Universität Zürich FS 2025 STA110 Introduction to Probability Michael Hediger

Mock Exam 3

Instructions

- The exam is open book — the use of AI or any form of communication is not allowed;
- Please make sure that every paper you need has your name and student number on it;
- Unless otherwise stated, full points will not be awarded for solutions without explanation;
- Do not use pencil, red or green pens.

Notation

We recall some of the terminology:

- Given a nonempty set Ω, $P(\Omega)$ is the power set on Ω.
- $B(R^k)$ denotes the Borel σ-field on R^k, $k \geq 1$.
- The measure μ$(A) = \begin{cases} \#A, & \text{if A is finite;} \\ \infty, & \text{otherwise,} \end{cases}$ $A \in P(\Omega)$ is referred to as the counting measure on $P(\Omega)$.
- Given a measurable space (Ω, F) and $x \in \Omega$, we write δ_x for the measure $F \ni A \mapsto \delta_x(A) = \begin{cases} 1, & \text{if x} \in \text{A;} \\ 0, & \text{otherwise.} \end{cases}$
- If not mentioned explicitly, a random vector is assumed to be defined on a probability space (Ω, F, P).

Exercise 1 (10 points)

(a) Definition of a Semiring

Given a nonempty set Ω, write down the definition of a semiring on Ω. [1 point]

> A semiring on Ω is a collection of subsets $S \subset P(\Omega)$ such that:
>
> 1. $\emptyset \in S$
> 2. S is closed under finite intersections, i.e., $A, B \in S \Rightarrow A \cap B \in S$

> 3. For any $A, B \in S$, there exists a finite sequence of disjoint sets $C_1, ..., C_n \in S$ such that $A \setminus B = \cup_{i=1}^{n} C_i$

(b) Measures on a Measurable Space

Given a measurable space (Ω, F), which of the following set functions μ_i, $i = 1, 2, 3$, is not a measure on F? [1.5 point — single choice, no explanation is needed to earn full points]

1. $F = P(\Omega)$ and $\mu_1(A) = 5\mu(A)$, where μ is the counting measure on $P(\Omega)$.
2. F any σ-field on Ω, μ any measure on F and $\mu_2(A) = \int_A f(\omega)\mu(d\omega)$, $A \in F$, where $f : \Omega \to R$ is nonnegative and F measurable.
3. $\Omega = N$, $F = P(N)$ and $\mu_3(A) = (\#A)^2$, $A \in P(N)$.

> The correct answer is 3.

(c) Probability Measures on a Measurable Space

Given a measurable space (E, B), which of the following set functions P_i, $i = 1, 2, 3$, is not a probability measure on B? [1.5 point — single choice, no explanation is needed to earn full points]

1. $E = N$, $B = P(N)$ and $P_1(A) = \sum_{n \in A \cap N} (1/2)^n$, $A \in P(N)$.
2. $E = R$, $B = B(R)$, and $P_2(A) = \lambda(A)$, $A \in B(R)$, where λ is the Lebesgue measure on $B(R)$.
3. $E = R$, $B = B(R)$ and $P_3(A) = \int_A (1/2) 1_{[0,1]}(x) dx$, $A \in B(R)$.

> The correct answer is 2 and 3.

(d) Integrals

Calculate the following integrals: [1 point each]

1. $\int_R x^2 1_{[-2,2]}(x) \lambda(dx)$, where λ is the Lebesgue measure on $B(R)$.

> $\int_{-2}^{2} x^2 dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{8}{3} - \frac{-8}{3} = \frac{16}{3}$

2. $\int_R |x| P(dx)$, where $P(A) = (1/3)\delta_{-1}(A) + (2/3)\delta_1(A)$, $A \in B(R)$.

> $\int |x| P(dx) = |-1| \cdot \frac{1}{3} + |1| \cdot \frac{2}{3} = \frac{1}{3} + \frac{2}{3} = 1$

3. $\int_N \frac{1}{x+1} \mu(dx)$, where μ is the counting measure on $P(N)$.

