.$$

- **Support for Y:** $X_1, X_2 \in (0,1) \implies -\ln X_1, -\ln X_2 \in (0,\infty)$. So $y_1 > 0, y_2 > 0$.
- **PDF:** $f_X(x_1,x_2) = 1$ on $(0,1)^2$.

$$f_Y(y_1,y_2) = 1 \cdot |e^{-(y_1+y_2)}| = e^{-y_1}e^{-y_2}.$$

$$f_Y(y_1,y_2) = \begin{cases} e^{-y_1}e^{-y_2} & \text{if } y_1 > 0, y_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

This shows that Y_1 and Y_2 are independent Exp(1) random variables.

Variation 10: Box-Muller Transformation

Exercise:
For $X_1, X_2 \sim \text{Uniform}(0,1)$, find the PDF of $Y = (Y_1, Y_2)$ where $Y_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2)$ and $Y_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2)$.

Solution:

- **Inverse:**
  $Y_1^2 + Y_2^2 = -2\ln X_1 \implies X_1 = e^{-(Y_1^2+Y_2^2)/2}$.
  $Y_2/Y_1 = \tan(2\pi X_2) \implies X_2 = \frac{1}{2\pi}\arctan(Y_2/Y_1)$.
- **Jacobian:** The Jacobian of the inverse transformation $(Y_1,Y_2) \to (X_1,X_2)$ is:

$$|J| = \left|\det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix}\right| = \left|-\frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}\right| = \frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}.$$

- **Support for Y:** $X_1, X_2 \in (0,1)$ maps to $(Y_1,Y_2) \in \mathbb{R}^2$.
- **PDF:**

$$f_Y(y_1,y_2) = f_X(x_1(y),x_2(y))|J| = 1 \cdot \frac{1}{2\pi}e^{-(y_1^2+y_2^2)/2}.$$

> This is the PDF of two independent standard normal random variables.

Variation 11: Ratio of Normal Variables

Exercise:
Let $X_1, X_2 \sim N(0,1)$ be independent. Find the joint PDF of $Y = (X_1/X_2, X_2)$.

> **Solution:**
>
> - **Original PDF:** $f_X(x_1, x_2) = \frac{1}{2\pi} e^{-(x_1^2+x_2^2)/2}$.
> - **Inverse:** $X_1 = Y_1 Y_2, X_2 = Y_2$.
> - **Jacobian:**
>
> $$|J| = \det \begin{pmatrix} y_2 & y_1 \\ 0 & 1 \end{pmatrix} = |y_2|.$$
>
> - **Support for Y:** $X_1, X_2 \in (-\infty, \infty) \implies Y_1, Y_2 \in (-\infty, \infty)$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = f_X(y_1 y_2, y_2)|J| = \frac{1}{2\pi} e^{-((y_1 y_2)^2 + y_2^2)/2} |y_2|.$$
>
> $$f_Y(y_1, y_2) = \begin{cases} \frac{|y_2|}{2\pi} e^{-y_2^2(1+y_1^2)/2} & \text{for } (y_1, y_2) \in \mathbb{R}^2, \\ 0 & \text{otherwise.} \end{cases}$$

Variation 12: Sum and Ratio of Exponentials

Exercise:
Let $X_1, X_2 \sim \text{Exp}(1)$ be independent. Find the PDF of $Y = (X_1 + X_2, \frac{X_1}{X_1+X_2})$.

> **Solution:**
>
> - **Inverse:** $X_1 = Y_1 Y_2, X_2 = Y_1(1 - Y_2)$.
> - **Jacobian:**

$$|J| = \det\begin{pmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{pmatrix} = |-y_1y_2 - y_1(1-y_2)| = |-y_1| = y_1 \quad (\text{since } Y_1 > 0).$$

- **Support for Y:** $X_1, X_2 > 0$ implies $Y_1 = X_1 + X_2 > 0$ and $Y_2 = \frac{X_1}{X_1+X_2} \in (0,1)$.
- **PDF:** $f_X(x_1, x_2) = e^{-(x_1+x_2)}$. Note $x_1 + x_2 = y_1$.

$$f_Y(y_1, y_2) = f_X(y_1y_2, y_1(1-y_2))|J| = e^{-y_1} \cdot y_1.$$

$$f_Y(y_1, y_2) = \begin{cases} y_1e^{-y_1} & \text{if } y_1 > 0, y_2 \in (0,1), \\ 0 & \text{otherwise.} \end{cases}$$

This shows $Y_1 \sim \text{Gamma}(2,1)$ is independent of $Y_2 \sim \text{Uniform}(0,1)$.

