Variation 1: Asymmetric Binary Support

Exercise: Let X be a discrete random variable with support $\{-1, 1\}$ and law:

$$P(X = -1) = \frac{1}{3}, \quad P(X = 1) = \frac{2}{3}.$$

Let $X_1, X_2, \ldots, X_n$ be n independent copies of X, and define the random vector $Y = (X_1, X_2, \ldots, X_n)$.

(a) $\mathbb{P}(X = -1)$ and $\mathbb{P}(X = 1)$?

Solution:

> $\mathbb{P}(X = -1) = \frac{1}{3}, \mathbb{P}(X = 1) = \frac{2}{3}$.

(b) Calculate $E[|X|^2]$.

Solution:

> $|X|^2 = X^2$.
> $X^2 = 1$ (since $(-1)^2 = 1, 1^2 = 1$).
> $E[X^2] = 1 \cdot P(X^2 = 1) = 1$.

(c) Find $E[X]$ and $\mathrm{Var}(X)$.

Solution:

> $E[X] = (-1) \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{1}{3}$.
> $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 1 - \left(\frac{1}{3}\right)^2 = 1 - \frac{1}{9} = \frac{8}{9}$.

(d) Law of $\frac{X+1}{2}$?

Solution:

> Let $T = \frac{X+1}{2}$.
>
> - If $X = -1, T = 0$ with $P = \frac{1}{3}$.
> - If $X = 1, T = 1$ with $P = \frac{2}{3}$.
>   Support: $\{0, 1\}$, law: $P(T = 0) = \frac{1}{3}, P(T = 1) = \frac{2}{3}$.

(e) Law of Y? Calculate $P(Y \in \{1\}^n)$.

Solution:

> Law of Y: product measure on $\{-1, 1\}^n$ with $P(Y = (y_1, \ldots, y_n)) = \prod_{i=1}^n P(X_i = y_i)$.
> $P(Y \in \{1\}^n) = P(X_1 = 1, \ldots, X_n = 1) = \left(\frac{2}{3}\right)^n$.

(f) Show $\|Y\|_2^2$ is constant $\mathbb{P}$-a.s.

Solution:

> $\|Y\|_2^2 = \sum_{i=1}^n X_i^2$. Since $X_i^2 = 1$ for all i, $\|Y\|_2^2 = n$. Constant everywhere.

(g) Law of $Z = X_1 + X_2$?

Solution:

> - $Z = -2$ if $X_1 = -1, X_2 = -1$: $P = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.
> - $Z = 0$ if mixed: $P = P(X_1 = -1, X_2 = 1) + P(X_1 = 1, X_2 = -1) = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$.
> - $Z = 2$ if both 1: $P = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$.
>   Law: $P(Z = -2) = \frac{1}{9}$, $P(Z = 0) = \frac{4}{9}$, $P(Z = 2) = \frac{4}{9}$.

Variation 2: Three-Valued Symmetric Support

Exercise: Let X be a discrete random variable with support $\{-1, 0, 1\}$ and law:

$$P(X = -1) = \frac{1}{4}, \quad P(X = 0) = \frac{1}{2}, \quad P(X = 1) = \frac{1}{4}.$$

Let $X_1, X_2, \dots, X_n$ be n independent copies of X, and define $Y = (X_1, X_2, \dots, X_n)$.

(a) $\mathbb{P}(X = -1), \mathbb{P}(X = 0), \mathbb{P}(X = 1)$?

Solution:

> $\mathbb{P}(X = -1) = \frac{1}{4}, \mathbb{P}(X = 0) = \frac{1}{2}, \mathbb{P}(X = 1) = \frac{1}{4}$.

(b) Calculate $E[|X|^2]$.

Solution:

> $|X|^2 = X^2$.
> $E[X^2] = (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{4} = \frac{1}{4} + 0 + \frac{1}{4} = \frac{1}{2}$.

