Here is an additional mock exam, designed to be more challenging, drawing on concepts and question styles from the provided sources.

Mock Exam 3 (More Difficult)

- The exam is open book — the use of AI or any form of communication is not allowed;
- Please make sure that every paper you need has your name and student number on it;
- Unless otherwise stated, full points will not be awarded for solutions without explanation;
- Do not use pencil, red or green pens.

Last Name:
First Name:
Student Number:

Notation: We recall some of the terminology:

- Given a nonempty set Ω, $\mathcal{P}(\Omega)$ is the power set on Ω.
- $\mathcal{B}(\mathbb{R}^k)$ denotes the Borel σ-field on $\mathbb{R}^k$, $k \geq 1$.
- The measure $\mu(A) = \#A$, if A is finite, and ∞, otherwise, for $A \in \mathcal{P}(\Omega)$, is referred to as the counting measure on $\mathcal{P}(\Omega)$.
- Given a measurable space $(\Omega, \mathcal{F})$ and $x \in \Omega$, we write δ_x for the measure $\mathcal{F} \ni A \mapsto \delta_x(A) = 1$, if $x \in A$, and 0, otherwise.
- If not mentioned explicitly, a random vector is assumed to be defined on a probability space $(\Omega, \mathcal{F}, P)$.

Exercise 1 (10 points).

(a) Given a measurable space $(\Omega, \mathcal{F})$, write down the definition of a **measure** μ on $\mathcal{F}$. [1 point]
(Inspired by Mock Exam 1 (a) and Mock Exam 2 (a), and Definition 5.1 in "The Metamorphosis")

(b) Given a measurable space $(\Omega, \mathcal{F})$, which of the following set functions $\mu_i, i = 1, 2, 3$, is not a measure on $\mathcal{F}$? [1.5 point — single choice, no explanation is needed to earn full points]

- $\mathcal{F} = \mathcal{P}(\mathbb{N})$ and $\mu_1(A) = \#A$, for $A \in \mathcal{P}(\mathbb{N})$.
- $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and $\mu_2(A) = \int_A (e^x + 1)\lambda(dx)$, $A \in \mathcal{B}(\mathbb{R})$, where λ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

- $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$ and $\mu_3(A) = \delta_0(A) + \delta_1(A) - \delta_2(A)$, where δ_x is the Dirac measure at x.

(c) Given a measurable space $(E, \mathcal{B})$, which of the following set functions $P_i, i = 1, 2, 3$, is not a probability measure on $\mathcal{B}$? [1.5 point — single choice, no explanation is needed to earn full points]

- $E = \mathbb{N}$, $\mathcal{B} = \mathcal{P}(\mathbb{N})$ and $P_1(A) = \sum_{n \in A \cap \mathbb{N}} 2^{-n}$, $A \in \mathcal{P}(\mathbb{N})$.
- $E = \mathbb{R}$, $\mathcal{B} = \mathcal{B}(\mathbb{R})$ and $P_2(A) = \int_A \frac{1}{\pi(1+x^2)} dx$, $A \in \mathcal{B}(\mathbb{R})$.
- $E = \mathbb{R}$, $\mathcal{B} = \mathcal{B}(\mathbb{R})$ and $P_3(A) = \frac{1}{2}\lambda(A)$, $A \in \mathcal{B}(\mathbb{R})$, where λ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

(d) Calculate the following integrals: [1 point each]

1. $\int_{\mathbb{N}} \mathbf{1}_{\{3,4,5\}}(x)\mu(dx)$, where μ is the counting measure on $\mathcal{P}(\mathbb{N})$.
2. $\int_{\mathbb{R}} e^x \mathbf{1}_{(-\infty,0]}(x)\lambda(dx)$, where λ is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.
3. $\int_{\mathbb{R}} x^2 \mu(dx)$, where $\mu(A) = \sum_{x=0}^{2}(x+1)\delta_x(A)$, $A \in \mathcal{B}(\mathbb{R})$.