Variation 13: Transformation on a Triangular Support

Exercise:
Let (X_1, X_2) be uniform on the triangle $S = \{(x_1, x_2) : x_1 > 0, x_2 > 0, x_1 + x_2 < 1\}$. Find the PDF of $Y = (X_1 + X_2, X_1 - X_2)$.

Solution:

- **Original PDF:** The area of S is $1/2$, so $f_X(x_1, x_2) = 2$ on S, and 0 otherwise.
- **Inverse:** $X_1 = \frac{Y_1+Y_2}{2}, X_2 = \frac{Y_1-Y_2}{2}$.
- **Jacobian:** $|J| = \det\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = |-1/4 - 1/4| = 1/2.$
- **Support for Y:** The vertices (0,0), (1,0), (0,1) of S map to (0,0), (1,1), (1,-1) in the Y-plane. The support is the triangle bounded by $y_1 > |y_2|$ and $y_1 < 1$.
- **PDF:**

$$f_Y(y_1, y_2) = f_X(\dots)|J| = 2 \cdot \frac{1}{2} = 1.$$

$$f_Y(y_1, y_2) = \begin{cases} 1 & \text{in the triangular region with vertices (0,0), (1,1), (1,-1),} \\ 0 & \text{otherwise.} \end{cases}$$

Variation 14: Product and Ratio Transformation

Exercise:
For $X_1, X_2 \sim \text{Uniform}(0, 1)$, find the PDF of $Y = (X_1 X_2, X_1/X_2)$.

> **Solution:**
>
> - **Inverse:** $X_1 = \sqrt{Y_1 Y_2}, X_2 = \sqrt{Y_1/Y_2}$.
> - **Jacobian:**
>
> $$|J| = \det \begin{pmatrix} \frac{\sqrt{y_2}}{2\sqrt{y_1}} & \frac{\sqrt{y_1}}{2\sqrt{y_2}} \\ \frac{1}{2\sqrt{y_1 y_2}} & \frac{-\sqrt{y_1}}{2y_2\sqrt{y_2}} \end{pmatrix} = |-\frac{1}{4y_2} - \frac{1}{4y_2}| = \frac{1}{2y_2}.$$
>
> - **Support for Y:** $0 < X_1, X_2 < 1$ implies $0 < Y_1 < Y_2$ and $Y_1 Y_2 < 1$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = f_X(\dots)|J| = 1 \cdot \frac{1}{2y_2}.$$
>
> $$f_Y(y_1, y_2) = \begin{cases} \frac{1}{2y_2} & \text{if } 0 < y_1 < y_2 \text{ and } y_1 y_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Variation 15: Beta Distribution from Gamma Variables

Exercise:
Let $X_1 \sim \text{Gamma}(2, 1)$ and $X_2 \sim \text{Gamma}(3, 1)$ be independent. Find the PDF of $Y = (X_1 + X_2, \frac{X_1}{X_1+X_2})$.

> **Solution:**
>
> - **Original PDF:** $f_X(x_1, x_2) = \frac{1}{\Gamma(2)\Gamma(3)} x_1^{2-1} e^{-x_1} x_2^{3-1} e^{-x_2} = \frac{1}{2} x_1 x_2^2 e^{-(x_1+x_2)}$.
> - **Inverse:** $X_1 = Y_1 Y_2, X_2 = Y_1(1 - Y_2)$.
> - **Jacobian:** $|J| = y_1$ (as in Var 12).
> - **Support for Y:** $X_1, X_2 > 0$ implies $Y_1 > 0$ and $Y_2 \in (0, 1)$.
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \frac{1}{2}(y_1 y_2)(y_1(1 - y_2))^2 e^{-y_1} \cdot y_1 = \frac{1}{2} y_1^4 y_2 (1 - y_2)^2 e^{-y_1}.$$

$$
f_Y(y_1, y_2) = \begin{cases} \frac{1}{2} y_1^4 y_2 (1-y_2)^2 e^{-y_1} & \text{if } y_1 > 0, y_2 \in (0,1), \\ 0 & \text{otherwise.} \end{cases}
$$

This can be factored to show $Y_1 \sim \text{Gamma}(5, 1)$ is independent of $Y_2 \sim \text{Beta}(2, 3)$.

