Cheat Sheet: Key Sum Operations in Probability & Analysis

1. Geometric Series

- **Convergence Condition**: Infinite geometric series converge **only if** $|r| < 1$.
- **Finite Geometric Series** (for $|r| \neq 1$):

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

- **Infinite Geometric Series**:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

(e.g., $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$)

- **Variants**:
  - Start at $k = m$: $\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$
  - Constant in numerator: $\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$ (for $|r| < 1$)

2. Arithmetic Series

- **Sum of First n Integers**:

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Sum of Squares**:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

- **Sum of Cubes**:

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

3. Special Infinite Series (for $|r| < 1$)

- **Linear multiplier**:

$$\sum_{k=1}^{\infty} k r^k = r\frac{d}{dr}\left(\sum_{k=0}^{\infty} r^k\right) = \frac{r}{(1-r)^2}$$

- **Squared multiplier**:

$$\sum_{k=1}^{\infty} k^2 r^k = \frac{r(1+r)}{(1-r)^3}$$

- **Exponential Series**:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

- **Harmonic Series** (divergent):

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

4. p-Series & Convergence Tests

- **p-Series Test**:
  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges** if $p > 1$ (e.g., $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$), **diverges** if $p \leq 1$.
- **Integral Test**:
  If $f(n) > 0$ and decreasing, $\sum_{n=1}^{\infty} f(n)$ converges $\iff \int_1^{\infty} f(x)dx$ converges.

5. Binomial & Taylor Series

- **Binomial Expansion**:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

For $|x| < 1$ and real α: $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

- **Sum of Binomial Coefficients**:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad \sum_{k=0}^{n} \binom{n}{k} r^k = (1+r)^n$$

6. Telescoping Series

- **Key Trick**: Partial fraction decomposition to cancel terms.
- **Example**:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

Rules of Thumb:

1. **Normalization Check**: For probability, always verify $\sum_{\text{all } \omega} P(\omega) = 1$.
2. **Convergence**: For any series, confirm convergence before applying closed-form formulas.
3. **Shift Indices**: Adjust indices to match standard forms (e.g., shift $n \to n+1$).
4. **Differentiation/Integration**: Useful to derive sums (e.g., differentiate $\sum r^k$ to get $\sum k r^{k-1}$).

Usage Example:
To compute $\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n$:

- Recognize $|r| = \frac{1}{2} < 1 \rightarrow$ series converges.
- Apply $\sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2}$ with $r = \frac{1}{2}$:

$$\text{Sum} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

> **Keep this sheet handy for probability normalizations, expectation calculations, and series convergence checks!**