EE 105: **Noise**
Resources and Worksheets

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April 29, 2025

Contents

1 Online resources

Below is a curated list of quality online resources that you may find helpful for learning, exploring, and practicing a variety of concepts related to circuits, electronics, and semiconductor devices.

1.1 Tutorials and reference websites

- **All About Circuits**: This website features extensive text-based tutorials that cover DC/AC circuits, semiconductor devices, and more. Every topic also includes a forum at the bottom of the page where students can ask questions and interact with the community to give and receive help. A comprehensive, hyperlinked table of contents on semiconductor devices is available here, and an online textbook on analog devices may be found here, with a corresponding table of contents toward the bottom of the page. For additional practice, select a topic to explore and challenge yourself in the Worksheets section.
- **Electronics-Tutorials** (*corrected hyperlink*): This is a fantastic organized collection of tutorials covering a myriad of topics in circuits with tons of diagrams and worked-out examples featuring step-by-step derivations. For semiconductor devices, relevant sections include semiconductor basics, diodes, and transistors.
- **Khan Academy**: Features a collection of short video lessons and practice problems on the fundamentals of circuit basics, signals, and systems with a user-friendly approach suitable for beginners. In regard to semiconductor devices, a set of video lectures on diodes may be found here.
- **LibreTexts**: LibreTexts is a fantastic online textbook resource for a variety of topics, including theory and worked out examples. A section on the basic physics concepts behind semiconductor devices may be found here, while a set of general textbook modules on semiconductors, including band theory, extrinsic vs. intrinsic semiconductors, metal-oxide-semiconductors (MOS), and diodes, may be found here.
- **MIT OpenCourseWare (OCW)**: Free, self-paced course materials available from MIT, including detailed lecture notes, problem sets, exams with solutions, and video lectures. For semiconductor devices, courses **6.002** (*Circuits and Electronics*, link here) and **6.012** (*Microelectronic Devices and Circuits*, links here, here, and here).
- **Selected problems by BarcelonaTech**: A collection of solved examples on semiconductor and electronic device physics, suitable for practice and deeper understanding.
- **Wikipedia**: A great resource for introductory and surface-level overviews of topics related to electronics and semiconductor devices. Examples of entries relevant to semiconductor devices include MOSFET, CMOS, and subthreshold conduction.

1.2 Simulation tools

- **Tinkercad Circuits**: A browser-based circuit design and simulation tool from Autodesk. This is a great tool for beginners to visualize and interact with circuits in a simulated fashion. It includes basic electronic components, measurement tools, and real-time simulation. For some introductory tutorials to get you started, check out the official playlist on YouTube.
- **Falstad Circuit Simulator**: A web-based real-time simulation interface with helpful visualizations of voltages and currents which allows students to see waveforms change dynamically. A large variety of preset examples (such as RLC circuits, op-amps, and digital logic circuits) are included for students to explore with a user-friendly drag-and-drop interface.
- **LTspice**: An excellent freeware circuit simulator developed by Analog Devices which is widely used in both academia and industry. It has a slightly steeper learning curve than the other simulators but it is capable of simulating many different types of complex circuits, making it a useful tool for learning schematic capture, waveform analysis, and advanced circuit behavior. For some introductory tutorials to get started, check out the following YouTube tutorial playlist.
- **PhET Simulations**: For absolute beginners who are particularly visual learners and are looking for hands-on learning to aid conceptual understanding, this website includes many interactive demos and simulation tools for DC and AC circuit basics. Students are able to manipulate variables in real time (e.g., resistor values, battery voltage, etc.) and see immediate effects.

- **CircuitLab**: Another browser-based circuit simulator with built-in analysis tools. Featuring a library of example circuits, it can be useful for practicing circuit design and learning how measurement tools work, such as oscilloscopes.

1.3 Discussion and forum communities

- **Electronics Stack Exchange**: A Q&A format discussion forum with a broad community of hobbyists, students, and professionals. It can be a great resource for well-structured answers to specific, technical circuit questions.
- **r/ElectricalEngineering**: An electrical engineering subreddit forum featuring discussions on coursework, textbooks, career advice, and circuit troubleshotting. It can be a good place to ask questions and see how others approach similar problems.

2 Overview of noise

2.1 Introduction

In electrical engineering, **noise** refers to random, unwanted perturbations that contaminate desired signals during generation, transmission, amplification, or measurement. Noise imposes a fundamental limit on system performance, affecting signal fidelity, error rates, and dynamic range. Understanding noise sources, their mathematical models, and methods for mitigation is crucial across all domains, from analog circuit design to high-speed digital systems and communication theory.