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Variation 6: Coupled Linear Transformation with Different Coefficients

Exercise:
Let $X_1, X_2 \sim \text{Uniform}(0, 1)$ be independent. Find the PDF of

$$
Y = \Big(2X_1 + X_2,\ X_1 - X_2\Big).
$$

Solution:

- **Inverse:**
  Write

$$
\begin{cases} y_1 = 2x_1 + x_2, \\ y_2 = x_1 - x_2. \end{cases}
$$

Solve for x_1 and x_2. From the second equation,

$$
x_1 = y_2 + x_2.
$$

Substitute into the first equation:

$$
y_1 = 2(y_2 + x_2) + x_2 = 2y_2 + 3x_2 \quad \Longrightarrow \quad x_2 = \frac{y_1 - 2y_2}{3}.
$$

Then,

$$x_1 = y_2 + \frac{y_1 - 2y_2}{3} = \frac{y_1 + y_2}{3}.$$

- **Jacobian:**
  Compute the partial derivatives:

$$\frac{\partial x_1}{\partial y_1} = \frac{1}{3}, \quad \frac{\partial x_1}{\partial y_2} = \frac{1}{3}, \quad \frac{\partial x_2}{\partial y_1} = \frac{1}{3}, \quad \frac{\partial x_2}{\partial y_2} = -\frac{2}{3}.$$

  Hence,

$$|J| = \left|\det\begin{pmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{pmatrix}\right| = \left|\frac{1}{3}\cdot\left(-\frac{2}{3}\right) - \frac{1}{3}\cdot\frac{1}{3}\right| = \left|\frac{-2-1}{9}\right| = \frac{1}{3}.$$

- **Support:**
  Since $x_1, x_2 \in [0,1]$, we require

$$0 \le \frac{y_1 + y_2}{3} \le 1 \quad \Longrightarrow \quad 0 \le y_1 + y_2 \le 3,$$

$$0 \le \frac{y_1 - 2y_2}{3} \le 1 \quad \Longrightarrow \quad 0 \le y_1 - 2y_2 \le 3.$$

- **PDF:**

$$f_Y(y_1, y_2) = \begin{cases} \frac{1}{3}, & \text{if } (y_1, y_2) \text{ satisfy } 0 \le y_1 + y_2 \le 3 \text{ and } 0 \le y_1 - 2y_2 \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

Variation 7: Non-Linear Transformation with Cube and Square Root

Exercise:
Let $X_1, X_2 \sim \text{Uniform}(0,1)$. Find the PDF of

$$Y = \left(X_1^3,\ \sqrt{X_2}\right).$$

Solution:

- **Inverse:**

$$x_1 = \sqrt[3]{y_1}, \quad x_2 = y_2^2.$$

- **Jacobian:**
  Compute the derivatives:

$$\frac{dx_1}{dy_1} = \frac{1}{3} y_1^{-2/3}, \quad \frac{dx_2}{dy_2} = 2y_2.$$

  Hence,

$$|J| = \frac{1}{3} y_1^{-2/3} \cdot 2y_2 = \frac{2y_2}{3y_1^{2/3}}.$$

- **Support:**
  Since $x_1, x_2 \in [0,1]$:

$$y_1 \in [0,1] \quad (\text{with } y_1 > 0 \text{ for the derivative to be defined}), \quad y_2 \in [0,1].$$

- **PDF:**

$$f_Y(y_1, y_2) = \begin{cases} \frac{2y_2}{3y_1^{2/3}}, & \text{if } y_1 \in (0,1] \text{ and } y_2 \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Variation 8: Logarithm and Square Root Transformation for Exponential Variables

Exercise:
Let $X_1, X_2 \sim \mathrm{Exp}(1)$. Find the PDF of

$$Y = \Big(\ln(1 + X_1),\ \sqrt{X_2}\Big).$$

Solution:

- **Inverse:**

$$x_1 = e^{y_1} - 1, \quad x_2 = y_2^2.$$

- **Jacobian:**

$$\frac{dx_1}{dy_1} = e^{y_1}, \quad \frac{dx_2}{dy_2} = 2y_2 \quad \Longrightarrow \quad |J| = 2y_2\, e^{y_1}.$$

- **Original PDF:**
  Since $f_X(x_1, x_2) = e^{-(x_1+x_2)}$, substituting the inverses gives

$$f_X\left(e^{y_1} - 1, y_2^2\right) = e^{-((e^{y_1}-1)+y_2^2)} = e^{\,1-e^{y_1}-y_2^2}.$$

- **PDF:**

$$f_Y(y_1, y_2) = 2y_2\, e^{y_1}\, e^{\,1-e^{y_1}-y_2^2} = 2y_2\, e^{\,1+y_1-e^{y_1}-y_2^2}.$$

- **Support:**
  $y_1 \geq 0$ and $y_2 \geq 0$.

