Exercises 1

1. $\int_{[-1,1]} x \, \lambda(dx)$, where λ is Lebesgue measure on $\mathbb{R}$.

> $\int_{-1}^{1} x \, dx = 0$ (odd function over symmetric interval).

2. $\int x^2 \, P(dx)$, where $P(A) = (1-p)\delta_0(A) + p\delta_1(A)$, $p \in (0,1)$.

> $(1-p)(0-p)^2 + p(1-p)^2 = p(1-p)$.

3. $\int_1^e \log(x) \, \lambda(dx)$, where λ is Lebesgue measure.

> $\int_1^e \log x \, dx = [x \log x - x]_1^e = (e \cdot 1 - e) - (1 \cdot 0 - 1) = 1$.

4. $\int_{\{1,2,3\}} x \, \mu(dx)$, where μ is counting measure on $\mathcal{P}(\mathbb{N})$.

> $\sum_{x=1}^{3} x = 6$.

5. $\int_{\mathbb{N}\times\mathbb{N}} 3^{-(x+y)} \, m(d(x,y))$, where m is counting measure on $\mathcal{P}(\mathbb{N}\times\mathbb{N})$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

> $\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} 3^{-(x+y)} = (\sum_{x=0}^{\infty} 3^{-x})^2 = \left(\frac{1}{1-\frac{1}{3}}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

6. $\int_{\mathbb{R}} x \, \mu(dx)$, where $\mu(A) = \sum_{x=0}^{\infty} \frac{e^{-2}2^x}{x!} \delta_x(A)$.

> Expectation of $\mathrm{Poisson}(2)$: 2.

7. $\int_{[0,2]} x^3 \, \lambda(dx)$.

> $\int_0^2 x^3 dx = \frac{2^4}{4} = 4$.

8. $\int (x-p)^3 \, P(dx)$, same P as exercise 2.

> $(1-p)(0-p)^3 + p(1-p)^3 = p(1-p)(1-2p)$.

9. $\int_0^1 \frac{1}{1+x^2} \, \lambda(dx)$.

> $\int_0^1 \frac{1}{1+x^2} dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}$.

10. $\sum_{k=1}^{\infty} \frac{k}{2^k}$, interpreting as $\int f(k) m(dk)$, m counting measure on $\mathbb{N} = \{1, 2, ...\}$.

> $\sum_{k=1}^{\infty} \frac{k}{2^k} = 2$.

11. $\int_{\mathbb{N}} (x!)^{-1} e^{-1}\, \mu(dx)$, μ as in exercise 6 with parameter 1 (i.e., $\mu(A) = \sum_{x=0}^{\infty} \frac{e^{-1}1^x}{x!} \delta_x(A)$), $\mathbb{N} = \{0, 1, 2, ...\}$.

> $\sum_{x=0}^{\infty} \frac{e^{-1}1^x}{x!} = 1$ (total probability of Poisson(1)). Note: the integral is $\int_{\mathbb{N}} \frac{1}{x!} e^{-1} \mu(dx)$. The question appears to imply the integrand term $\frac{e^{-1}\lambda^x}{x!}$ is part of the measure. Re-reading exercise 6, $\mu(A)$ is the measure $P(X \in A)$ for Poisson. So here we integrate $g(x) = (x!)^{-1} e^{-1}$ against $dP(x)$, where P is Poisson(1). This would be $\sum_{x=0}^{\infty} (x!)^{-1} e^{-1} \frac{e^{-1}1^x}{x!} = e^{-2} \sum_{x=0}^{\infty} \frac{1}{(x!)^2}$, which is $e^{-2} I_0(2)$ where I_0 is modified Bessel func.
> The likely intended question is $\int_{\mathbb{N}} 1\, \mu(dx)$ where μ is Poisson(1) measure, which sums to 1. OR $\int_{\mathbb{N}} x \cdot \frac{e^{-1}}{x!} \mu_c(dx)$ where μ_c is counting measure, and this would be $E[X] = 1$.
> Assuming the question means: given Poisson measure $P_\lambda(A) = \sum_{x \in A \cap \mathbb{N}_0} \frac{e^{-\lambda}\lambda^x}{x!}$, calculate $\int_{\mathbb{N}_0} g(x) P_1(dx)$ with $g(x) = 1$.
> **Solution:** The sum of probabilities for a Poisson(1) distribution is $\sum_{x=0}^{\infty} \frac{e^{-1}1^x}{x!} = e^{-1} \sum_{x=0}^{\infty} \frac{1}{x!} = e^{-1} e = 1$. If the question meant $\int_{\mathbb{N}_0} \frac{(x!)^{-1} e^{-1}}{\text{term from measure}} (\text{term from measure } dP(x))$, this is likely asking for $\sum_{x=0}^{\infty} (x!)^{-1} e^{-1} P(x)$ where $P(x)$ is Poisson probability for x. This would be $e^{-1} \sum((x!)^{-2} e^{-1}) = e^{-2} \sum (x!)^{-2}$. The phrasing is a bit ambiguous but often μ as in ex 6 implies μ IS the Poisson measure. If $\mu(dx) = \frac{e^{-1}1^x}{x!} \delta_x(dx)$ then $\int (x!)^{-1} e^{-1} \mu(dx) = \sum_{x=0}^{\infty} (x!)^{-1} e^{-1} \frac{e^{-1}1^x}{x!} = e^{-2} \sum_{x=0}^{\infty} \frac{1}{(x!)^2}$.
> If it means
>
> $$\int_{\mathbb{N}_0} \frac{e^{-1}}{x!} \text{counting\_measure}(dx)$$
>
> this is $e^{-1} \sum (x!)^{-1} = e^{-1} e = 1$. Given the solution in the original is `1`, this last interpretation is most likely, where μ is counting measure.
> **Original Solution Interpretation:** $\sum_{x=0}^{\infty} \frac{e^{-1}1^x}{x!} = 1$ (total probability of Poisson(1)). This implies the integrand is 1 and the measure is Poisson(1).