2.2 Fundamental Types of Noise

Several dominant noise mechanisms arise from basic physical principles. These include, but are not limited to:

- Thermal (Johnson-Nyquist) noise
- Shot noise
- Flicker ($1/f$) noise
- Quantization noise
- Clock jitter noise

Each of these are briefly described in the following sections.

2.2.1 Thermal Noise (Johnson-Nyquist Noise)

Thermal noise is generated by the random thermal motion of charge carriers within any resistive element at temperature T.

Voltage noise power spectral density (PSD):

$$S_v(f) = 4k_{\mathrm{B}}TR \quad [\mathrm{V}^2/\mathrm{Hz}] \tag{1}$$

where k is Boltzmann's constant (1.38×10^{-23} J/K), R is resistance in ohms, and T is the absolute temperature in kelvin. Thermal noise is *white*, meaning it is uniformly distributed across frequency.

2.2.2 Shot Noise

Shot noise arises due to the discrete nature of electric charge when carriers cross a potential barrier (e.g., in diodes or photodetectors).

Current noise PSD:

$$S_i(f) = 2qI \quad [\mathrm{A}^2/\mathrm{Hz}] \tag{2}$$

where q is the elementary charge (1.602×10^{-19} C) and I is the average direct current.

2.2.3 Flicker Noise (1/f Noise)

Flicker noise, or $1/f$ noise, originates from traps and imperfections within materials and device interfaces.

Power spectral density:

$$S(f) \propto \frac{1}{f^\alpha}, \quad \alpha \approx 1 \tag{3}$$

Flicker noise dominates at low frequencies (< 10 kHz) and is material- and process-dependent.

2.2.4 Quantization Noise

When analog signals are digitized, finite bit resolution introduces **quantization noise**.

Quantization noise power for a uniform n-bit ADC:

$$P_{\mathrm{quant}} = \frac{\Delta^2}{12} \tag{4}$$

where Δ is the voltage step size (LSB).

2.2.5 Clock Jitter and Aperture Uncertainty

Sampling a signal with imperfect clock timing introduces timing jitter, leading to amplitude errors, especially for high-frequency inputs.

2.3 Noise Propagation in Real-World Systems

In a real-world signal chain, such as:

$$\text{Computer} \to \text{DAC} \to \text{Driver Circuit} \to \text{Source (LED)} \to \text{Channel} \to \text{Photodetector} \to \text{Amplifier} \to \text{ADC} \to \text{Computer},$$

noise from each subsystem adds to the total system noise.

Uncorrelated noise sources combine as:

$$v_{n,\text{total}} = \sqrt{\sum_i v_{n,i}^2} \tag{5}$$

where $v_{n,i}$ is the RMS voltage noise contribution from each source.

Noise power integrates over system bandwidth Δf:

$$P_n = S_n \Delta f \tag{6}$$

Thus, minimizing bandwidth is an effective strategy to reduce noise.

2.4 Core Performance Metrics

2.4.1 Signal-to-Noise Ratio (SNR)

The Signal-to-Noise Ratio (SNR) quantifies how much stronger the desired signal is compared to the background noise. It is defined as teh ratio of signal power P_{sig} to noise power P_{noise}, and is often expressed in decibels (dB):

$$\text{SNR} = \frac{P_{\text{signal}}}{P_{\text{noise}}} \quad \text{or} \quad \text{SNR}_{\text{dB}} = 10 \log_{10}\left(\frac{P_{\text{signal}}}{P_{\text{noise}}}\right) \tag{7}$$

A higher SNR means the signal stands out more clearly above the noise floor, leading to better fidelity and lower error rates in communication or measurement systems.

2.4.2 Noise Figure (NF)

The Noise Figure (NF) measures how much an active device (e.g., amplifier, mixer, etc.) degrades the SNR of a signal passing through it. It is defined as the ratio of input SNR ot output SNR, and is oftentimes also expressed in decibels (dB):

$$\text{NF} = 10 \log_{10}(F) \quad \text{where} \quad F = \frac{\text{SNR}_{\text{in}}}{\text{SNR}_{\text{out}}} \tag{8}$$

For cascaded stages (Friis' Formula):

$$F_{\text{total}} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \cdots \tag{9}$$

where G_i are the power gains of each stage. A lower Noise Figure indicates a cleaner device that adds less extra noise, which is especially critical for first stages in low-signal-level receivers.

2.5 Advanced Topics in Noise Engineering

When considered all together, the following advanced topics in noise represent the frontier of noise control in modern electrical and photonic systems, from pushing ADC dynamic range into the 120 dB regime to detecting minute quantum signals at the edge of our understanding of physics.

