Chapter 4: Measurable Spaces

4.1 Families of Measurable Sets

4.1.1 σ-fields

Definition 4.1: σ-field

Let Ω be a non-empty set. A family of subsets $\mathcal{F}$ of Ω is called a **σ-field** (or σ-algebra) on Ω if it satisfies:
(i) The entire set is in the family: $\Omega \in \mathcal{F}$.
(ii) It is closed under complementation: If $A \in \mathcal{F}$, then its complement $A^c = \Omega \setminus A$ is also in $\mathcal{F}$.
(iii) It is closed under countable unions: If $\{A_i : i \in \mathbb{N}\}$ is a collection of sets where each $A_i \in \mathcal{F}$, then their union $\cup_{i \in \mathbb{N}} A_i$ is also in $\mathcal{F}$.

Remarks & Properties:

- From the definition, a σ-field $\mathcal{F}$ must also contain the empty set $\emptyset$, since $\Omega \in \mathcal{F}$ and $\emptyset = \Omega^c$.
- A σ-field is closed under countable intersections. If $\{A_i : i \in \mathbb{N}\} \subset \mathcal{F}$, then $\cap_{i \in \mathbb{N}} A_i = (\cup_{i \in \mathbb{N}} A_i^c)^c$, which is in $\mathcal{F}$.

Examples of σ-fields:

1. **Trivial σ-field:** $\mathcal{F} = \{\emptyset, \Omega\}$. This is the smallest possible σ-field on any set Ω.
2. **Power Set:** $\mathcal{F} = \mathcal{P}(\Omega)$ (the set of all subsets of Ω). This is the largest possible σ-field.

Definition 4.2: Measurable Space & Measurable Set

- A pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ-field on Ω, is called a **measurable space**.
- A set A is said to be **measurable** if $A \in \mathcal{F}$.

4.1.2 σ-fields Generated by Families of Sets

Proposition 4.1: Generated σ-field

Let $\mathcal{G}$ be any family of subsets of Ω. There exists a unique, smallest σ-field, denoted $\sigma(\mathcal{G})$, that contains $\mathcal{G}$. It is defined as the intersection of all σ-fields on Ω that contain $\mathcal{G}$.

$$\sigma(\mathcal{G}) = \bigcap\{\mathcal{U} : \mathcal{U} \text{ is a } \sigma\text{-field on } \Omega \text{ and } \mathcal{G} \subset \mathcal{U}\}$$

$\sigma(\mathcal{G})$ is called the **σ-field generated by** $\mathcal{G}$.

4.1.3 Borel Sets of Real Coordinate Spaces

Definition 4.3 & 4.4: Borel σ-field

- The **Borel σ-field on the real numbers**, denoted $\mathcal{B}(\mathbb{R})$, is the σ-field generated by the family of all open (or closed, or half-open) intervals in $\mathbb{R}$.
- The **Borel σ-field on** $\mathbb{R}^k$, denoted $\mathcal{B}(\mathbb{R}^k)$, is the σ-field generated by the family of all open rectangles in $\mathbb{R}^k$.
- The elements of a Borel σ-field are called **Borel sets**.

Chapter 5: Measure Spaces

5.1 The Notion of a Measure

Definition 5.1: Measure
Let $(\Omega, \mathcal{F})$ be a measurable space. A function $\mu : \mathcal{F} \to [0, \infty]$ is a **measure** if it satisfies:
(i) The measure of the empty set is zero: $\mu(\emptyset) = 0$.
(ii) **Countable Additivity (or σ-additivity):** For any countable collection of pairwise disjoint sets $\{A_i : i \in \mathbb{N}\} \subset \mathcal{F}$, the measure of their union is the sum of their measures:

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

Definition 5.2: Measure Space
A triple $(\Omega, \mathcal{F}, \mu)$ consisting of a set Ω, a σ-field $\mathcal{F}$ on Ω, and a measure μ on $\mathcal{F}$, is called a **measure space**.

Examples of Measures

1. **Point (Dirac) Measure:** For a given point $x \in \Omega$, $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise.
2. **Counting Measure:** Defined on $(\Omega, \mathcal{P}(\Omega))$, $\mu(A) = \#A$ (the number of elements in A).
3. **Lebesgue Measure (λ):** A unique measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that assigns to any interval its length.

5.1.3 Properties of a Measure

- **Monotonicity:** If $A \subset B$, then $\mu(A) \leq \mu(B)$.
- **Continuity from Below:** If $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.
- **Continuity from Above:** If $A_n \downarrow A$ and $\mu(A_1) < \infty$, then $\mu(A_n) \downarrow \mu(A)$.
- **Countable Subadditivity:** For any collection $\{A_i\}$, $\mu(\cup_i A_i) \leq \sum_i \mu(A_i)$.

