Cheat Sheet: Convergence of Random Vectors

(Based on Chapter 12 of Probability Theory Notes)

Key Definitions

2. Modes of Convergence (strongest to weakest)

Let $(X_n)_{n\in\mathbb{N}}$ be random vectors on $(\Omega, \mathcal{F}, P)$, and X a random vector.

Almost Sure (a.s.) Convergence

Definition 12.1:

$$P\left(\lim_{n\to\infty} X_n = X\right) = 1.$$

Notation: $X_n \xrightarrow{\text{a.s.}} X$.

L_p Convergence

Definition 12.1:
For $p = 1, 2$, assume $E[\|X_n\|^p], E[\|X\|^p] < \infty$. Then,

$$\lim_{n\to\infty} E\left[\|X_n - X\|^p\right] = 0.$$

Notation: $X_n \xrightarrow{L_p} X$.

Convergence in Probability

Definition 12.1:
For every $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\|X_n - X\| > \varepsilon\right) = 0.$$

Notation: $X_n \xrightarrow{P} X$.

3. Borel-Cantelli Lemma

Proposition 12.1:
For events $\{A_n\}_{n\in\mathbb{N}}$:

- **Part (i)**: If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P\left(\limsup A_n\right) = 0$.
- **Part (ii)**: If $\{A_n\}$ is independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P\left(\limsup A_n\right) = 1$.

Intuition:

- $\limsup A_n$ = "infinitely many A_n occur."
- Finite sum ⇒ finitely many events occur almost surely.
- Infinite sum + independence ⇒ infinitely many events occur almost surely.

4. Hierarchy of Convergence

1. **Strongest** → **Weakest**:
   - $X_n \xrightarrow{\text{a.s.}} X \Rightarrow X_n \xrightarrow{P} X$ (Prop 12.3).
   - $X_n \xrightarrow{L_p} X \Rightarrow X_n \xrightarrow{P} X$ (Prop 12.2).
   - $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$
   - **No reverse implications hold** (see Remark 12.3).
2. **Uniqueness of Limits** (Prop 12.4):
   If $X_n \xrightarrow{P} X$ and $X_n \xrightarrow{P} Y$, then $X = Y$ almost surely.

5. Key Results

Subsequence Extraction (Prop 12.5)

If $X_n \xrightarrow{P} X$, there exists a subsequence $\left(X_{s(n)}\right)$ such that $X_{s(n)} \xrightarrow{\text{a.s.}} X$.

Dominated Convergence for L_1 (Prop 12.6)

If $X_n \xrightarrow{P} X$ and $|X_n| \leq Y$ (with $E[|Y|] < \infty$), then $X_n \xrightarrow{L_1} X$.

Scheffé's Lemma (Prop 12.8)

If $X_n \xrightarrow{P} X$, $E[|X_n|] \to E[|X|]$, and $E[|X|] < \infty$, then $X_n \xrightarrow{L_1} X$.

6. Law of Large Numbers (LLN)

Strong LLN (Prop 12.9):
For i.i.d. $(X_n)_{n\in\mathbb{N}}$ with $E[|X_1|] < \infty$:

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{\text{a.s.}} E[X_1].$$

Example 12.4:
For i.i.d. X_n with mean μ and variance σ^2:

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \xrightarrow{\text{a.s.}} \sigma^2.$$

7. Convergence in Distribution

Definition 12.2:
$X_n \xrightarrow{d} X$ if for every continuous bounded φ:

$$E[\varphi(X_n)] \to E[\varphi(X)].$$

Properties (Remark 12.7):

1. Weakest form of convergence: $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$ (Prop 12.10).
2. **Lévy's Theorem**: $X_n \xrightarrow{d} X$ iff $\Phi_{X_n}(v) \to \Phi_X(v)$ for all $v \in \mathbb{R}^k$.
3. **Central Limit Theorem (CLT)**:
   For i.i.d. X_n with finite variance:

$$\frac{S_n - n\mu}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

Additional Notes

- **Continuous Mapping Theorem**: If $X_n \xrightarrow{\text{a.s.}/P/d} X$ and g is continuous, then $g(X_n) \xrightarrow{\text{a.s.}/P/d} g(X)$ (Prop 12.7, 12.10).
- **Edge Cases**: See Examples