2.5.1 Noise Shaping and Delta-Sigma Modulation

Oversampling and noise shaping techniques push quantization noise out of the signal band, enabling high dynamic range in ADCs and DACs. Delta-sigma ($\Delta\Sigma$) modulators use oversampling and feedback to shape quantization noise, moving it out of the signal band. A first-order loop has an integrator and 1-bit quantizer in the feedback path:

$$Y(z) = X(z) \cdot [1] + Q(z) \cdot (1 - z^{-1}), \tag{10}$$

where [1] is the signal transfer function (for example), and $(1 - z^{-1})$ is the noise transfer function (NTF),

$$\mathrm{NTF}(z) = 1 - z^{-1} \approx j\omega \quad (\omega \to 0), \tag{11}$$

so that low-frequency (in-band) noise is attenuated by ω, while high-frequency noise is amplified.

The *oversampling ratio* (OSR) is $\mathrm{OSR} = \frac{f_s/2}{B}$, where f_s is the modulator clock and B is the signal bandwidth. In an mth-order modulator, in-band noise scales as $(\mathrm{OSR})^{-(2m+1)}$.

There are several practical impacts, such as audio ADCs achieving > 100 dB SNR with only 16 bits of internal quantization. Additionally, in instrumentation, $\Delta\Sigma$ front-ends allow ppm-level resolution without ultra-precise resistor matching.

2.5.2 Phase Noise and Oscillator Stability

Phase noise quantifies short-term frequency fluctuations of an oscillator as sideband power around its carrier. Phase noise models rapid frequency fluctuations in oscillators, critical for RF communication links. Phase noise $L(f)$ measures carrier sideband power (expressed in dBc/Hz at offset f from carrier):

$$L(f) = 10 \log_{10} \frac{S_\phi(f)}{2} \text{ [dBc/Hz]}, \tag{12}$$

where $S_\phi(f)$ is the one-sided phase modulation PSD.

Leeson's model approximates:

$$L(f) \approx 10 \log_{10} \Big[\frac{FkT}{2P_{\mathrm{osc}}} \big(1 + (\tfrac{f_c}{2Qf})^2\big) \Big], \tag{13}$$

with $f_L = f_c/2Q$, Q the resonator quality factor, and F a device noise factor.

Key regions include the *flicker* region ($\propto 1/f^3$) below f_L, the *white* region ($\propto 1/f^2$) for $f > f_L$, and the *floor* set by thermal noise and resonator losses. Practical considerations include the fact that high-Q resonators (SAW, crystal) reduce close-in phase noise. In addition, power-supply and buffer design must minimize AM-to-PM conversion.

2.5.3 Low-Noise Amplifier (LNA) Design

Designing LNAs involves noise impedance "noise matching" rather than pure power matching to minimize system noise figure.

Noise in each transistor is specified by $(F_{\min}, R_n, \Gamma_{\mathrm{opt}})$:

$$F(\Gamma_S) = F_{\min} + \frac{4R_n}{Z_0} \frac{|\Gamma_S - \Gamma_{\mathrm{opt}}|^2}{(1 - |\Gamma_S|^2)|1 + \Gamma_{\mathrm{opt}}|^2}. \tag{14}$$

Design uses noise and gain circles on the Smith chart to choose input match Γ_S. These are represented by contours of constant NF in the Smith chart, in which the designer picks Γ_S near Γ_{opt} for minimum noise. There are trade-offs, including the fact that maximum gain often occurs at a different Γ_S rather than minimum NF. In addition, stability, linearity, and input matching must be balanced. Designing LNAs typically involve the following steps:

1. Extract noise parameters via measurement or foundry data.
2. Plot noise circles and gain circles on Smith chart.
3. Choose matching network to achieve Γ_S at RF input.

2.5.4 Cross-Spectrum Measurement

Two-channel cross-spectrum methods enable measurement of extremely low noise floors by rejecting uncorrelated instrument noise. The goal is to measure extremely low-level noise (down to nV/$\sqrt{\text{Hz}}$) by averaging out uncorrelated instrument self-noise.

The principle involves acquiring Device-Under-Test (DUT) output with two separate, nominally identical channels:

$$V_1(t) = V_{\text{DUT}}(t) + N_1(t), \tag{15}$$
$$V_2(t) = V_{\text{DUT}}(t) + N_2(t). \tag{16}$$

The cross-spectral density (cross-spectrum) $S_{12}(f)$ is computed, which averages out uncorrelated N_1, N_2, revealing $S_{\text{DUT}}(f)$ with $1/\sqrt{M}$ noise suppression after M averages. Setup considerations require ensuring isolations to prevent cross-talk, using identical low-noise preamps on both channels, and synchronizing sampling clocks for coherent FFT. Cross-spectrum measurement applications include characterizing ultra-low-noise amplifiers, voltage references, and MEMS sensors.