12. $\int_{\mathbb{N} \times \{0,1\}} 4^{-(x+y)}\, m(d(x, y))$, m counting measure, $\mathbb{N} = \{0, 1, 2, ...\}$.

> $\sum_{x=0}^{\infty} \sum_{y=0}^{1} 4^{-(x+y)} = (\sum_{x=0}^{\infty} 4^{-x}) \left(\sum_{y=0}^{1} 4^{-y}\right) = \frac{1}{1-1/4} \times (4^0 + 4^{-1}) = \frac{4}{3} \times (1 + \frac{1}{4}) = \frac{4}{3} \times \frac{5}{4} = \frac{5}{3}$.

13. $\int_0^{\infty} x e^{-x}\, \lambda(dx)$.

> $\int_0^{\infty} x e^{-x} dx = 1$ (Gamma(2) distribution mean, or $\Gamma(2)$ function).

14. $\int_{\{0,1\}^n} \sum_{i=1}^{n} x_i\, \mu(dx)$, μ counting measure on $\{0, 1\}^n$.

> $\sum_{x\in\{0,1\}^n}\sum_{i=1}^n x_i = n\cdot 2^{n-1}$. For a fixed i, $x_i = 1$ in half of the 2^n configurations, so 2^{n-1} times. Summing over n coordinates gives $n2^{n-1}$.

15. $\int_{\mathbb{R}} e^{-x^2}\,\lambda(dx)$ over $\mathbb{R}$.

> No elementary closed form for the antiderivative, but the definite integral is known as $\sqrt{\pi}$ (Gaussian integral).

16. $\int_{\mathbb{N}_0} x^2 P_\lambda(dx)$, where $P_\lambda(dx)$ denotes integrating against the Poisson measure with parameter λ, i.e. $P_\lambda(A) = \sum_{x\in A\cap\mathbb{N}_0}\frac{e^{-\lambda}\lambda^x}{x!}$. (The original text says $\delta_x(dx)$ which is confusing).

> This is $E[X^2]$ for $X\sim\mathrm{Poisson}(\lambda)$. $E[X^2] = \mathrm{Var}(X) + (E[X])^2 = \lambda+\lambda^2$.

17. $\int_0^1 x\log(x)\,\lambda(dx)$.

> $\int_0^1 x\log x dx = [\frac{x^2}{2}\log x - \frac{x^2}{4}]_0^1 = (0-\frac{1}{4}) - \lim_{x\to 0}(\frac{x^2}{2}\log x - \frac{x^2}{4}) = -\frac{1}{4}$.

18. $\int_{[0,1]^2}(x^2+y^2)\,\lambda(dxdy)$.

> $\int_0^1\int_0^1(x^2+y^2)dydx = \int_0^1[x^2y+\frac{y^3}{3}]_0^1 dx = \int_0^1(x^2+\frac{1}{3})dx = [\frac{x^3}{3}+\frac{x}{3}]_0^1 = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$.