> $\sum_{n \in N} \frac{1}{n+1}$. Since N is the natural numbers, this sum diverges to infinity. Thus, $\int_N \frac{1}{x+1} \mu(dx) = \infty$

(e) Discrete Laws

Which of the following laws P_i, $i = 1, 2, 3$, is not discrete? [1.5 point — single choice, no explanation is needed to earn full points]

1. The law P_1 of a random variable X with distribution function $F_X(t) = \begin{cases} 0, & \text{if t} < 0; \\ 0.3, & \text{if } 0 \leq \text{t} < 1; \\ 0.7, & \text{if } 1 \leq \text{t} < 2; \\ 1, & \text{if t} \geq 2 \end{cases}$.
2. The law P_2 of a random variable X with distribution function $F_X(t) = \begin{cases} 0, & \text{if t} < 0; \\ t, & \text{if } 0 \leq \text{t} < 1; \\ 1, & \text{if t} \geq 1 \end{cases}$.
3. The law P_3 of the random variable X defined on $\Omega = a, b, c$ with $P(X = a) = 1/3$, $P(X = b) = 1/3$, $P(X = c) = 1/3$.

> The correct answer is 2.

(f) True or False

Decide whether the following statements are true or false: [0.5 point each — no explanation is needed to earn full points]

1. The family of all subsets of Ω, denoted $P(\Omega)$, is the largest possible σ-field on Ω.

> True

2. If $f : R \to R$ is continuous, then f is measurable $B(R)/B(R)$.

> True

3. If X and Y are two independent random variables, then $Cov(X, Y) = 0$.

> True

4. Convergence in probability implies almost sure convergence.

> False

Exercise 2 (13 points)

Let X be a discrete random variable with support $-2, 2$ and law $P_X(A) = (1/2)\delta_{-2}(A) + (1/2)\delta_2(A)$, $A \in B(R)$.

(a) Probabilities

What are $P(X = -2)$ and $P(X = 2)$? [1 point]

> $P(X = -2) = \frac{1}{2}$ and $P(X = 2) = \frac{1}{2}$

(b) Expected Value

Calculate $E[|X|^2]$. [1.5 point]

> $E[|X|^2] = \frac{1}{2}|-2|^2 + \frac{1}{2}|2|^2 = \frac{1}{2}(4) + \frac{1}{2}(4) = 4$

(c) Expected Value and Variance

Find $E[X]$ and $Var(X)$. [2 points]

> $E[X] = \frac{1}{2}(-2) + \frac{1}{2}(2) = 0$
> $Var(X) = E[X^2] - (E[X])^2 = \frac{1}{2}(-2)^2 + \frac{1}{2}(2)^2 - 0^2 = 4$

(d) Law of $(X/2)^2$

What is the law of $(X/2)^2$? [1.5 points]

> $(X/2)^2 = 1$ with probability 1. The law is $P((X/2)^2 = 1) = 1$.

Exercise 3 (18 points)

Let $\phi(x) = \begin{cases} x, & \text{if } 0 \leq \text{x} < 1; \\ 2 - x, & \text{if } 1 \leq \text{x} < 2; \\ 0, & \text{otherwise} \end{cases}$.

(a) Integral of $\phi(x)$

Verify that $\int_R \phi(x)dx = 1$. [2 points]

> $\int_0^1 xdx + \int_1^2 (2-x)dx = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{2} + (4-2) - (2 - \frac{1}{2}) = \frac{1}{2} + 2 - \frac{3}{2} = 1$

Exercise 4 (6 points)

Let X_1 and X_2 be two random variables that are independent with common law that is continuous uniform on the interval $[0, 1]$. What is the probability density function of the random vector $Y =$

$(X_1 + X_2, X_1 - X_2)$?

Exercise 5 (6 points)

Let X be a discrete random variable with support $0, 1, ..., M$, where M is a positive integer. Suppose that X has law defined upon: $P(X = k) = C(k + 1)$, $k = 0, 1, ..., M$, where $C \in R$. Find C.

> To find C, we need $\sum_{k=0}^{M} P(X = k) = 1$. Thus, $\sum_{k=0}^{M} C(k + 1) = 1$.
> $C \sum_{k=0}^{M} (k + 1) = C \sum_{j=1}^{M+1} j = C \frac{(M+1)(M+2)}{2} = 1$.
> $C = \frac{2}{(M+1)(M+2)}$