2.5.5 Quantum-Limited Noise

In optical and microwave detection, the **Standard Quantum Limit** (SQL) arises because photon (or phonon) arrival is quantized. At extremely low signal levels, therefore, shot noise and quantum noise define the ultimate limit to sensitivity.

The minimum detectable power fluctuation is set by photon statistics and is given by:

$$\Delta P_{\text{min}} = \sqrt{2h\nu PB}, \tag{17}$$

where $h\nu$ is photon energy, P is optical power, and B is measurement bandwidth. To achieve noise floors beyond SQL, squeezing and homodyne methods can surpass the classical shot-noise limit. *Squeezed states* involve reducing noise in one quadrature below shot noise at the expense of increased noise in the orthogonal quadrature, while *homodyne detection* involves interfering a signal with a strong local oscillator to access phase-quadrature information with sub-shot-noise sensitivity.

The practical impacts from achieving sub-SQL noise floors involve metrology applications typically used in research. Two examples include gravitational-wave detectors (e.g., LIGO) which use squeezed light to beat SQL, and quantum communication which relies on single-photon-level detection with superconducting nanowire detectors achieving ~ 0.1 photon/$\sqrt{\text{Hz}}$ noise floors.

2.5.6 Random Number Generation and Stochastic Computing

Noise may also be leveraged to perform computing as well.[1] In *stochastic computing*, numbers are represented as probabilities p of a binary 1 or 0 signal in a clocked bit-stream of length n_{bits}. As $n_{\text{bits}} \to \infty$, the average value of the signal is p distributed in the interval [0,1]. A circuit that can perform multiplication followed by addition on stochastic data is illustrated in Fig.1.

Since basic linear algebra operations involve multiplication of matrix $\mathbf{A}$ with vector $\mathbf{s}$, stochastic computing might be tasked with computing both $\mathbf{x} = \mathbf{A}\mathbf{s}$ and $\mathbf{s} = \mathbf{A}^{-1}\mathbf{x}$. For the simplest 2×2 matrix, $\mathbf{x} = \mathbf{A}\mathbf{s}$ may be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \tag{18}$$

so that evaluation of $x_1 = a_{11} \times s_1 + a_{12} \times s_2$ requires multiplication and addition (also known as multiply and accumulate, or MAC). In the limit $n_{\text{bits}} \to \infty$, the output of the AND function on stochastic input streams a_{11} and s_1 is the product of probabilities $p(a_{11})$ and $p(s_1)$. The sum of the two AND outputs is found by multiplexing using a select (SEL) that has random value of binary 1 or 0 each clock cycle. In this way, the average value of the MAC output is scaled to fall in the interval [0,1].

This use of random bit streams to represent numbers has the advantage that the multiplication and addition circuits are very simple to implement. There is also some inherent robustness to random errors in the bit

[1] A. F. J. Levi, Applied Quantum Mechanics, 3rd ed. Cambridge: Cambridge University Press, 2023.

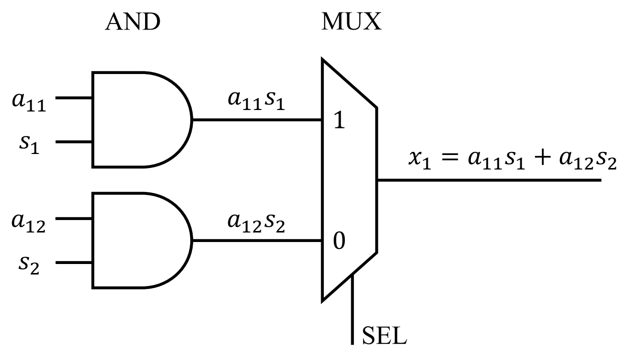

Figure 1: Illustration of the AND and MUX functions to perform matrix element multiplication followed by addition in stochastic computing. Numbers are represented as probabilities of a binary 1 or 0 signal in a clocked bit-stream of length n_{bits}. Select (SEL) has a random value of binary 1 or 0 each clock cycle and the output is scaled to fall in the interval [0,1].

stream. However, accuracy of the calculation is sensitive to unintended correlations between random number generators, the finite number of bits used to represent a number, and the condition of the matrix. In principle, while the use of a quantum source, such as a single photon generator, can be used to physically guarantee random number generation, the number of bits, n_{bits}, and matrix condition number are also important considerations when solving $\mathbf{s} = \mathbf{A}^{-1}\mathbf{x}$ for which the determinant of matrix $\mathbf{A}$ must be calculated (see Challenge Problems 3.1-3.4).