19. $\int_{\mathbb{N}\times\mathbb{N}}(xy)^{-1}2^{-(x+y)}\,m(d(x,y))$, where $\mathbb{N}=\{1,2,...\}$ and m is counting measure.

> $\sum_{x=1}^\infty\sum_{y=1}^\infty\frac{2^{-x}}{x}\frac{2^{-y}}{y} = \left(\sum_{k=1}^\infty\frac{(1/2)^k}{k}\right)^2 = (-\log(1-1/2))^2 = (\log 2)^2$.

20. $\int_0^\infty\frac{\sin x}{x}\,\lambda(dx)$.

> $\int_0^\infty\frac{\sin x}{x}dx = \frac{\pi}{2}$ (Dirichlet integral).

Exercises 2

1. $\int_{[0,1]} d\lambda(x)$, where λ is Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

> **Solution:** 1
>
> *(Simple Lebesgue measure of an interval.)*

2. $\int_{\mathbb{R}}|x|\delta_0(dx)$, where δ_0 is the Dirac measure at 0.

> **Solution:** 0
>
> *(Dirac measure evaluation at a point.)*

3. $\int_A d\mu(x)$, where $A = \{1, 2, 3\}$ and μ is the counting measure on $\mathcal{B}(\mathbb{Z})$.

> **Solution:** 3
> *(Counting measure on a finite set.)*

4. $\int_{\mathbb{R}} x^2(\delta_2 + \delta_3)(dx)$.

> **Solution:** 13 (since $2^2 + 3^2 = 4 + 9$)
> *(Sum of Dirac measures.)*

5. $\int_{[0,\infty)} e^{-x}\lambda(dx)$, where λ is Lebesgue measure.

> **Solution:** 1
> *(Exponential integral over Lebesgue measure.)*

6. $\int_{\mathbb{R}} (x - p)^2 P(dx)$, where $P = (1 - p)\delta_0 + p\delta_1$ for $p \in (0, 1)$.

> **Solution:** $p(1 - p)$
> *(Variance of a Bernoulli distribution.)*

7. $\int_{\mathbb{N}} e^{-x}\mu(dx)$, where μ is the counting measure on $\mathbb{N} = \{0, 1, 2, \ldots\}$.

> **Solution:** $\sum_{x=0}^{\infty} e^{-x} = \sum_{x=0}^{\infty} (1/e)^x = \frac{1}{1-1/e} = \frac{e}{e-1}$
> *(Geometric series under counting measure.)*

8. $\int_{[0,1]\times[0,1]} (x + y)(\lambda \times \lambda)(d(x, y))$, where $\lambda \times \lambda$ is the product measure.

> **Solution:** $\int_0^1 \int_0^1 (x + y)dxdy = \int_0^1 (\frac{1}{2} + y)dy = \frac{1}{2} + \frac{1}{2} = 1$
> *(Double integral with linear function.)*

9. $\int_{\mathbb{R}} x\mu(dx)$, where $\mu = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}}\delta_k$.

> **Solution:** $\sum_{k=0}^{\infty} k\frac{1}{2^{k+1}} = \frac{1}{2}\sum_{k=0}^{\infty} k(\frac{1}{2})^k = \frac{1}{2}\frac{1/2}{(1-1/2)^2} = \frac{1}{2}\frac{1/2}{1/4} = 1$
> *(Expected value of a discrete distribution (a geometric distribution variant starting at 0, $P(X = k) = (1/2)^{k+1}$).)*

10. $\int_{\mathbb{N}\times\mathbb{N}} e^{-(x+y)}(\mu \times \mu)(d(x, y))$, where μ is the counting measure on $\mathbb{N} = \{0, 1, 2, \ldots\}$.

> **Solution:** $(\sum_{x=0}^{\infty} e^{-x}) \left(\sum_{y=0}^{\infty} e^{-y}\right) = \left(\frac{e}{e-1}\right)^2$
> *(Product measure and separable function, using result from ex. 7.)*

11. $\int_{\mathbb{R}} x\nu(dx)$, where $d\nu = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$ (standard Gaussian measure).

> **Solution:** 0
> *(Symmetry of Gaussian measure; expected value of N(0,1).)*

12. $\int_{[0,\infty)} x^2\mu(dx)$, where $d\mu = 2e^{-2x}dx$. (This is an Exponential distribution with rate 2).