5.2 Semirings

Definition 5.3: Semiring

A family of subsets $\mathcal{A}$ of Ω is a **semiring** if it satisfies:
(i) $\emptyset \in \mathcal{A}$.
(ii) It is closed under finite intersections: If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.
(iii) The difference of two sets can be written as a finite disjoint union of sets in the family.

Chapter 6: From Outer Measure to Measure

This chapter describes how to construct a measure on a σ-field, like the Lebesgue measure.

Key Concepts

1. **Outer Measure (μ^*):** A function on $\mathcal{P}(\Omega)$ that is non-negative, zero for $\emptyset$, monotone, and countably subadditive.
2. **Carathéodory's Criterion:** An outer measure μ^* can be restricted to a σ-field of "**μ^*-measurable sets**" to become a true measure. A set A is μ^*-measurable if $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ for all $E \subset \Omega$.
3. **Carathéodory's Extension Theorem:** A finitely additive and countably subadditive function ρ on a semiring $\mathcal{A}$ can be extended to a unique measure on the generated σ-field $\sigma(\mathcal{A})$.
4. **Uniqueness of Lebesgue Measure:** The Lebesgue measure λ is the unique measure on $\mathcal{B}(\mathbb{R})$ that satisfies $\lambda((a, b]) = b - a$.

Chapter 7: Measurable Functions

7.1 The Concept of Measurable Functions

Definition 7.1: Measurable Function
Let $(\Omega, \mathcal{F})$ and $(\Omega^*, \mathcal{F}^*)$ be two measurable spaces. A function $f : \Omega \to \Omega^*$ is **measurable** (specifically, $\mathcal{F}/\mathcal{F}^*$ measurable) if the pre-image of every measurable set in the target space is a measurable set in the domain space.

$$\forall A^* \in \mathcal{F}^*, \quad f^{-1}(A^*) = \{\omega \in \Omega : f(\omega) \in A^*\} \in \mathcal{F}$$

Proposition 7.1: A Simpler Criterion for Measurability
If $\mathcal{F}^* = \sigma(\mathcal{G})$ for some family of sets $\mathcal{G}$, then f is measurable if and only if $f^{-1}(G) \in \mathcal{F}$ for all $G \in \mathcal{G}$.

Key Types of Measurable Functions

- **Borel Function:** A function $f : \mathbb{R}^m \to \mathbb{R}^k$ that is $\mathcal{B}(\mathbb{R}^m)/\mathcal{B}(\mathbb{R}^k)$ measurable.
- **Real-Valued Measurable Function:** A function $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. It is sufficient to check that sets of the form $\{\omega : f(\omega) \le x\}$ are in $\mathcal{F}$ for all $x \in \mathbb{R}$.
- **Indicator Function:** The function $1_A(\omega)$ is $\mathcal{F}$-measurable if and only if the set $A \in \mathcal{F}$.

Properties of Measurable Functions

- **Compositions:** The composition of measurable functions is measurable.
- **Continuous Functions:** Any continuous function $f : \mathbb{R}^m \to \mathbb{R}^k$ is a Borel function.
- **Algebraic Operations:** If $f, g : \Omega \to \mathbb{R}$ are measurable, then so are $f + g$, fg, cf (for $c \in \mathbb{R}$), $\max\{f, g\}$, $\min\{f, g\}$, and $|f|$.
- **Limits:** If $\{f_n\}$ is a sequence of measurable functions, then $\sup f_n$, $\inf f_n$, $\limsup f_n$, and $\liminf f_n$ are all measurable. If $f_n \to f$ pointwise, then f is also measurable.

7.1.1 Simple Functions

Definition 7.3: Simple Function
A function $f : \Omega \to \mathbb{R}$ is **simple** if it takes only a finite number of values. It can always be written as a finite linear combination of indicator functions:

$$f(\omega) = \sum_{i=1}^{n} \alpha_i 1_{A_i}(\omega)$$

A simple function is measurable if all its carrier sets A_i are measurable.

Proposition 7.10 & 7.11: Approximation by Simple Functions

- Any non-negative measurable function f is the pointwise limit of a monotonically increasing sequence of non-negative simple measurable functions.
- Any measurable function $f : \Omega \to \mathbb{R}$ is the pointwise limit of a sequence of simple measurable functions.

7.3 Minimal Measurability

σ-field Generated by a Function
The σ-field generated by a function $f : \Omega \to \mathbb{R}^k$ is the smallest σ-field on Ω that makes f measurable. It is given by:

$$\sigma(f) = \{f^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^k)\}$$

Proposition 7.13: Doob-Dynkin Lemma
Let $f : \Omega \to \mathbb{R}^k$ be a function. A function $h : \Omega \to \mathbb{R}$ is $\sigma(f)$-measurable if and only if there exists a Borel measurable function $g : \mathbb{R}^k \to \mathbb{R}$ such that $h = g(f)$.