2.6 Conclusion

Noise is an intrinsic and therefore an unavoidable and fundamental aspect of all electrical systems. Mastery of its sources, mathematical models, and mitigation techniques enables designs that approach—but respect—the fundamental limits imposed by physics. By understanding its physical origins, mathematical behavior, and engineering implications, designers can optimize circuits, communication links, and measurement systems to operate close to the theoretical limits set by nature. General mastery of noise analysis is thus foundational to high-performance electrical and electronic engineering.

Click here to access a first set of computational problems on the basics of noise and noise propagation. Click here to access a second set of conceptual problems on the basics of noise and noise propagation. Click here to access a third set of analytic and computational challenge problems on the topics of stochastic computing and noise cancellation.

3 Problems

3.1 Noise: Problem Set 1 (Numeric/Computational) [return to TOC]

The following first set of problems focus on using Python to calculate and plot a variety of quantities and measures relating to noise.

3.1.1 Problem 1.1: Thermal Noise Calculation

Compute the RMS thermal-noise voltage of a $10\,\mathrm{k}\Omega$ resistor at $T = 300\,\mathrm{K}$ over $\Delta f = 1\,\mathrm{MHz}$.

3.1.2 Problem 1.2: Shot Noise Current

A photodiode draws $I = 100\,\mu\mathrm{A}$. Calculate its shot-noise RMS current over $\Delta f = 100\,\mathrm{kHz}$.

3.1.3 Problem 1.3: Comparing Noise PSDs

Plot the thermal-noise PSD of a $1\,\mathrm{k}\Omega$ resistor and the shot-noise PSD for $I = 10\,\mu\mathrm{A}$ on the same log–log axis over $f = 1$–$10^6\,\mathrm{Hz}$.

3.1.4 Problem 1.4: Quantization Noise of an n-bit ADC

A full-scale range of ±5 V is quantized by a 12-bit ADC. Compute the quantization-noise RMS voltage.

3.1.5 Problem 1.5: Total Noise from Multiple Densities

An amplifier has voltage-noise density $e_n = 2\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$ and is driven by a source resistor with $4\,\mathrm{nV}/\sqrt{\mathrm{Hz}}$. Over $\Delta f = 100\,\mathrm{kHz}$, compute total RMS noise.

3.1.6 Problem 1.6: Noise Through an RC Filter

Generate white noise, pass it through a 1 kHz low-pass RC filter ($R = C = 1/(2\pi \times 10^3)$), and plot time- and frequency-domain outputs.

3.1.7 Problem 1.7: Signal-to-Noise Ratio (SNR)

A 1 kHz sinewave of amplitude 100 mV is contaminated by white noise of spectral density $S_v = 10^{-12}\,\mathrm{V}^2/\mathrm{Hz}$ over $\Delta f = 20\,\mathrm{kHz}$. Compute SNR in dB.

3.1.8 Problem 1.8: Cascaded Noise Figure (Friis)

Three amplifier stages have gains $G_1 = 10$, $G_2 = 5$, $G_3 = 4$ (linear) and noise figures $F_1 = 2$, $F_2 = 3$, $F_3 = 4$. Compute overall noise factor F_{tot}.

3.1.9 Problem 1.9: Flicker Noise PSD Plot

Assume $S(f) = S_0/f$ with $S_0 = 10^{-18}\,\mathrm{V}^2$. Plot from $f = 1$ Hz to 1 MHz on log axes.

3.1.10 Problem 1.10: ADC Resolution for Target SNR

Find the minimum number of bits n so that an ideal ADC's SNR exceeds 60 dB. (Use $\mathrm{SNR}_{\mathrm{dB}} \approx 6.02\,n + 1.76$.)

3.2 Noise: Problem Set 2 (Analytic/Conceptual)

[return to TOC]

The following second set of problems also focus on the fundamentals of noise and noise propagation, but concentrate on analytic derivations and conceptual questions.

3.2.1 Problem 2.1: Thermal Noise Derivation

Show that a resistor R at temperature T produces a white voltage-noise power spectral density given by $S_v(f) = 4k_\mathrm{B}TR$ by appealing to the Johnson–Nyquist argument or the fluctuation–dissipation theorem.

3.2.2 Problem 2.2: Shot Noise Formula

Starting from a Poisson process of charge arrivals, derive that a DC current I across a barrier produces a current-noise PSD given by $S_i(f) = 2qI$.

3.2.3 Problem 2.3: Flicker Noise Origins

Explain why defect/trap dynamics in semiconductors give rise to a $1/f$ power spectral density, and state the typical frequency exponent range.

