Mock Exam 3

Notation

We recall some of the terminology:

- Given a nonempty set Ω, $P(\Omega)$ is the power set on Ω;
- $B(\mathbb{R}^k)$ denotes the Borel σ-field on $\mathbb{R}^k$, $k \geq 1$;
- The measure $\mu(A) = \begin{cases} \#A, & \text{if } A \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$, $A \in P(\Omega)$, is referred to as the counting measure on $P(\Omega)$;
- Given a measurable space $(\Omega, \mathcal{F})$ and $x \in \Omega$, we write δ_x for the measure $\mathcal{F} \ni A \mapsto \delta_x(A) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$;
- If not mentioned explicitly, a random vector is assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 1 (10 points)

(a) Given a nonempty set Ω, write down the definition of an outer measure μ^* on $P(\Omega)$. [1 point]

> **Exercise 1(a) Solution**
> An outer measure μ^* on $P(\Omega)$ is a function $\mu^* : P(\Omega) \to [0, \infty]$ satisfying:
>
> 1. $\mu^*(\emptyset) = 0$
> 2. Monotonicity: $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$
> 3. Countable subadditivity: $\mu^*\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

(b) Given a measurable space $(\Omega, \mathcal{F})$, which of the following set functions μ_i, $i = 1, 2, 3$, is not a measure on $\mathcal{F}$? [1.5 point — single choice, no explanation is needed to earn full points]
• $\mu_1(A) = \sqrt{\lambda(A)}$, $A \in B(\mathbb{R})$, where λ is the Lebesgue measure on $B(\mathbb{R})$.
• $\mu_2(A) = \sum_{n \in A \cap \mathbb{N}} n$, $A \in B(\mathbb{R})$.
• $\mu_3(A) = \mu(A) + \nu(A)$, $A \in \mathcal{F}$, where μ and ν are measures on $\mathcal{F}$.

> **Exercise 1(b) Solution**
> $\mu_2(A)$

(c) Given a measurable space $(E, \mathcal{B})$, which of the following set functions P_i, $i = 1, 2, 3$, is not a probability measure on $\mathcal{B}$? [1.5 point — single choice, no explanation is needed to earn full points]
- $E = \mathbb{R}$, $\mathcal{B} = B(\mathbb{R})$ and $P_1(A) = \int_A x^2 e^{-x^3/3} dx$, $A \in B(\mathbb{R})$.
- $E = \mathbb{N}$, $\mathcal{B} = P(\mathbb{N})$ and $P_2(A) = \sum_{n \in A \cap \mathbb{N}} (1/2)^n$, $A \in P(\mathbb{N})$.
- $E = \mathbb{R}$, $\mathcal{B} = B(\mathbb{R})$ and $P_3(A) = \int_A e^{-|x|} dx$, $A \in B(\mathbb{R})$.

(d) Calculate the following integrals: [1 point each]

1. $\int_{\mathbb{R}} |x| \mathbf{1}_{[-1,1]} \lambda(dx)$, where λ is the Lebesgue measure on $B(\mathbb{R})$.

> **Solution**:
> The indicator function $\mathbf{1}_{[-1,1]}$ restricts the domain to $[-1, 1]$, and $|x|$ is symmetric. Using Lebesgue measure properties:
>
> $$\int_{\mathbb{R}} |x| \mathbf{1}_{[-1,1]} \lambda(dx) = \int_{-1}^{1} |x| dx = 2 \int_0^1 x dx = 2 \left[\frac{x^2}{2} \right]_0^1 = 1.$$

2. $\int_{\mathbb{R}} e^{-|x|} P(dx)$, where $P(A) = (1/2)\delta_{-1}(A) + (1/2)\delta_1(A)$, $A \in B(\mathbb{R})$.

> **Solution**:
> P is a discrete measure concentrated at $\{-1, 1\}$. The integral simplifies to:
>
> $$\int_{\mathbb{R}} e^{-|x|} P(dx) = \frac{1}{2} e^{-|-1|} + \frac{1}{2} e^{-|1|} = \frac{1}{2} e^{-1} + \frac{1}{2} e^{-1} = e^{-1}.$$

3. $\int_{\mathbb{N}} (1/n^2) \mu(dn)$, where μ is the counting measure on $P(\mathbb{N})$.