> **Solution:** $\int_0^\infty x^2 2e^{-2x}dx$. For $\text{Exp}(\beta)$ with density $\beta e^{-\beta x}$, $E[X^2] = 2/\beta^2$. Here $\beta = 2$, so $E[X^2] = 2/2^2 = 1/2$.
> *(Exponential distribution second moment.)*

13. $\int_{[-2,2]} |x|\lambda(dx)$, where λ is Lebesgue measure.

> **Solution:** $2\int_0^2 xdx = 2[\frac{x^2}{2}]_0^2 = 2 \cdot \frac{4}{2} = 4$
> *(Integral of absolute value over symmetric interval.)*

14. $\int_{[-1,1]} 1\mu(dx)$, where $d\mu = |x|dx$.

> **Solution:** $\int_{-1}^1 |x|dx = 2\int_0^1 xdx = 2 \cdot \frac{1}{2} = 1$
> *(Integration against a weighted Lebesgue measure.)*

15. $\int_{\mathbb{R}} e^{-|x|}\lambda(dx)$.

> **Solution:** $2\int_0^\infty e^{-x}dx = 2[-e^{-x}]_0^\infty = 2(0-(-1)) = 2$
> *(Double-sided exponential integral.)*

16. $\int_{\mathbb{N}_0} (x-1)^2 P(dx)$, where P is Poisson measure with mean 1: $P(\{k\}) = \frac{e^{-1}}{k!}$ for $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

> **Solution:** This is $E[(X-1)^2]$ for $X \sim \text{Poisson}(1)$. Mean $E[X] = \lambda = 1$. So $E[(X - E[X])^2] = \text{Var}(X) = \lambda = 1$.
> *(Variance of Poisson distribution.)*

17. $\int_{[0,1]} x\nu(dx)$, where $\nu = g_*\lambda$ is the pushforward of Lebesgue measure λ on $[0,1]$ under $g(x) = x^2$.

> **Solution:** By change of variables, $\int_{[0,1]} g(x)\lambda(dx) = \int_0^1 x^2dx = \frac{1}{3}$.
> *(Pushforward measure and change of variables. The integral becomes $\int_S yd\nu(y)$. If $Y = g(X)$, then $E[Y] = \int xd\nu(x)$ means we are finding $E[X]$ where X has measure ν. The wording here is $\int x\nu(dx)$ where ν is defined on range of g. So we integrate $\int_{[0,1]} yd\nu(y)$. Using transformation: if $y = x^2$, then pdf of Y is $f_Y(y) = f_X(\sqrt{y})|\frac{d}{dy}\sqrt{y}| = 1 \cdot \frac{1}{2\sqrt{y}}$ for $y \in [0,1]$. So $\int_0^1 y\frac{1}{2\sqrt{y}}dy = \int_0^1 \frac{\sqrt{y}}{2}dy = [\frac{1}{2}\frac{y^{3/2}}{3/2}]_0^1 = \frac{1}{3}$.
> Original interpretation: $\int x\nu(dx)$. If ν is on the target space of g, then the variable should be y,

> not x. If it means $\int xd(g_*\lambda)(x)$, then it's $\int x^2 d\lambda(x)$. This means x is a dummy variable representing values in range of g. So $\int_0^1 yf_Y(y)dy$. This is what was computed. Alternatively $\int_{g([0,1])} id(y)d\nu(y) = \int_{[0,1]} id(g(x))d\lambda(x) = \int_0^1 x^2 dx = 1/3$. This is standard notation. The x in $\int x\nu(dx)$ is the variable of integration over the space where ν is defined.
> *(Pushforward measure and change of variables.)*

18. $\int_{\mathbb{Z}} |x|e^{-|x|}\mu(dx)$, where μ is the counting measure on $\mathbb{Z}$.

> **Solution:** $\sum_{x\in\mathbb{Z}} |x|e^{-|x|} = 0 \cdot e^0 + 2\sum_{x=1}^{\infty} xe^{-x} = 2\sum_{x=1}^{\infty} x(1/e)^x = 2\frac{1/e}{(1-1/e)^2} = 2\frac{e^{-1}}{(1-e^{-1})^2} = 2\frac{e^{-1}}{((e-1)/e)^2} = 2\frac{e^{-1}}{(e-1)^2/e^2} = \frac{2e}{(e-1)^2}$.
> *(Series involving absolute values and exponential decay.)*

19. $\int_{\mathbb{R}} x\mu(dx)$, where $\mu = \delta_0 + \lambda|_{[1,2]}$ (Lebesgue measure restricted to $[1,2]$).