(c) Find $E[X]$ and $\mathrm{Var}(X)$.

Solution:
$E[X] = (-1) \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = 0$.
$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - 0 = \frac{1}{2}$.

(d) Law of $\frac{X+1}{2}$?

Solution:

> Let $T = \frac{X+1}{2}$.
>
> - $X = -1$: $T = 0$, $P = \frac{1}{4}$.
> - $X = 0$: $T = 0.5$, $P = \frac{1}{2}$.
> - $X = 1$: $T = 1$, $P = \frac{1}{4}$.
>   Support: $\{0, 0.5, 1\}$, law: $P(T = 0) = \frac{1}{4}$, $P(T = 0.5) = \frac{1}{2}$, $P(T = 1) = \frac{1}{4}$.

(e) Law of Y? Calculate $P(Y \in \{1\}^n)$.

Solution:

> Law of Y: product measure on $\{-1,0,1\}^n$ with $P(Y=(y_1,\dots,y_n))=\prod_{i=1}^n P(X_i=y_i)$.
> $P(Y\in\{1\}^n)=P(X_1=1,\dots,X_n=1)=\left(\frac{1}{4}\right)^n$.

(f) Show $\|Y\|_2^2$ is **not** constant $\mathbb{P}$-a.s. and find $P(\|Y\|_2^2=0)$ and $P(\|Y\|_2^2=n)$.

Solution:

> $\|Y\|_2^2=\sum_{i=1}^n X_i^2$. Since $X_i^2=0$ if $X_i=0$ and 1 otherwise:
>
> - $P(\|Y\|_2^2=0)=P(\text{all } X_i=0)=\left(\frac{1}{2}\right)^n$.
> - $P(\|Y\|_2^2=n)=P(\text{no } X_i=0)=\left(\frac{1}{2}\right)^n$ (as $P(X_i\neq 0)=\frac{1}{2}$).
>   Since $0<\left(\frac{1}{2}\right)^n<1$ for $n\geq 1$, $\|Y\|_2^2$ is not constant.

(g) Law of $Z=X_1+X_2$?

Solution:

> Support: $\{-2,-1,0,1,2\}$. Calculations:
>
> - $Z=-2$: $X_1=-1, X_2=-1$, $P=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$.
> - $Z=-1$: mixed $(-1,0)$ or $(0,-1)$, $P=\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{4}$.
> - $Z=0$: could be $(-1,1)$, $(1,-1)$, or $(0,0)$, $P=\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{3}{8}$.
> - $Z=1$: mixed $(1,0)$ or $(0,1)$, $P=\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{4}$.
> - $Z=2$: $(1,1)$, $P=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$.
>   Law: $P(Z=-2)=\frac{1}{16}$, $P(Z=-1)=\frac{4}{16}$, $P(Z=0)=\frac{6}{16}$, $P(Z=1)=\frac{4}{16}$, $P(Z=2)=\frac{1}{16}$.

Variation 3: Asymmetric Binary Support in $\{0,1\}$

Exercise: Let X be a discrete random variable with support $\{0,1\}$ and law:

$$P(X=0)=\frac{1}{3},\quad P(X=1)=\frac{2}{3}.$$

Let $X_1,X_2,\dots,X_n$ be n independent copies of X, and define $Y=(X_1,X_2,\dots,X_n)$.

(a) $\mathbb{P}(X=0)$ and $\mathbb{P}(X=1)$?

Solution:

> $\mathbb{P}(X=0)=\frac{1}{3}$, $\mathbb{P}(X=1)=\frac{2}{3}$.

(b) Calculate $E[|X|^2]$.

Solution:

> $|X|^2=X^2=X$ (since $0^2=0, 1^2=1$)
> $E[|X|^2]=E[X]=0\cdot\frac{1}{3}+1\cdot\frac{2}{3}=\frac{2}{3}$.

(c) Find $E[X]$ and $\mathrm{Var}(X)$.