3.2.4 Problem 2.4: Uncorrelated Noise Addition

Show that if two uncorrelated noise sources v_{n1} and v_{n2} have RMS voltages σ_1 and σ_2, the total RMS is $\sqrt{\sigma_1^2 + \sigma_2^2}$.

3.2.5 Problem 2.5: Bandwidth and Integrated Noise

Given a white-noise PSD S_v and a filter with ideal bandwidth Δf, derive the total noise variance $v_n^2 = S_v\,\Delta f$.

3.2.6 Problem 2.6: Quantization Noise vs. Jitter

Compare the frequency dependence of quantization noise and clock-jitter–induced noise in an ADC.

3.2.7 Problem 2.7: Noise Figure Interpretation

Define Noise Figure (NF) and explain its significance in amplifier chains.

3.2.8 Problem 2.8: Impedance Noise Matching

Why is "noise-match" impedance in an LNA often different from the conjugate-match for maximum power transfer?

3.2.9 Problem 2.9: Cross-Spectrum Noise Measurement

Outline how a two-channel cross-spectrum analyzer can measure nV/Hz noise floors.

3.2.10 Problem 2.10: Noise Shaping in Modulators

Briefly describe how modulators push quantization noise out of band.

3.2.11 Problem 2.11: Noise Through a First-Order RC Filter

A resistor R at temperature T drives a first-order low-pass RC filter (series resistor R, shunt capacitor C). The resistor's thermal-noise PSD is

$$S_{v,\mathrm{in}}(f) \;=\; 4k_\mathrm{B}TR.$$

The filter's transfer function is

$$H(j\omega) \;=\; \frac{1}{1 + j\omega RC}\,, \qquad \omega = 2\pi f.$$

(a) Show that the output-noise variance is

$$v_{\mathrm{n,out}}^2 \;=\; \int_0^\infty S_{v,\mathrm{in}}(f)\,\bigl|H(j2\pi f)\bigr|^2\,df \;=\; \frac{k_\mathrm{B}T}{C}.$$

(b) Hence explain why the RMS noise of an ideal RC integrator is independent of R.

3.3 Noise: Problem Set 3 (Challenge Problems) [return to TOC]

The following set of problems include topics relating to stochastic computing, as well as the novel cancellation of thermal noise. The first four problems in particular reinforce the core stochastic-computing primitives (AND, MUX), their convergence properties, and practical simulation in Python.

3.3.1 Problem 3.1 (Analytic/Conceptual): Stochastic Multiplication by AND

In stochastic computing, a real number $p \in [0, 1]$ is represented by a random bit-stream $b[n]$ of length N, where

$$P\{b[n] = 1\} = p, \quad P\{b[n] = 0\} = 1 - p.$$

Show that if two *independent* bit-streams $a[n]$ and $s[n]$ represent probabilities $p(a)$ and $p(s)$, then the bit-wise AND

$$y[n] = a[n] \text{ AND } s[n]$$

has

$$P\{y[n] = 1\} = p(a)\, p(s).$$

3.3.2 Problem 3.2 (Analytic/Conceptual): Stochastic Addition by Multiplexing

Consider two independent stochastic product streams

$$u[n] = a_{11}[n] \text{ AND } s_1[n], \quad v[n] = a_{12}[n] \text{ AND } s_2[n],$$

with means $p_u = P\{u[n] = 1\}$ and $p_v = P\{v[n] = 1\}$. A 2-to-1 multiplexer (MUX) controlled by an independent stream $\text{SEL}[n]$ with $P\{\text{SEL} = 1\} = P\{\text{SEL} = 0\} = 1/2$ outputs

$$x[n] = \begin{cases} u[n], & \text{SEL}[n] = 1, \\ v[n], & \text{SEL}[n] = 0. \end{cases}$$

Show that the expected value of $x[n]$ is

$$P\{x[n] = 1\} = \tfrac{1}{2}\, p_u + \tfrac{1}{2}\, p_v.$$

3.3.3 Problem 3.3 (Numeric/Computational): Simulation of Stochastic Multiplication

Let $p = 0.3$ and $q = 0.6$. Write a Python script to generate two independent stochastic bit-streams of length N, compute their bit-wise AND, and estimate the product pq by the fraction of 1's in the AND output. Repeat for $N \in \{10^2, 10^3, 10^4, 10^5\}$. Plot the estimated product versus $1/\sqrt{N}$ and comment on the convergence.