> **Solution:** $\int_{\mathbb{R}} x\delta_0(dx) + \int_{\mathbb{R}} x\lambda|_{[1,2]}(dx) = 0 \cdot 1 + \int_1^2 xdx = 0 + [\frac{x^2}{2}]_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$.
> *(Mixed discrete-continuous measure.)*

20. $\int_{[0,\infty)\times[0,\infty)} |x-y|e^{-x}e^{-y}(\lambda\times\lambda)(d(x,y))$.

> **Solution:** $\int_0^\infty \int_0^\infty |x-y|e^{-x}e^{-y}dxdy$. This is $E[|X-Y|]$ for i.i.d $X, Y \sim \text{Exp}(1)$. The probability $P(X>Y) = 1/2$.
> $2\int_0^\infty \int_y^\infty (x-y)e^{-x}e^{-y}dxdy = 2\int_0^\infty e^{-y}\left(\int_y^\infty xe^{-x}dx - y\int_y^\infty e^{-x}dx\right)dy$.
> $\int_y^\infty e^{-x}dx = [-e^{-x}]_y^\infty = e^{-y}$.
> $\int_y^\infty xe^{-x}dx = [-xe^{-x} - e^{-x}]_y^\infty = (0-0) - (-ye^{-y} - e^{-y}) = ye^{-y} + e^{-y}$.
> So integral becomes $2\int_0^\infty e^{-y}(ye^{-y} + e^{-y} - ye^{-y})dy = 2\int_0^\infty e^{-2y}dy = 2[-\frac{1}{2}e^{-2y}]_0^\infty = 2(0 - (-\frac{1}{2})) = 1$.
> *(Integral involving absolute difference of exponentials.)*

Exercises 3

1. $\int_{\mathbb{R}} \mathbf{1}_{[0,1]}(x)\cos(\pi x)\,d\lambda(x)$, where λ is Lebesgue measure.

> **Solution:** $\int_0^1 \cos(\pi x)dx = \left[\frac{1}{\pi}\sin(\pi x)\right]_0^1 = \frac{1}{\pi}(\sin(\pi) - \sin(0)) = 0$.
> *(Standard definite integral of a trigonometric function.)*

2. $\int_{\mathbb{R}^2} \mathbf{1}_{B(0,1)}(x,y)\,d(\lambda\times\lambda)(x,y)$, where $B(0,1) = \{(x,y) : x^2+y^2 \le 1\}$ is the unit disk and λ is Lebesgue measure.

> **Solution:** This is the Lebesgue measure of the unit disk, which is its area: $\pi(1)^2 = \pi$.
> *(Lebesgue measure of a geometric shape in $\mathbb{R}^2$.)*

3. $\int_{\{0,1,2\}} (x^2 - 1)\, d\mu(x)$, where $\mu = \delta_0 + 2\delta_1 + \delta_2$.

> **Solution:** $(0^2 - 1) \cdot 1 + (1^2 - 1) \cdot 2 + (2^2 - 1) \cdot 1 = (-1) + (0) \cdot 2 + (3) \cdot 1 = -1 + 0 + 3 = 2$.
> *(Linearity of integral for a sum of Dirac measures.)*

4. $\int_{[0,\pi/2]} \sin(x)\, d\nu(x)$, where $\nu(A) = \lambda(A \cap [0, \pi/4]) + \delta_{\pi/2}(A)$, and λ is Lebesgue measure.

> **Solution:** $\int_0^{\pi/4} \sin(x) dx + \sin(\pi/2) \cdot 1 = [-\cos(x)]_0^{\pi/4} + 1 = (-\cos(\pi/4) - (-\cos(0))) + 1 = (-\frac{\sqrt{2}}{2} + 1) + 1 = 2 - \frac{\sqrt{2}}{2}$.
> *(Integral with respect to a mixed measure: part Lebesgue, part Dirac.)*

5. $\int_{\mathbb{N}_0} x\, dP(x)$ where $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and P is the Geometric distribution measure $P(\{k\}) = (1-p)^k p$ for $k \in \mathbb{N}_0, p \in (0, 1)$.

> **Solution:** This is the expected value of a geometric random variable starting at 0. $E[X] = \sum_{k=0}^{\infty} kp(1-p)^k = \frac{1-p}{p}$.
> *(Expected value of a geometric distribution.)*

6. $\int_{(0,1)} x^{-1/2}\, d\lambda(x)$, where λ is Lebesgue measure.