Solution:

> $E[X] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$.
> $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$.

(d) Law of $\frac{X+1}{2}$?
Solution:

> Let $T = \frac{X+1}{2}$.
>
> - $X = 0$: $T = 0.5$, $P = \frac{1}{3}$.
> - $X = 1$: $T = 1$, $P = \frac{2}{3}$.
>   Support: $\{0.5, 1\}$, law: $P(T = 0.5) = \frac{1}{3}$, $P(T = 1) = \frac{2}{3}$.

(e) Law of Y? Calculate $P(Y \in \{1\}^n)$.
Solution:

> Law of Y: product measure on $\{0,1\}^n$ with $P(Y = (y_1, \dots, y_n)) = \prod_{i=1}^n P(X_i = y_i)$.
> $P(Y \in \{1\}^n) = P(X_1 = 1, \dots, X_n = 1) = \left(\frac{2}{3}\right)^n$.

(f) Is $\|Y\|_2^2$ constant? If not, find its law.
Solution:

> $\|Y\|_2^2 = \sum_{i=1}^n X_i^2 = \sum_{i=1}^n X_i$ (since $X_i^2 = X_i$).
> $\sum_{i=1}^n X_i \sim \mathrm{Binomial}(n, p = \frac{2}{3})$ (number of 1's).
> Thus, for $k = 0, 1, \dots, n$:
> $P\left(\|Y\|_2^2 = k\right) = \binom{n}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{n-k}$.
> Not constant (e.g., for $n = 1$, takes 0 and 1).

(g) Law of $Z = X_1 + X_2$?
Solution:

> Support: $\{0, 1, 2\}$.
>
> - $Z = 0$: $X_1 = 0, X_2 = 0$, $P = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.
> - $Z = 1$: mixed, $P = P(0,1) + P(1,0) = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9}$.
> - $Z = 2$: $X_1 = 1, X_2 = 1$, $P = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$.
>   Law: $P(Z = 0) = \frac{1}{9}$, $P(Z = 1) = \frac{4}{9}$, $P(Z = 2) = \frac{4}{9}$.

Variation 4: Non-Integer Support

Exercise: Let X be a discrete random variable with support $\{3, 5\}$ and law:

$$P(X = 3) = \frac{2}{5}, \quad P(X = 5) = \frac{3}{5}.$$

Let $X_1, X_2, \dots, X_n$ be n independent copies of X, and define $Y = (X_1, X_2, \dots, X_n)$.

(a) $\mathbb{P}(X=3)$ and $\mathbb{P}(X=5)$?

Solution:

> $\mathbb{P}(X=3)=\frac{2}{5}$, $\mathbb{P}(X=5)=\frac{3}{5}$.

(b) Calculate $E[|X|^2]$.

Solution:

> $|X|^2=X^2$.
> $E[X^2]=3^2\cdot\frac{2}{5}+5^2\cdot\frac{3}{5}=9\cdot\frac{2}{5}+25\cdot\frac{3}{5}=18/5+75/5=93/5=18.6.$

(c) Find $E[X]$ and $\mathrm{Var}(X)$.

Solution:

> $E[X]=3\cdot\frac{2}{5}+5\cdot\frac{3}{5}=6/5+15/5=21/5=4.2.$
> $\mathrm{Var}(X)=E[X^2]-(E[X])^2=\frac{93}{5}-\left(\frac{21}{5}\right)^2=\frac{93}{5}-\frac{441}{25}=\frac{465}{25}-\frac{441}{25}=\frac{24}{25}=0.96.$

(d) Law of $\frac{X+1}{2}$?

Solution:

> Let $T=\frac{X+1}{2}$.
>
> - $X=3$: $T=\frac{4}{2}=2$, $P=\frac{2}{5}$.
> - $X=5$: $T=\frac{6}{2}=3$, $P=\frac{3}{5}$.
>   Support: $\{2,3\}$, law: $P(T=2)=\frac{2}{5}$, $P(T=3)=\frac{3}{5}$.