3.3.4 Problem 3.4 (Numeric/Computational): Stochastic Multiply-Accumulate (MAC)

Using Python, implement the stochastic MAC for

$$x_1 = \frac{a_{11}s_1 + a_{12}s_2}{2}$$

using bit-streams of length $N = 10^5$, where

$$a_{11} = 0.2,\ a_{12} = 0.7,\ s_1 = 0.5,\ s_2 = 0.8.$$

Use AND gates for each product, then a MUX with 50% select probability to average. Estimate $\widehat{x}_1$ and compare to the theoretical value $\frac{1}{2}(0.2 \cdot 0.5 + 0.7 \cdot 0.8) = 0.34$. Plot the distribution (histogram) of the MUX output bits.

3.3.5 Problem 3.5 (Analytic/Conceptual): RMS Voltage Noise Cancellation

Johnson[2] and Nyquist[3] showed that thermal fluctuations (whose cause is fundamentally due to interactions between quantized particle states) create RMS voltage noise $V_{\mathrm{RMS}} = \sqrt{4Rk_{\mathrm{B}}T\Delta f}$ in a macroscopic resistor of value R (Ohms) at absolute temperature T (Kelvin) measured over a frequency bandwidth Δf, so long as the frequencies considered $f \ll k_{\mathrm{B}}T/(2\pi\hbar)$. This noise can limit sensitivity of an RF receiver. Bruccoleri et al.[4] showed how the following circuit, in which the current-source transconductance amplifier (g_{m} cell) is an inverter, could be used to cancel thermal noise generated by the input load resistor R.

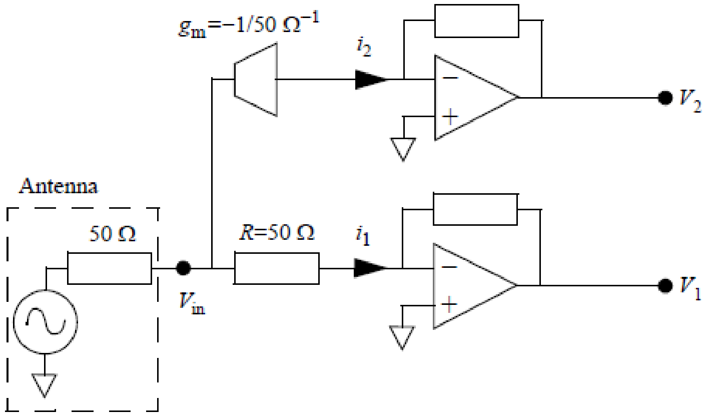

Figure 2: Schematic for a circuit that could be used to cancel thermal noise generated by the input load resistor R, in which the current-source transconductance amplifier (g_{m} cell) is an inverter.

Explain how this noise cancellation works by evaluating the current i_1 and i_2 for a voltage signal V_{in} at the input and voltage noise V_{n} generated in the resistor R. What physical principals and conservation laws do you exploit to analyze the circuit? What limits the performance of the noise cancellation circuit?

[2] Johnson, J. B., *Phys. Rev.* **32**, 97 (1928)
[3] Nyquist, H., *Phys. Rev.* **32**, 110 (1928)
[4] *IEEE J. Solid-State Circuits*, **39**, 275 (2004)

4 Solutions to problems

4.1 Noise: Solution Set 1 (Computational)

4.1.1 Solution 1.1: Thermal Noise Calculation

$$v_n = \sqrt{4k_\mathrm{B}TR\Delta f} = \sqrt{4(1.38\times10^{-23})(300)(10^4)(10^6)} \approx 12.87\ \mu\mathrm{V}_\mathrm{rms}.$$

4.1.2 Solution 1.2: Shot Noise Current

$$i_n = \sqrt{2qI\,\Delta f} = \sqrt{2(1.602\times10^{-19})(100\times10^{-6})(1\times10^5)} \approx 1.79\ \mathrm{nA}_\mathrm{rms}.$$

4.1.3 Solution 1.3: Comparing Noise PSDs

Solution sketch (resulting plot is shown in Fig. 3):

- Thermal PSD: $S_{v,T} = 4k_\mathrm{B}TR$ (flat).
- Shot PSD: convert to voltage PSD via an assumed transimpedance $Z = 1\,\mathrm{k}\Omega$: $S_{v,S} = (2qI)\,Z^2$.