> **Solution:** $\int_0^1 x^{-1/2} dx = [2x^{1/2}]_0^1 = 2(1)^{1/2} - \lim_{a \to 0} 2a^{1/2} = 2 - 0 = 2$.
> *(Integral of a function with an integrable singularity at the boundary.)*

7. $\int_{\mathbb{R}} \mathbf{1}_{\mathbb{Q}}(x)\, d\lambda(x)$, where $\mathbb{Q}$ is the set of rational numbers and λ is Lebesgue measure.

> **Solution:** Since $\mathbb{Q}$ is a set of Lebesgue measure zero, $\lambda(\mathbb{Q}) = 0$. The integral of the indicator function is $\int \mathbf{1}_{\mathbb{Q}} d\lambda = \lambda(\mathbb{Q}) = 0$.
> *(Integral of an indicator function of a null set.)*

8. $\int_{[0,1]} x\, d\mu_C(x)$, where μ_C is the standard Cantor measure on $[0, 1]$ (normalized so $\mu_C([0, 1]) = 1$).

> **Solution:** By symmetry of the Cantor set and the Cantor measure construction around $1/2$, the expected value $\int_{[0,1]} x\, d\mu_C(x)$ is $\frac{1}{2}$.
> *(Integral with respect to the Cantor measure.)*

9. $\int_{[0,1]^2} \min(x,y)\, d(\lambda \times \lambda)(x,y)$, where λ is Lebesgue measure.

> **Solution:** $\int_0^1 \left(\int_0^x y\, dy + \int_x^1 x\, dy \right) dx = \int_0^1 \left(\frac{x^2}{2} + x(1-x) \right) dx =$
> $\int_0^1 \left(\frac{x^2}{2} + x - x^2 \right) dx = \int_0^1 \left(x - \frac{x^2}{2} \right) dx = \left[\frac{x^2}{2} - \frac{x^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{3-1}{6} = \frac{1}{3}.$
> *(Double integral involving the min function, solved using Fubini's theorem.)*

10. $\int_{\mathbb{N}} \frac{1}{x(x+1)}\, d\mu(x)$, where μ is the counting measure on $\mathbb{N} = \{1, 2, ...\}$.

> **Solution:** $\sum_{x=1}^{\infty} \frac{1}{x(x+1)} = \sum_{x=1}^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots = 1$. (Telescoping series).
> *(Summation as an integral with counting measure; telescoping series.)*

11. $\int_{[0,1]} x^2\, d(g_*\lambda)(x)$, where $g_*\lambda$ is the pushforward of Lebesgue measure λ on $[0,1]$ by $g(t) = t^3$. (The integration variable x lives in the target space of g, which is also $[0,1]$).

> **Solution:** Using the change of variables formula for pushforward measures:
> $\int_{[0,1]} x^2\, d(g_*\lambda)(x) = \int_{[0,1]} (g(t))^2\, d\lambda(t) = \int_0^1 (t^3)^2\, dt = \int_0^1 t^6\, dt = \left[\frac{t^7}{7} \right]_0^1 = \frac{1}{7}.$
> *(Pushforward measure and change of variables theorem.)*

12. $\int_{[0,\infty)} x e^{-x^2}\, d\lambda(x)$, where λ is Lebesgue measure.

> **Solution:** Let $u = x^2$, then $du = 2x dx$, so $x dx = \frac{1}{2} du$. The limits $x = 0 \Rightarrow u = 0$ and $x \to \infty \Rightarrow u \to \infty$.
> $\int_0^{\infty} e^{-u} \frac{1}{2} du = \frac{1}{2} [-e^{-u}]_0^{\infty} = \frac{1}{2}(0 - (-1)) = \frac{1}{2}.$
> *(Gaussian-related integral, solvable with u-substitution.)*

13. $\int_{\mathbb{R}} \cos(x)\, d\mu(x)$, where $d\mu(x) = \frac{1}{\pi(1+x^2)} dx$ (Standard Cauchy distribution measure).

> **Solution:** This is the real part of the characteristic function $\phi(t) = E[e^{itX}]$ of the standard Cauchy distribution, evaluated at $t = 1$. The characteristic function is $\phi(t) = e^{-|t|}$. So, $\int_{\mathbb{R}} \cos(x) \frac{1}{\pi(1+x^2)} dx = \mathrm{Re}(e^{-|1|}) = e^{-1}$. (Since $\sin(x)$ is odd, its integral against the even measure is 0).
> *(Integral related to the characteristic function of the Cauchy distribution.)*

14. $\int_{\{(x,y) \in \mathbb{N}_0^2 : x+y=N\}} 1\, d\mu(x,y)$, where $N \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and μ is the counting measure on $\mathbb{N}_0^2$.