> **Solution**:
> The counting measure μ assigns mass 1 to each $n \in \mathbb{N}$. Thus:
>
> $$\int_{\mathbb{N}} \frac{1}{n^2} \mu(dn) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(e) Which of the following laws P_i, $i = 1, 2, 3$, is not discrete? [1.5 point — single choice, no explanation is needed to earn full points]

1. The law P_1 where $P_1(X = k) = (1/e)/k!$ for $k = 0, 1, 2, \ldots$
2. The law P_2 of a random variable X with probability density function $\phi(x) = \cos(x)$ for $x \in [0, \pi/2]$, and 0 otherwise.

3. The law P_3 where $P_3(X=k) = (1/3)^k/(2-1/3)$ for $k = 0, 1, 2, \ldots$

> **Solution**:
> 2

(f) Decide whether the following statements are true or false: [0.5 point each — no explanation is needed to earn full points]

1. If X_1 and X_2 are two independent Gaussian random variables, then (X_1, X_2) is a Gauss vector.
2. If a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ converges to X in probability, then there exists a subsequence $(X_{s(n)})_{n\in\mathbb{N}}$ that converges almost surely to X.
3. The union of two σ-fields on Ω is always a σ-field on Ω.

> **Solution**:
> T,T,F

Exercise 2 (13 points)
Let X be a discrete random variable with support $\{0, 1, 2\}$ and law $\mathbb{P}(X=0) = 1/4$, $\mathbb{P}(X = 1) = 1/2$, $\mathbb{P}(X=2) = 1/4$.
(a) What are $\mathbb{P}(X=0)$, $\mathbb{P}(X=1)$ and $\mathbb{P}(X=2)$? [1 point]

> **Solution**
> **(a)**
> $\mathbb{P}(X=0) = \frac{1}{4}$, $\mathbb{P}(X=1) = \frac{1}{2}$, $\mathbb{P}(X=2) = \frac{1}{4}$

(b) Calculate $\mathbb{E}[|X|^2]$. [1.5 point]

> **(b)**
> $\mathbb{E}[|X|^2] = \mathbb{E}[X^2] = \sum x^2\mathbb{P}(X=x) = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = 0 + 0.5 + 1 = 1.5$

(c) Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. [2 points]

> **(c)**
> $\mathbb{E}[X] = \sum x\mathbb{P}(X=x) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$
> $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 1.5 - 1^2 = 0.5$

(d) What is the law of $|X-1|$? [1.5 points]
Let $X_1, \ldots, X_n$ be n independent copies of X, i.e., for any $i = 1, \ldots, n$, X_i has law P_X and $X_1, \ldots, X_n$ are independent. Define the random vector $Y = (X_1, \ldots, X_n)$.

> **(d)**
> $|X-1|$ takes values:
>
> - 0 when $X=1$ ($\mathbb{P}=\frac{1}{2}$)
> - 1 when $X=0$ or $X=2$ ($\mathbb{P}=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$)
>   Law: $\mathbb{P}(|X-1|=0)=\frac{1}{2}$, $\mathbb{P}(|X-1|=1)=\frac{1}{2}$

Let $X_1,\ldots,X_n$ be n independent copies of X, i.e., for any $i=1,\ldots,n$, X_i has law P_X and $X_1,\ldots,X_n$ are independent. Define the random vector $Y=(X_1,\ldots,X_n)$.
(e) Calculate $\mathbb{P}(Y=(0,1,0,1,\ldots,0))$ assuming n is an even integer. [1 point]

> **(e)**
> For $Y=(0,1,0,1,\ldots,0)$ (n even):
> Each pair $(0,1)$ has probability $\mathbb{P}(X_i=0)\mathbb{P}(X_{i+1}=1)=\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{8}$
> With $n/2$ such pairs: $\left(\frac{1}{8}\right)^{n/2}=8^{-n/2}$

(f) Calculate the characteristic function of X, $\Phi_X(v)$. [3 points]

> **(f)**
> $\Phi_X(v)=\mathbb{E}[e^{ivX}]=\sum e^{ivx}\mathbb{P}(X=x)$
> $=e^{iv\cdot 0}\cdot\frac{1}{4}+e^{iv\cdot 1}\cdot\frac{1}{2}+e^{iv\cdot 2}\cdot\frac{1}{4}=\frac{1}{4}+\frac{1}{2}e^{iv}+\frac{1}{4}e^{2iv}$

