Notation

We recall some of the terminology:

- Given a nonempty set Ω, $\mathcal{P}(\Omega)$ is the power set on Ω.
- $\mathcal{B}(\mathbb{R}^k)$ denotes the Borel σ-field on $\mathbb{R}^k$, $k \geq 1$.
- The measure $\mu(A) = \#A$ if A is finite, and ∞ otherwise, for $A \in \mathcal{P}(\Omega)$, is the **counting measure** on $\mathcal{P}(\Omega)$.
- Given a measurable space $(\Omega, \mathcal{F})$ and $x \in \Omega$, we write δ_x for the measure $\mathcal{F} \ni A \mapsto \delta_x(A) = \mathbf{1}_A(x)$.
- The Lebesgue measure on $\mathcal{B}(\mathbb{R}^k)$ is denoted by λ_k. When $k = 1$, we write λ.
- If not mentioned explicitly, a random vector is assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 1 (10 points)

(a) Borel-Cantelli Lemmas [2 points]
State the first and second Borel-Cantelli Lemmas.

(b) Not a Measure [1.5 points — single choice, no explanation needed]
Given the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which of the following set functions μ_i is **not** a measure?

- $\mu_1(A) = \sum_{q \in \mathbb{Q} \cap A} 1$, where $\mathbb{Q}$ is the set of rational numbers.
- Let C be the standard Cantor set. $\mu_2(A) = \lambda(A \cap C^c)$, where $C^c = \mathbb{R} \setminus C$.
- $\mu_3(A) = \begin{cases} 0 & \text{if } \lambda(A) = 0 \\ \infty & \text{if } \lambda(A) > 0 \end{cases}$

(c) Not a Probability Measure [1.5 points — single choice, no explanation needed]
Which of the following definitions for $\mathbb{P}_i$ does **not** describe a valid probability measure?

- Let $X \sim \mathrm{Exp}(1)$. $\mathbb{P}_1(A) = \mathbb{P}(X^2 \in A)$ for $A \in \mathcal{B}(\mathbb{R})$.
- $\mathbb{P}_2(A) = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \delta_n(A)$ for $A \in \mathcal{P}(\mathbb{N})$.
- Let $F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x)$. $\mathbb{P}_3((-\infty, x]) = F(x)$ for all $x \in \mathbb{R}$.

(d) Calculate the following integrals: [1 point each]

1. $\int_{\mathbb{N}} \frac{1}{x!} \mu(dx)$, where μ is the counting measure on $\mathcal{P}(\mathbb{N})$.
2. $\int_{\mathbb{R}} \sin^2(\pi x) \mathbb{P}(dx)$, where $\mathbb{P}$ is the law of a random variable $X \sim \mathrm{Uniform}[0, 2]$.

3. $\int_{\mathbb{R}^2} \frac{1}{y} \mathbb{P}(d(x, y))$, where $\mathbb{P}$ is the law of a random vector (X, Y) with density $f(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$.

(e) Modes of Convergence [1.5 points — single choice, no explanation needed]

Let X_n be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Which statement is **false**?

1. If $X_n \to X$ almost surely, then $X_n \to X$ in probability.
2. If $X_n \to X$ in L^2, then $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$.
3. If $X_n \to c$ in probability, where c is a constant, then $X_n \to c$ in distribution.

(f) True or False? [0.5 point each — no explanation needed]

1. Let X, Y be random variables. If $\mathbb{E}[X|Y] = \mathbb{E}[X]$ almost surely, then X and Y are independent.
2. The characteristic function $\phi_X(v) = \cos(v)$ corresponds to a random variable X with a well-defined probability density function.

Exercise 2 (13 points)

Let $a \in (0, 1)$ be a fixed parameter. Consider a discrete random variable X with support on the set of all integers $\mathbb{Z}$ and a probability mass function (PMF) given by:

$$\mathbb{P}(X = k) = C_a \cdot a^{|k|}, \quad k \in \mathbb{Z}$$

where C_a is a normalization constant.

(a) Find the constant C_a. [2 points]
(b) Calculate $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [3 points]
(c) Let $Y = X^2$. Find the law of Y, i.e., its PMF. [2 points]
(d) Let X_1, X_2 be two independent copies of X. Find the probability $\mathbb{P}(X_1 + X_2 = 1)$. [3 points]
(e) Calculate the conditional probability $\mathbb{P}(X_1 = 1 | X_1 + X_2 = 0)$. [3 points]

Exercise 3 (18 points)

Let $\alpha > 0$ and $\beta > 0$ be parameters. A random variable X has a probability density function (PDF) f_X given by:

$$f_X(x) = \begin{cases} Cx^{\alpha-1}(1-x)^{\beta-1} & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

This defines the Beta distribution, $X \sim \mathrm{Beta}(\alpha, \beta)$. The normalization constant is given by $C = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$, where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function. You may use the property $\Gamma(z+1) = z\Gamma(z)$ for $z > 0$.

(a) For the specific case $\alpha = 2, \beta = 3$, verify that $C = 12$. [2 points]

(b) For the general case, calculate $\mathbb{E}[X]$. [4 points]

(c) For the general case, calculate $\mathbb{E}[X^2]$ and find $\mathrm{Var}(X)$. [5 points]

(d) Let $X \sim \mathrm{Beta}(1, 3)$. Find the distribution function $F_X(t)$ of X. [4 points]

(e) Let $X \sim \mathrm{Beta}(\alpha, \beta)$. Find the density of the random variable $Y = -\ln(X)$. You do not need to identify the name of the resulting distribution. [3 points]

Exercise 4 (6 points)

Let X and Y be independent random variables with $X \sim \mathrm{Exponential}(\lambda)$ and $Y \sim \mathrm{Exponential}(\mu)$. Their respective PDFs are $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f_Y(y) = \mu e^{-\mu y}$ for $y > 0$.

Find the probability density function of the random variable $Z = X - Y$.

Hint: Consider the cases $Z > 0$ and $Z < 0$ separately. This may involve a convolution-style integral.

Exercise 5 (6 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed random variables with $\mathbb{P}(X_n = 1) = p$ and $\mathbb{P}(X_n = -1) = 1 - p$, where $p \in (0, 1)$ and $p \neq 1/2$. Let $S_n = \sum_{k=1}^n X_k$ be the simple random walk starting at $S_0 = 0$. Define the random variable $M_n = \left(\frac{1-p}{p}\right)^{S_n}$.

Show that $(M_n)_{n\in\mathbb{N}}$ is a martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

Recall: To show M_n is a martingale, you must verify three conditions:

1. M_n is $\mathcal{F}_n$-measurable for all n.
2. $\mathbb{E}[|M_n|] < \infty$ for all n.

3. $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$ for all n.