> **Solution:** This counts the number of pairs of non-negative integers (x, y) such that $x + y = N$. The possible values for x are $0, 1, \dots, N$, each determining a unique y. Thus there are $N+1$

> such pairs.
> *(Counting measure on a specific discrete set, a diagonal slice of $\mathbb{N}_0^2$.)*

15. $\int_{[0,1]} x\, dF(x)$, where $F(x)$ is the cumulative distribution function (CDF) of a uniform $U(0,1)$ random variable.

> **Solution:** For $X \sim U(0,1)$, $F(x) = x$ for $x \in [0,1]$, and $dF(x) = f(x)dx = 1 \cdot dx$ for $x \in (0,1)$. The integral is interpreted as $\int_{[0,1]} x\lambda(dx)$, where λ is Lebesgue measure.
> $\int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$.
> *(Connection to probability theory (expected value) and Riemann-Stieltjes integrals for absolutely continuous CDFs.)*

16. $\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, d(\lambda \times \lambda)(x,y)$, where λ is Lebesgue measure.

> **Solution:** Convert to polar coordinates: $x = r\cos\theta, y = r\sin\theta, dxdy = rdrd\theta$. $x^2 + y^2 = r^2$.
> $\int_0^{2\pi} \int_0^\infty e^{-r^2} r\, drd\theta = 2\pi \int_0^\infty e^{-r^2} r\, dr$. From problem #12, $\int_0^\infty e^{-r^2} r\, dr = 1/2$.
> So the integral is $2\pi \cdot \frac{1}{2} = \pi$.
> *(Double integral evaluated using polar coordinates, related to Gaussian integrals.)*

17. $\int_{[0,1]} x\, d\mu(x)$, where $\mu = \frac{1}{2}\delta_{1/4} + \frac{1}{2}\lambda|_{[0,1]}$ (a mixture of a Dirac measure at $1/4$ and scaled Lebesgue measure on $[0,1]$). The integration variable x lives on $[0,1]$.

> **Solution:** $\int_{[0,1]} x\, d\left(\frac{1}{2}\delta_{1/4}\right)(x) + \int_{[0,1]} x\, d\left(\frac{1}{2}\lambda|_{[0,1]}\right)(x) = \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2}\int_0^1 x\, dx = \frac{1}{8} + \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$.
> *(Integral with respect to a linear combination of measures (mixed measure).)*

18. $\int_0^\infty \frac{\sin^2(x)}{x^2}\, d\lambda(x)$, where λ is Lebesgue measure.

> **Solution:** This is a known definite integral. $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$. It can be solved using Parseval's theorem with the Fourier transform of $\mathbf{1}_{[-1,1]}(t)$, or complex integration.
> *(Known definite integral related to the sinc function.)*

19. $\int_{T_2} (x+y)\, d(\lambda \times \lambda)(x,y)$ where $T_2 = \{(x,y) \in [0,1]^2 : x + y \leq 1\}$ (a triangle) and λ is Lebesgue measure.

> **Solution:** $\int_0^1 \int_0^{1-x} (x+y)\, dydx = \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx = \int_0^1 \left(x(1-x) + \frac{(1-x)^2}{2}\right) dx$
> $= \int_0^1 \left(x - x^2 + \frac{1-2x+x^2}{2}\right) dx = \int_0^1 \left(x - x^2 + \frac{1}{2} - x + \frac{x^2}{2}\right) dx$

> $= \int_0^1 \left(\frac{1}{2} - \frac{x^2}{2}\right) dx = \left[\frac{x}{2} - \frac{x^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{3-1}{6} = \frac{1}{3}.$
> *(Double integral over a triangular region.)*

20. $\int_{\mathbb{R}} \frac{1}{(x-a)^2+b^2}\, d\lambda(x)$, for $b > 0$, λ Lebesgue measure.

