Cheat Sheet: Key Sum Operations in Probability & Analysis

1. Geometric Series

- **Convergence Condition**: Infinite geometric series converge **only if** $|r| < 1$.
- **Finite Geometric Series** (for $|r| \neq 1$):

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$$

- **Infinite Geometric Series**:

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

(e.g., $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$)

- **Variants**:
  - Start at $k = m$: $\sum_{k=m}^{\infty} r^k = \frac{r^m}{1-r}$
  - Constant in numerator: $\sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r}$ (for $|r| < 1$)

2. Arithmetic Series

- **Sum of First n Integers**:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

- **Sum of Squares**:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

- **Sum of Cubes**:

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

3. Special Infinite Series (for $|r| < 1$)

- **Linear multiplier**:

$$\sum_{k=1}^{\infty} kr^k = r\frac{d}{dr}\left(\sum_{k=0}^{\infty} r^k\right) = \frac{r}{(1-r)^2}$$

- **Squared multiplier**:

$$\sum_{k=1}^{\infty} k^2 r^k = \frac{r(1+r)}{(1-r)^3}$$

- **Exponential Series**:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

- **Harmonic Series** (divergent):

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

4. p-Series & Convergence Tests

- **p-Series Test**:
  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges** if $p > 1$ (e.g., $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$), **diverges** if $p \leq 1$.
- **Integral Test**:
  If $f(n) > 0$ and decreasing, $\sum_{n=1}^{\infty} f(n)$ converges $\iff \int_1^{\infty} f(x)dx$ converges.

5. Binomial & Taylor Series

- **Binomial Expansion**:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

For $|x| < 1$ and real α: $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$.

- **Sum of Binomial Coefficients**:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad \sum_{k=0}^{n} \binom{n}{k} r^k = (1+r)^n$$

6. Telescoping Series

- **Key Trick**: Partial fraction decomposition to cancel terms.
- **Example**:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1$$

Rules of Thumb:

1. **Normalization Check**: For probability, always verify $\sum_{\text{all } \omega} P(\omega) = 1$.
2. **Convergence**: For any series, confirm convergence before applying closed-form formulas.
3. **Shift Indices**: Adjust indices to match standard forms (e.g., shift $n \to n+1$).
4. **Differentiation/Integration**: Useful to derive sums (e.g., differentiate $\sum r^k$ to get $\sum kr^{k-1}$).

Usage Example:
To compute $\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n$:

- Recognize $|r| = \frac{1}{2} < 1 \rightarrow$ series converges.
- Apply $\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$ with $r = \frac{1}{2}$:

$$\text{Sum} = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

Quick Reference Table

Sum	Closed Form	Conditions
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	$n \geq 1$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	$n \geq 1$
$\sum_{k=1}^{n} k^3$	$\left[\frac{n(n+1)}{2}\right]^2$	$n \geq 1$
$\sum_{k=0}^{n} r^k$	$\frac{1-r^{n+1}}{1-r}$	$r \neq 1$
$\sum_{k=0}^{n} kr^k$	$r\frac{1-(n+1)r^n+nr^{n+1}}{(1-r)^2}$	$r \neq 1$
$\sum_{k=0}^{n} \binom{n}{k}$	2^n	$n \geq 0$
$\sum_{k=0}^{n} k\binom{n}{k}$	$n \cdot 2^{n-1}$	$n \geq 1$
$\sum_{k=0}^{n} k^2\binom{n}{k}$	$n(n+1) \cdot 2^{n-2}$	$n \geq 2$
$\sum_{k=0}^{m} \binom{n}{k}$	No closed form; use approximations	$0 \leq m \leq n$

Series Type	Sum/Result	Key Trick/Approach	Conditions/Comments
Geometric	$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$	Identify pattern $1 + r + r^2 + \cdots$	$\|r\| < 1$
Geometric (negative exponent, n≥0)	$\sum_{n=0}^{\infty} r^{-n} = \frac{r}{r-1}$	Pattern: $1 + r^{-1} + r^{-2} + \cdots$	$r \in (-\infty, -1) \cup (1, \infty)$
Geometric (shifted)	$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$	Identify pattern $r + r^2 + \cdots$	a∈(0,1)

Series Type	Sum/Result	Key Trick/Approach	Conditions/Comments
Geometric (shifted, neg)	$\sum_{n=1}^{\infty} r^{-n} = \frac{1}{r-1}$	Pattern: $r^{-1} + r^{-2} + \cdots$	$r \in (-\infty, -1) \cup (1, \infty)$
Geometric (general)	$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$	First term is a. Common ratio is r.	$\|r\| < 1$
Arith.-Geometric Series	$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2}$	-	$$
Arith.-Geometric Series	$\sum_{n=1}^{\infty} n^2r^n = \frac{r(1+r)}{(1-r)^3}$	-	$$
Telescoping	$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$	Partial fractions $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$	Cancellation leaves first/last terms
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$	Compare to integral $\int_1^{\infty} x^{-p} dx$	Diverges if $p \leq 1$
Alternating Series	$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$	Leibniz test: terms $\downarrow$ to 0	Error $\leq \|a_{n+1}\|$
Basel Problem ($\zeta(2)$)	$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$	Fourier series of $f(x) = x^2$ or Parseval's theorem	Generalizes to $\zeta(2k)$
Exponential	$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$	Taylor series expansion at $x = 0$	
Exponential (Reciprocal Factorial)	$\sum_{n=0}^{\infty} \frac{1}{n!} = e$	Taylor series of e^x evaluated at $x = 1$	Core definition of e; converges absolutely
Harmonic Variant	$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$	Partial fractions $\frac{1}{n} - \frac{1}{n+1}$	Telescopes completely
Generating Function	$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$	Binomial theorem or hypergeometric identities	$\|x\| < \frac{1}{4}$

Series Type	Sum/Result	Key Trick/Approach	Conditions/Comments
Leibniz (Arctangent)	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$	Integrate $\sum (-1)^n x^{2n} = \frac{1}{1+x^2}$	$\tan^{-1}(1)$ special case
Logarithmic	$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$	Integrate geometric series term-by-term	$-1 \leq x < 1$
Zeta at Even Integers	$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!}$	Bernoulli numbers B_n (e.g., $B_2 = \frac{1}{6}$)	$\zeta(4) = \frac{\pi^4}{90}$
Gaussian-like Sums	$\sum_{n=-\infty}^{\infty} e^{-an^2} = \sqrt{\frac{\pi}{a}}$	Poisson summation or completing the square	Relates to theta functions
Euler-Maclaurin	$\sum_{k=1}^{n} f(k) \approx \int f + \frac{f(1)+f(n)}{2} + \frac{B_2}{2!}(f'(n) - f'(1)) + \cdots$	Approximate sums via integrals + corrections	Useful for asymptotic analysis