Variation 9: Ratio and Product Transformation

Exercise:
Let $X_1, X_2 \sim \mathrm{Uniform}(0,1)$. Find the PDF of

$$Y = \left(\frac{X_1}{X_2},\ X_1 X_2\right).$$

Solution:

- **Inverse:**
  From $y_1 = \frac{x_1}{x_2}$ we get $x_1 = y_1 x_2$. Then, since

$$y_2 = x_1 x_2 = y_1 x_2^2, \quad \text{it follows that} \quad x_2 = \sqrt{\frac{y_2}{y_1}},$$

  and so

$$x_1 = \sqrt{y_1 y_2}.$$

- **Jacobian:**
  A careful computation (omitting intermediate algebra) shows that

$$|J| = \frac{1}{2y_1}.$$

- **Support:**
  For $x_1, x_2 \in [0,1]$ we require

$$\sqrt{y_1 y_2} \le 1 \quad \Longrightarrow \quad y_1 y_2 \le 1,$$

  and

$$\sqrt{\frac{y_2}{y_1}} \le 1 \quad \Longrightarrow \quad y_2 \le y_1.$$

  Thus, the support is

$$y_1 > 0, \quad 0 \le y_2 \le \min\Bigl(y_1,\ \frac{1}{y_1}\Bigr).$$

- **PDF:**

$$f_Y(y_1, y_2) = \begin{cases} \dfrac{1}{2y_1}, & \text{if } y_1 > 0,\ 0 \le y_2 \le \min\left(y_1,\ \frac{1}{y_1}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Variation 10: Multiplicative Transformation

Exercise:
Let $X_1, X_2 \sim \text{Uniform}(0,1)$. Find the PDF of

$$Y = \Bigl(X_1,\ X_1 X_2\Bigr).$$

Solution:

- **Inverse:**
  Clearly,

$$x_1 = y_1, \quad x_2 = \frac{y_2}{y_1} \quad (\text{with } y_1 > 0).$$

- **Jacobian:**
  The Jacobian matrix is

$$J = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{pmatrix}.$$

Hence,

$$|J| = 1 \cdot \frac{1}{y_1} - 0 = \frac{1}{y_1}.$$

- **Support:**
$x_1 \in [0,1]$ implies $y_1 \in (0,1]$ and $x_2 \in [0,1]$ requires

$$0 \le \frac{y_2}{y_1} \le 1 \quad \Longrightarrow \quad 0 \le y_2 \le y_1.$$

- **PDF:**

$$f_Y(y_1, y_2) = \begin{cases} \frac{1}{y_1}, & \text{if } y_1 \in (0,1] \text{ and } 0 \le y_2 \le y_1, \\ 0, & \text{otherwise.} \end{cases}$$

Variation 11: Logarithmic Transformation for Exponential Variables

Exercise:
Let $X_1, X_2 \sim \mathrm{Exp}(1)$. Find the PDF of

$$Y = \Big(\ln(1 + X_1),\ \ln(1 + X_2)\Big).$$

Solution:

- **Inverse:**

$$x_1 = e^{y_1} - 1, \quad x_2 = e^{y_2} - 1.$$

- **Jacobian:**

$$\frac{dx_1}{dy_1} = e^{y_1}, \quad \frac{dx_2}{dy_2} = e^{y_2} \quad \Longrightarrow \quad |J| = e^{y_1 + y_2}.$$

- **Original PDF:**

$$f_X(x_1, x_2) = e^{-(x_1+x_2)} = e^{-\left[(e^{y_1}-1)+(e^{y_2}-1)\right]} = e^{\,2-e^{y_1}-e^{y_2}}.$$

- **PDF:**

$$f_Y(y_1, y_2) = e^{y_1+y_2}\, e^{\,2-e^{y_1}-e^{y_2}} = e^{\,2+y_1+y_2-e^{y_1}-e^{y_2}}.$$

- **Support:**
  $y_1, y_2 \geq 0.$

Variation 12: Sum and Component Transformation

Exercise:
Let $X_1, X_2 \sim \mathrm{Uniform}(0,1)$. Find the PDF of

$$Y = \Big(X_1 + X_2,\ X_1\Big).$$

Solution:

- **Inverse:**

$$x_1 = y_2, \quad x_2 = y_1 - y_2.$$

- **Jacobian:**
  The transformation is linear with

$$|J| = \left|\det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\right| = 1.$$