> **Solution:** Let $u = \frac{x-a}{b}$, so $x = bu + a$ and $dx = b\,du$.
> $\int_{-\infty}^{\infty} \frac{1}{b^2u^2+b^2} b\,du = \int_{-\infty}^{\infty} \frac{b}{b^2(u^2+1)}\, du = \frac{1}{b} \int_{-\infty}^{\infty} \frac{1}{u^2+1}\, du$
> $= \frac{1}{b}[\arctan(u)]_{-\infty}^{\infty} = \frac{1}{b}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{1}{b}(\pi) = \frac{\pi}{b}.$
> *(Integral of a Lorentzian/Cauchy profile.)*
>
> *(Expected absolute difference of two i.i.d. exponential variables.)*

Measure-Theoretic Integral Patterns Cheat Sheet

Basic Measure Integrals

Measure Type	Domain	Integrand	Result	Notes
Lebesgue (λ)	[-a, a]	Odd f(x)	0	f(-x) = -f(x) symmetry
	[a, b]	1	b - a	Interval length
	[0, ∞)	e^{-x}	1	Exponential decay
	ℝ	e^{-x2}	√π	Gaussian integral
	[0, 1]	$1/(1+x^2)$	π/4	Arctangent primitive
	[1, e]	ln x	1	Integration by parts
	[0, 1]	x ln x	-1/4	Gamma function relation
	[0, ∞)	x e^{-x}	1	Γ(2) = 1!
	[0, ∞)	sin(x)/x	π/2	Dirichlet integral
	[-a, a]	|x|	a^2	
	[0,1]×[0,1]	$x^2 + y^2$	2/3	Fubini's theorem
Dirac (δ)	Any	f(x)	f(a)	Point evaluation at a
	ℝ	|x|	|a|	
	ℝ	x^2 (cδ_a + dδ_b)	$ca^2 + db^2$	Linear combination

Measure Type	Domain	Integrand	Result	Notes
Counting (μ)	$\{a_1..a_n\}$	1	n	Cardinality
	$\mathbb{N}$	r^x	$(1-r)^{-1}$	Geometric series (|r|<1)
	$\mathbb{N}$	$x \cdot r^x$	$r(1-r)^{-2}$	Differentiated series
	$\{0,1\}^n$	Σx_i	$n \cdot 2^{n-1}$	Linear function sum
	{1,2,3}	x	6	Finite sum
Bernoulli (P)	{0,1}	x	p	Mean
	{0,1}	$(x-p)^2$	p(1-p)	Variance
	{0,1}	$(x-p)^3$	p(1-p)(1-2p)	Skewness
Poisson (P)	$\mathbb{N}_0$	1	1	Normalization
	$\mathbb{N}_0$	x	λ	Mean
	$\mathbb{N}_0$	$(x-\lambda)^2$	λ	Variance
	$\mathbb{N}_0$	x^2	$\lambda + \lambda^2$	Second moment

Advanced Measures

Measure Type	Domain	Integrand	Result	Notes
Product Measures	$\mathbb{N} \times \mathbb{N}$	f(x)g(y)	[∑f][∑g]	Separable functions
	$[0,\infty)^2$	$\|x-y\|e^{-x}e^{-y}$	1	Exp(1) i.i.d. absolute difference
	[0,1]×[0,1]	x + y	1	Fubini's theorem
	$\mathbb{N} \times \{0,1\}$	$4^{-(x+y)}$	5/3	Mixed geometric series

Measure Type	Domain	Integrand	Result	Notes
Weighted Measures	$\mathbb{R}$	$e^{-\|x\|}$	2	Double-sided exponential
	[-1,1]	1 (dμ = |x|dx)	1	Weighted Lebesgue

Measure Type	Domain	Integrand	Result	Notes
	$[0,\infty)$	$x^2(2e^{-2x}\,dx)$	1/2	Exponential distribution moment
	$\mathbb{Z}$	$\lvert x\rvert e^{-\lvert x\rvert}$	$2e(e-1)^{-2}$	Discrete Laplace
Pushforward	[0,1] $g(x)=x^2$	x	1/3	$\int x\cdot d(g^*\lambda) = \int g(t)d\lambda(t)$

Key Properties

Linearity:
$\int(af + bg)d\mu = a\int fd\mu + b\int gd\mu$

Product Measures:
$\int_{X\times Y} h(x,y)d(\mu\times\nu) = \int_X \int_Y h(x,y)d\nu\, d\mu$

Change of Variables:
For T measurable: $\int_{T(\Omega)} f\, d\nu = \int_\Omega f\circ T\, d\mu$

Convergence Tools:
Monotone Convergence, Dominated Convergence, Fubini-Tonelli

Series Relations:
$\sum_{k=1}^\infty k2^{-k} = 2$, $\sum_{k=0}^\infty k!^{-1}e^{-1} = 1$