Chapter 8: Integration (First Part)

8.1 The Integral for Nonnegative Functions

Definition 8.3: The Integral
For a non-negative measurable function $f : (\Omega, \mathcal{F}, \mu) \to [0, \infty]$, the integral is defined as:

$$\int_{\Omega} f(\omega)\mu(d\omega) = \sup \left\{ \sum_{A \in \xi} \inf_{\omega \in A} f(\omega)\mu(A) : \xi \text{ is a finite measurable partition of } \Omega \right\}$$

Integral of a Simple Function: For a non-negative simple function $f = \sum_{i=1}^{N} \alpha_i 1_{A_i}$ in standard form, the integral is:

$$\int_\Omega f\,d\mu = \sum_{i=1}^{N} \alpha_i \mu(A_i)$$

Proposition 8.2: Monotone Convergence Theorem (MCT)
If $\{f_n\}$ is a sequence of non-negative measurable functions such that $f_n(\omega) \uparrow f(\omega)$ for all ω, then:

$$\lim_{n\to\infty} \int f_n\,d\mu = \int f\,d\mu$$

Proposition 8.3: Linearity
For non-negative measurable functions f, g and $\alpha, \beta \geq 0$:

$$\int (\alpha f + \beta g)\,d\mu = \alpha \int f\,d\mu + \beta \int g\,d\mu$$

Definition 8.4: Almost Everywhere (a.e.)
A property holds **μ-almost everywhere** if the set of points where it fails has measure zero.

8.2 Integrable Functions

Positive and Negative Parts: $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Definition 8.5: Integral of a Real-Valued Function
For a measurable function $f : \Omega \to \mathbb{R}$, its integral is defined as:

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,d\mu$$

provided at least one of the integrals on the right is finite. The function f is **integrable** if both are finite.

Proposition 8.6: Integrability Criterion
A measurable function f is integrable if and only if $\int |f|\,d\mu < \infty$.

8.3 Limit Theorems

Proposition 8.9: Fatou's Lemma
For any sequence $\{f_n\}$ of non-negative measurable functions:

$$\int (\liminf_{n\to\infty} f_n)\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu$$

Proposition 8.10: Lebesgue's Dominated Convergence Theorem (DCT)
Let $\{f_n\}$ be a sequence of measurable functions such that:

1. $f_n(\omega) \to f(\omega)$ a.e.
2. There exists an integrable function g such that $|f_n(\omega)| \leq g(\omega)$ a.e. for all n.
   Then f is integrable and:

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu = \int (\lim_{n\to\infty} f_n) \, d\mu$$

8.4 Integration over Measurable Sets

Definition 8.6: The integral of f over a set $A \in \mathcal{F}$ is defined as:

$$\int_A f \, d\mu = \int f \cdot 1_A \, d\mu$$

Chapter 9: Integration (Second Part)

9.1 Pushforward Measure

Definition 9.1: Pushforward Measure
Let μ be a measure on $(\Omega, \mathcal{F})$ and $g : (\Omega, \mathcal{F}) \to (\Omega^*, \mathcal{F}^*)$ be a measurable function. The **pushforward measure** $\mu_{g^{-1}}$ (or simply μ_g) on $(\Omega^*, \mathcal{F}^*)$ is defined by:

$$\mu_{g^{-1}}(A^*) = \mu(g^{-1}(A^*)) \quad \text{for any } A^* \in \mathcal{F}^*$$

Proposition 9.1: Change of Variables Formula
Let $f : (\Omega^*, \mathcal{F}^*) \to \mathbb{R}$ be a measurable function that is either non-negative or integrable with respect to $\mu_{g^{-1}}$. Then $f \circ g$ is either non-negative or integrable with respect to μ, and:

$$\int_{\Omega^*} f(y) \, \mu_{g^{-1}}(dy) = \int_{\Omega} f(g(\omega)) \, \mu(d\omega)$$

9.2 Densities

Definition 9.2: Density
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A non-negative measurable function $\phi : \Omega \to \mathbb{R}$ is a **density** (or Radon-Nikodym derivative) of a measure ν with respect to a measure μ if:

$$\nu(A) = \int_A \phi(\omega)\, \mu(d\omega) \quad \text{for all } A \in \mathcal{F}$$

This is often written as $d\nu = \phi\, d\mu$ or $\frac{d\nu}{d\mu} = \phi$.

Proposition 9.3: Integration with a Density
If ν has density ϕ with respect to μ, then for any non-negative or ν-integrable function f:

$$\int f\, d\nu = \int f \cdot \phi\, d\mu$$