(e) Law of Y? Calculate $P(Y\in\{5\}^n)$.

Solution:

> Law of Y: product measure on $\{3,5\}^n$ with $P(Y=(y_1,\dots,y_n))=\prod_{i=1}^n P(X_i=y_i)$.
> $P(Y\in\{5\}^n)=P(X_1=5,\dots,X_n=5)=\left(\frac{3}{5}\right)^n$.

(f) Is $\|Y\|_2^2$ constant? Justify and find its range.

Solution:

> $\|Y\|_2^2=\sum_{i=1}^n X_i^2$, where $X_i^2\in\{9,25\}$ (since $3^2=9,5^2=25$).
> Range: from $9n$ (all 3) to $25n$ (all 5).
> Since $n\geq 1$ and $9<25$, unless $n=0$, it takes at least two values if $n\geq 1$. Thus, not constant (e.g., for $n=1$, is 9 or 25).

(g) Law of $Z=X_1+X_2$?

Solution:

> Support: $\{6,8,10\}$ (since $3+3=6, 3+5=8, 5+3=8, 5+5=10$).
>
> - $Z=6$: $X_1=3, X_2=3$, $P=\frac{2}{5}\cdot\frac{2}{5}=\frac{4}{25}$.
> - $Z=8$: mixed $(3,5)$ or $(5,3)$, $P=\frac{2}{5}\cdot\frac{3}{5}+\frac{3}{5}\cdot\frac{2}{5}=\frac{12}{25}$.
> - $Z=10$: $X_1=5, X_2=5$, $P=\frac{3}{5}\cdot\frac{3}{5}=\frac{9}{25}$.
>   Law: $P(Z=6)=\frac{4}{25}$, $P(Z=8)=\frac{12}{25}$, $P(Z=10)=\frac{9}{25}$.

Variation 5: Asymmetric Three-Valued Support

Exercise: Let X be a discrete random variable with support $\{-1, 0, 1\}$ and law:

$$P(X = -1) = \frac{1}{2}, \quad P(X = 0) = \frac{1}{3}, \quad P(X = 1) = \frac{1}{6}.$$

Let $X_1, X_2, \ldots, X_n$ be n independent copies of X, and define $Y = (X_1, X_2, \ldots, X_n)$.

(a) $\mathbb{P}(X = -1)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X = 1)$?

Solution:

> $\mathbb{P}(X = -1) = \frac{1}{2}$, $\mathbb{P}(X = 0) = \frac{1}{3}$, $\mathbb{P}(X = 1) = \frac{1}{6}$.

(b) Calculate $E[|X|^2]$.

Solution:

$E[X^2] = (-1)^2 \cdot \frac{1}{2} + 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{6} = 1 \cdot \frac{1}{2} + 0 + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$.

(c) Find $E[X]$ and $\mathrm{Var}(X)$.

Solution:

> $E[X] = (-1) \cdot \frac{1}{2} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{6} = -\frac{1}{2} + \frac{1}{6} = -\frac{1}{3}$.
> $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{3} - \left(-\frac{1}{3}\right)^2 = \frac{2}{3} - \frac{1}{9} = \frac{5}{9}$.

(d) Law of $\frac{X+1}{2}$?

Solution:

> Let $T = \frac{X+1}{2}$.
>
> - $X = -1$: $T = 0$, $P = \frac{1}{2}$.
> - $X = 0$: $T = 0.5$, $P = \frac{1}{3}$.
> - $X = 1$: $T = 1$, $P = \frac{1}{6}$.
>   Support: $\{0, 0.5, 1\}$, law: $P(T = 0) = \frac{1}{2}$, $P(T = 0.5) = \frac{1}{3}$, $P(T = 1) = \frac{1}{6}$.

(e) Law of Y? Calculate $P(Y \in \{1\}^n)$.