- **Support:**
  $x_1 \in [0,1]$ gives $y_2 \in [0,1]$. Also, $x_2 \in [0,1]$ implies

$$0 \leq y_1 - y_2 \leq 1 \quad \Longrightarrow \quad y_2 \leq y_1 \leq y_2 + 1.$$

- **PDF:**

$$f_Y(y_1, y_2) = \begin{cases} 1, & \text{if } y_2 \in [0,1] \text{ and } y_2 \leq y_1 \leq y_2 + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Variation 13: Quadratic Addition Transformation

Exercise:
Let $X_1, X_2 \sim \mathrm{Uniform}(0,1)$. Find the PDF of

$$Y = \Big(X_1,\ X_1^2 + X_2\Big).$$

> **Solution:**
>
> - **Inverse:**
>
> $$x_1 = y_1, \quad x_2 = y_2 - y_1^2.$$
>
> - **Jacobian:**
> The Jacobian is
>
> $$J = \begin{pmatrix} 1 & 0 \\ -2y_1 & 1 \end{pmatrix} \quad \Longrightarrow \quad |J| = 1.$$
>
> - **Support:**
> Since $x_1 \in [0,1]$ implies $y_1 \in [0,1]$, and $x_2 \in [0,1]$ gives
>
> $$0 \le y_2 - y_1^2 \le 1 \quad \Longrightarrow \quad y_2 \in [y_1^2,\, y_1^2 + 1].$$
>
> - **PDF:**
>
> $$f_Y(y_1, y_2) = \begin{cases} 1, & \text{if } y_1 \in [0,1] \text{ and } y_2 \in [y_1^2,\, y_1^2+1], \\ 0, & \text{otherwise.} \end{cases}$$

Variation 14: Absolute Difference and Sum Transformation

Exercise:
Let $X_1, X_2 \sim \mathrm{Uniform}(0,1)$. Find the PDF of

$$Y = \Big(|X_1 - X_2|,\ X_1 + X_2\Big).$$

> **Solution:**
> This transformation has two branches. For the branch where $X_1 \ge X_2$:
>
> - **Inverse (Branch 1):**

$$x_1 = \frac{y_2 + y_1}{2}, \quad x_2 = \frac{y_2 - y_1}{2}.$$

The Jacobian is

$$\left| \det \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \right| = \frac{1}{2}.$$

For the branch $X_2 > X_1$, a similar computation yields the same absolute Jacobian. Summing the contributions from both branches gives an overall factor of 1.

- **Support:**
  The conditions $x_1, x_2 \in [0,1]$ imply

$$y_1 \in [0,1] \quad \text{and} \quad y_2 \in [y_1,\, 2 - y_1].$$

- **PDF:**

$$f_Y(y_1, y_2) = \begin{cases} 1, & \text{if } y_1 \in [0,1] \text{ and } y_2 \in [y_1,\, 2 - y_1], \\ 0, & \text{otherwise.} \end{cases}$$

Variation 15: Mixed Transformation with Uniform and Exponential Variables

Exercise:
Let $X_1 \sim \text{Uniform}(0,1)$ and $X_2 \sim \text{Exp}(1)$ be independent. Find the PDF of

$$Y = \left(e^{X_1} - 1,\ 2X_2\right).$$

Solution:

- **Inverse:**

$$x_1 = \ln(1 + y_1), \quad x_2 = \frac{y_2}{2}.$$

- **Jacobian:**

$$\frac{dx_1}{dy_1} = \frac{1}{1 + y_1}, \quad \frac{dx_2}{dy_2} = \frac{1}{2} \quad \Longrightarrow \quad |J| = \frac{1}{2(1 + y_1)}.$$

- **Support:**
  Since $x_1 \in [0,1]$ we have

$$y_1 \in \left[e^0 - 1,\ e^1 - 1\right] = [0, e-1],$$

  and $x_2 \geq 0$ implies $y_2 \geq 0$.
- **Original PDF:**
  $f_{X_1}(x_1) = 1$ for $x_1 \in [0,1]$ and $f_{X_2}(x_2) = e^{-x_2}$ for $x_2 \geq 0$. Substituting $x_2 = \frac{y_2}{2}$ yields

$$f_{X_2}\left(\frac{y_2}{2}\right) = e^{-y_2/2}.$$

- **PDF:**

$$f_Y(y_1, y_2) = \frac{1}{2(1+y_1)}\, e^{-y_2/2},$$

  for $y_1 \in [0, e-1]$ and $y_2 \geq 0$; zero otherwise.