Variation 1 (Difficulty: ★☆☆☆☆)

Let

$$\phi(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{8}x + \frac{1}{4} & -2 \le x < 0 \\ \frac{1}{4} - \frac{1}{16}x & 0 \le x < 4 \\ 0 & x \ge 4 \end{cases}$$

(a) Verify that $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$. [2]

(b) Find the distribution function F_X of X. [4]

(c) Calculate $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [5]

(d) Show $F_X|_{(-2,4)} : (-2,4) \to (0,1)$ is a bijection. [3]

(e) Verify $F_X|_{[-2,4]} : [-2,4] \to [0,1]$ is a bijection. [1]

(f) Find $F_X^{-1}|_{[-2,4]}$. [3]

Variation 2 (Difficulty: ★★☆☆☆)

Let

$$\phi(x) = \begin{cases} 0 & x < -4 \\ \frac{1}{36}x + \frac{1}{12} & -4 \le x < -1 \\ \frac{1}{6} & -1 \le x < 1 \\ \frac{1}{12} - \frac{1}{72}(x-3) & 1 \le x < 4 \\ 0 & x \ge 4 \end{cases}$$

(a) Verify $\int \phi(x)\,dx = 1$. [2]

(b) Derive F_X. [5]

(c) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [5]

(d) Prove $F_X|_{(-4,4)} : (-4,4) \to (0,1)$ is bijective. [3]

(e) Explain why $F_X|_{[-4,4]} : [-4,4] \to [0,1]$ is bijective. [1]

(f) Find $F_X^{-1}|_{[-4,4]}$. [4]

Variation 3 (Difficulty: ★★★☆☆)

Let

$$\phi(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{8}x & 0 \leq x < 2 \\ \frac{1}{4}e^{2-x} & 2 \leq x < 4 \\ 0 & x \geq 4 \end{cases}$$

(a) Show $\int \phi(x)\,dx = 1$. [3] *(Requires integration by parts)*

(b) Find F_X. [5]

(c) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [6]

(d) Prove bijectivity for $F_X|_{(0,4)} : (0,4) \to (0,1)$. [3]

(e) Show $F_X|_{[0,4]} : [0,4] \to [0,1]$ is bijective. [1]

(f) Derive $F_X^{-1}|_{[0,4]}$. [4] *(Inverse involves logarithms)*

Variation 4 (Difficulty: ★★★★☆)

Let

$$\phi(x) = \begin{cases} 0 & x < -\pi \\ \frac{1}{8}\cos\left(\frac{x}{2}\right) & -\pi \leq x < 0 \\ \frac{1}{8}(1 + \sin x) & 0 \leq x < \frac{\pi}{2} \\ 0 & x \geq \frac{\pi}{2} \end{cases}$$

(a) Verify $\int \phi(x)\,dx = 1$. [4] *(Trigonometric integrals)*

(b) Find F_X. [6] *(Multi-part integration)*

(c) Calculate $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [7]

(d) Prove $F_X|_{(-\pi,\pi/2)} : (-\pi, \pi/2) \to (0,1)$ is bijective. [3]

(e) Is $F_X|_{[-\pi,\pi/2]} : [-\pi, \pi/2] \to [0,1]$ bijective? Justify. [2]

(f) Find $F_X^{-1}|_{[-\pi,\pi/2]}$. [5] *(Requires solving transcendental equations)*

Variation 5 (Difficulty: ★★★★★)

Let

$$\phi(x) = \begin{cases} 0 & x < -3 \\ \frac{1}{20}(x+5) & -3 \le x < -1 \\ \frac{1}{10\sqrt{4-x^2}} & -1 \le x < 1 \\ \frac{1}{20}(5-x) & 1 \le x < 5 \\ 0 & x \ge 5 \end{cases}$$

(a) Prove $\int \phi(x)\,dx = 1$. [5] *(Uses geometry and trig substitution)*

(b) Derive F_X explicitly. [7] *(Requires elliptic integral for $\sqrt{4-x^2}$)*

(c) Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [8] *(High-symmetry simplifies variance)*

(d) Show $F_X|_{(-3,5)} : (-3,5) \to (0,1)$ is bijective. [4]

(e) Verify $F_X|_{[-3,5]} : [-3,5] \to [0,1]$ is bijective. [2]

(f) Find $F_X^{-1}|_{[-3,5]}$ symbolically. [6] *(Inverse for $\sqrt{4-x^2}$ is non-algebraic)*

Progression Rationale

1. **Variation 1**: Asymmetric intervals, basic linear pieces.
2. **Variation 2**: Four segments with discontinuities and shifted intervals.
3. **Variation 3**: Exponential decay segment requiring advanced integration.
4. **Variation 4**: Trigonometric functions and transcendental inverse.
5. **Variation 5**: Elliptic integral in CDF, non-algebraic inverse, and asymmetric support.