Solution:

Law of Y: product measure on $\{-1, 0, 1\}^n$.
$P(Y \in \{1\}^n) = P(X_1 = 1, \ldots, X_n = 1) = \left(\frac{1}{6}\right)^n$.

(f) Find $E[\|Y\|_2^2]$ and $\mathrm{Var}(\|Y\|_2^2)$.

Solution:

> $\|Y\|_2^2 = \sum_{i=1}^n X_i^2$.
> Let $W_i = X_i^2$, so $\|Y\|_2^2 = \sum_{i=1}^n W_i$.
> W_i is discrete:
>
> - $P(W_i = 0) = P(X_i = 0) = \frac{1}{3}$.

> - $P(W_i = 1) = P(X_i = -1 \text{ or } 1) = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$.
>   Thus, $E[W_i] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$.
> $\mathrm{Var}(W_i) = E[W_i^2] - (E[W_i])^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}$ (since $W_i^2 = W_i$).
>   By independence:
> $E[\|Y\|_2^2] = n \cdot E[W_i] = n \cdot \frac{2}{3}$.
> $\mathrm{Var}(\|Y\|_2^2) = n \cdot \mathrm{Var}(W_i) = n \cdot \frac{2}{9}$.

(g) Law of $Z = X_1 + X_2$?

Solution:

> Support: $\{-2, -1, 0, 1, 2\}$ (all combinations).
> Calculations:
>
> - $Z = -2$: $(-1, -1)$, $P = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.
> - $Z = -1$: $(-1, 0)$, $(0, -1)$, $P = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3}$.
> - $Z = 0$: $(-1, 1)$, $(0, 0)$, $(1, -1)$, $P = \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12} + \frac{1}{9} + \frac{1}{12} = \frac{1}{4} + \frac{1}{9}\ (\text{scaling via } 36!) = \frac{9}{36} + \frac{4}{36} = \frac{13}{36}$? Let's calculate accurately:
> $(-1, 1)$: $P = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$,
> $(0, 0)$: $P = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} = \frac{4}{36}$,
> $(1, -1)$: $P = \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12} = \frac{3}{36}$,
>   Total $= \frac{1}{12} + \frac{1}{9} + \frac{1}{12} = \frac{3}{36} + \frac{4}{36} + \frac{3}{36} = \frac{10}{36} = \frac{5}{18}$.
> - $Z = 1$: $(0, 1)$, $(1, 0)$, $P = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{3} = \frac{1}{18} + \frac{1}{18} = \frac{1}{9}$.
> - $Z = 2$: $(1, 1)$, $P = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.
>   Law:
> $P(Z = -2) = \frac{9}{36}$,
> $P(Z = -1) = \frac{12}{36}$,
> $P(Z = 0) = \frac{10}{36}$,
> $P(Z = 1) = \frac{4}{36}$,
> $P(Z = 2) = \frac{1}{36}$.
>   Simplify: $\frac{9}{36} = \frac{1}{4}$, but better reduced:
>   Sum: $9 + 12 + 10 + 4 + 1 = 36/36$, so:
>
> $$P(Z = -2) = \frac{9}{36}, \quad P(Z = -1) = \frac{12}{36}, \quad P(Z = 0) = \frac{10}{36}, \quad P(Z = 1) = \frac{4}{36}, \quad P(Z = 2) = \frac{1}{36}.$$
>
>   Simplify fractions: $\frac{3}{12}, \frac{1}{3}, \frac{5}{18}, \frac{1}{9}, \frac{1}{36}$? Or leave as is.

Progression:

- Variation 1: Basic asymmetric binary with clear constant norm.
- Variation 2: Triple-valued symmetric, introduces more calculations.
- Variation 3: $\{0,1\}$ support, requires handling sums as binomials.
- Variation 4: Non-integer support, expands to multiplicative values.
- Variation 5: Asymmetric triple-valued, most complex distributions and variances.