(e) Which of the following laws $P_i, i = 1, 2, 3$, is not discrete? [1.5 point — single choice, no explanation is needed to earn full points]

1. The law P_1 of a random variable X such that $P_1(\mathbb{Z}) = 1$.
2. The law P_2 of a random variable X with probability density function $\phi(x) = \mathbf{1}(x)$.
3. The law P_3 of a random variable X with distribution function $F_X(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1/4, & \text{if } 0 \le t < 1 \\ 3/4, & \text{if } 1 \le t < 2 \\ 1, & \text{if } t \ge 2 \end{cases}$.

(f) Decide whether the following statements are true or false: [0.5 point each — no explanation is needed to earn full points]

1. The set $\mathcal{F} = \{\emptyset, \{1\}, \{1,2,3\}\}$ is a σ-field on $\Omega = \{1,2,3\}$.
2. If A and B are two sets, then $A \cup (B \setminus A) = A \cup B$.
3. If $X_n \to_P X$, then $X_n \to_{L^1} X$.
4. If X_1 and X_2 are two independent random variables, then $E[X_1 X_2] = E[X_1]E[X_2]$.

Exercise 2 (13 points).

Let X be a discrete random variable with support $\{-1, 0, 1\}$ and law $P_X(A) = \frac{1}{4}\delta_{-1}(A) + \frac{1}{2}\delta_0(A) + \frac{1}{4}\delta_1(A)$, $A \in \mathcal{B}(\mathbb{R})$.

(a) What are $P(X=-1)$, $P(X=0)$ and $P(X=1)$? [1 point]
(b) Calculate $E[|X|^3]$. [1.5 point]
(c) Find $E[X]$ and $\mathrm{Var}(X)$. [2 points]
(d) What is the law of $(X+1)^2$? [1.5 points]

Let $X_1,\dots,X_n$ be n independent copies of X, i.e., for any $i=1,\dots,n$, X_i has law P_X and $X_1,\dots,X_n$ are independent. Define the random vector $Y=(X_1,\dots,X_n)$.

(e) What is the law of Y? Calculate $P(Y\in\{0\}^n)$. [1 point]
(f) Find $E[\sum_{i=1}^n X_i^2]$ and $\mathrm{Var}(\sum_{i=1}^n X_i^2)$. [3 points]
(g) What is the law of $Z=X_1+X_2$? [3 points]

Exercise 3 (18 points).

Let $\phi(x)=\begin{cases} 0 & x<-1 \\ \frac{3}{2}(1+x)^2 & -1\le x<0 \\ \frac{3}{2}(1-x)^2 & 0\le x<1 \\ 0 & x\ge 1 \end{cases}$.

(a) Verify that $\int_{\mathbb{R}}\phi(x)dx=1$. [2 points]
Let X be a random variable with law $P_X(dx)=\phi(x)dx$.

(b) Find the distribution function F_X of X. [4 points]
(c) Calculate the expected value and the variance of X. [4.5 points]
(d) Show that $F_X|_{(-1,1)}:(-1,1)\to(0,1)$ is a bijection. [3 points]
(e) Verify that $F_X|_{[-1,1]}:[-1,1]\to$ is a bijection. [1.5 points]
(f) Calculate the inverse $F_X^{-1}|_{[-1,1]}$ of $F_X|_{[-1,1]}$. [3 points]
Note: You can use the fact that:

- $p=\frac{1}{2}(1+t)^3 \implies t=(2p)^{1/3}-1$
- $p=1-\frac{1}{2}(1-t)^3 \implies t=1-(2(1-p))^{1/3}$

Exercise 4 (6 points).

Let X_1 and X_2 be two random variables that are independent with common law that is continuous uniform on the interval $[0,1]$. What is the probability density function of the random vector $Y=(X_1+X_2,X_1-X_2)$?

Exercise 5 (6 points).

Let X be a discrete random variable with support $\{0, 1, \dots, N\}$, where $N \geq 1$. Suppose that X has law defined upon:
$P(X = k) = C_N k(N - k)$, for $k = 0, \dots, N$, where $C_N \in \mathbb{R}$. Find C_N.