(g) What is the law of $Z=X_1+X_2$? [3 points]

> **(g)**
> $Z=X_1+X_2$ takes values $\{0,1,2,3,4\}$:
>
> | z | (X_1,X_2) | $\mathbb{P}(Z=z)$ |
> |---|---|---|
> | 0 | (0,0) | $\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$ |
> | 1 | (0,1),(1,0) | $2\cdot\frac{1}{4}\cdot\frac{1}{2}=\frac{1}{4}$ |
> | 2 | (0,2),(1,1),(2,0) | $2\cdot\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}+\frac{1}{4}=\frac{3}{8}$ |
> | 3 | (1,2),(2,1) | $2\cdot\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{4}$ |
> | 4 | (2,2) | $\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}$ |

Exercise 3 (18 points)
Let X be a random variable with law $P_X(dx)=\phi(x)dx$, where

$$\phi(x) = \begin{cases} 3x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that $\int_{\mathbb{R}} \phi(x)dx = 1$. [2 points]

> $\int_{\mathbb{R}} \phi(x)dx = \int_0^1 3x^2dx = \left[x^3\right]_0^1 = 1^3 - 0^3 = 1$

(b) Find the distribution function F_X of X. [4 points]

> $F_X(x) = P(X \le x) = \int_{-\infty}^{x} \phi(t)dt$
>
> - For $x < 0$: $\int_{-\infty}^{x} 0dt = 0$
> - For $0 \le x \le 1$: $\int_0^x 3t^2dt = \left[t^3\right]_0^x = x^3$
> - For $x > 1$: $\int_0^1 3t^2dt = 1$
>
> $$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

(c) Calculate the expected value $\mathbb{E}[X]$ and the variance $\mathrm{Var}(X)$. [4.5 points]

> $\mathbb{E}[X] = \int_{\mathbb{R}} x\phi(x)dx = \int_0^1 x \cdot 3x^2dx = 3\int_0^1 x^3dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$
> $\mathbb{E}[X^2] = \int_0^1 x^2 \cdot 3x^2dx = 3\int_0^1 x^4dx = 3\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{5}$
> $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{48}{80} - \frac{45}{80} = \frac{3}{80}$

(d) Show that $F_X \restriction_{(0,1)} : (0,1) \to (0,1)$ is a bijection. [3 points]

> - **Injective**: Let $a, b \in (0,1)$ with $F_X(a) = F_X(b)$. Then $a^3 = b^3 \implies a = b$.
> - **Surjective**: For any $y \in (0,1)$, let $x = y^{1/3} \in (0,1)$. Then $F_X(x) = (y^{1/3})^3 = y$.
>   Thus F_X is bijective on $(0,1)$.

(e) Verify that $F_X \restriction_{\{1\}} : \{1\} \to \{1\}$ is a bijection. [1.5 points]

> $F_X(1) = 1^3 = 1$, so $F_X|_{\{1\}}(1) = 1$.
>
> - **Injective**: Only one element in domain/codomain.
> - **Surjective**: 1 maps to 1, covering the codomain.
>   Thus it is a bijection.

(f) Calculate the inverse $F_X^{-1} \restriction_{\{1\}}$ of $F_X \restriction_{\{1\}}$. [3 points]

> $F_X|_{\{1\}} : \{1\} \to \{1\}$ is defined by $1 \mapsto 1$.
> Its inverse is $F_X^{-1}|_{\{1\}} : \{1\} \to \{1\}$ defined by $1 \mapsto 1$.

Exercise 4 (6 points)

Let X_1 and X_2 be two random variables that are independent with common law that is exponential with parameter $\lambda = 1$. That is, $P_{X_1}(dx) = e^{-x}\mathbf{1}_{[0,\infty)}(x)dx$ and $P_{X_2}(dx) = e^{-x}\mathbf{1}_{[0,\infty)}(x)dx$.
What is the probability density function of the random vector $Y = (X_1 + X_2, X_1)$?

Exercise 5 (6 points)

Let X be a discrete random variable with support $\{0, 1, 2, \dots\}$. Suppose that X has law defined upon: $\mathbb{P}(X = k) = C/k!$, $k = 0, 1, 2, \dots$
Find C.