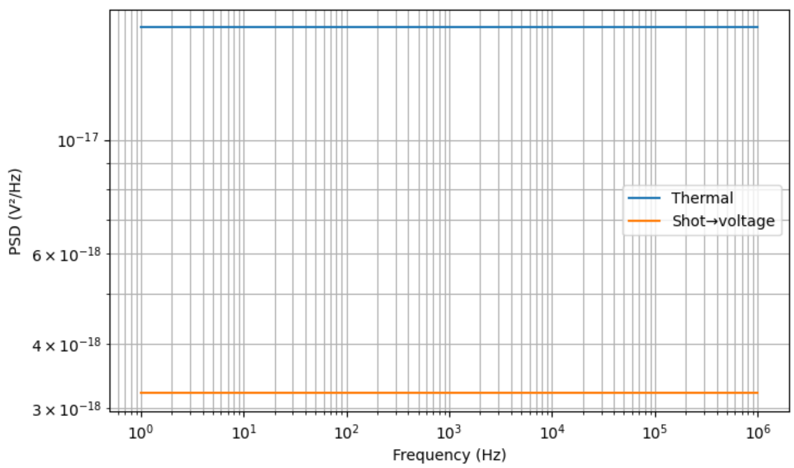

Figure 3: Solution for problem 3. See script `set1_prob3.py` for solution details.

4.1.4 Solution 1.4: Quantization Noise of an n-bit ADC

$$\Delta = \frac{10\,\mathrm{V}}{2^{12}} \approx 2.441\,\mathrm{mV}, \quad v_{n,\mathrm{q}} = \frac{\Delta}{\sqrt{12}} \approx 0.705\,\mathrm{mV}.$$

4.1.5 Solution 1.5: Total Noise from Multiple Densities

$$e_{n,\mathrm{tot}} = \sqrt{(2)^2 + (4)^2}\,\mathrm{nV}/\sqrt{\mathrm{Hz}} = \sqrt{20} \approx 4.472\,\mathrm{nV}/\sqrt{\mathrm{Hz}},$$

$$v_n = 4.472\times10^{-9}\sqrt{10^5} \approx 1.4142\ \mu\mathrm{V}$$

4.1.6 Solution 1.6: Noise Through an RC Filter

Solution sketch: Simulate with `scipy.signal.lfilter` or FFT.

Solution plot is shown in Fig. 4.

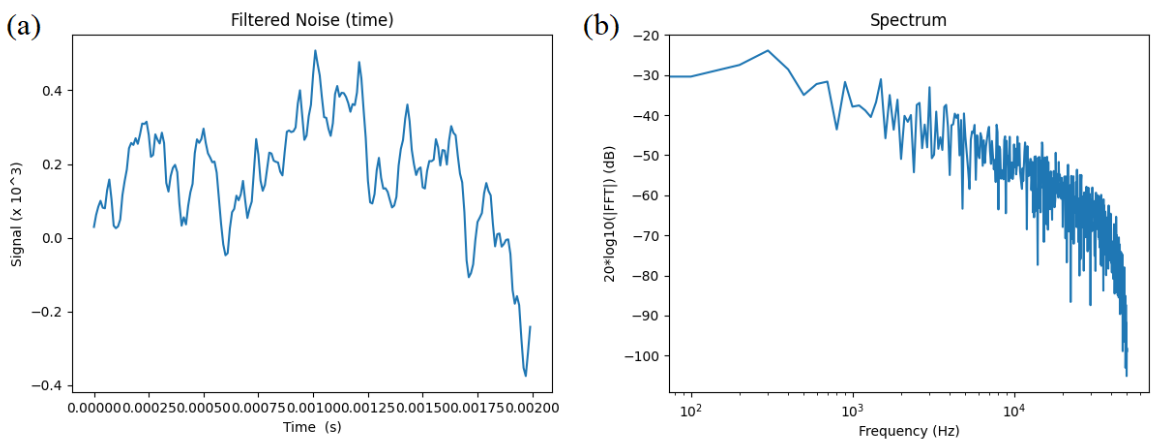

Figure 4: Solution for problem 6. (a) Time series. (b) Fast Fourier Transform (FFT) magnitude in log scale (dB). See script `set1_prob6.py` for solution details.

4.1.7 Solution 1.7: Signal-to-Noise Ratio (SNR)

$$P_s = \frac{(0.1)^2}{2} = 5 \times 10^{-3}\,\mathrm{W}, \quad P_n = S_v\,\Delta f = 10^{-12} \times 2 \times 10^4 = 2 \times 10^{-8}\,\mathrm{W},$$

$$\mathrm{SNR}_{\mathrm{dB}} = 10 \log_{10} \frac{5 \times 10^{-3}}{2 \times 10^{-8}} \approx 54\,\mathrm{dB}.$$

4.1.8 Solution 1.8: Cascaded Noise Figure (Friis)

$$F_{\mathrm{tot}} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} = 2 + \frac{2}{10} + \frac{3}{10 \cdot 5} = 2 + 0.2 + 0.06 = 2.26.$$

4.1.9 Solution 1.9: Flicker Noise PSD Plot

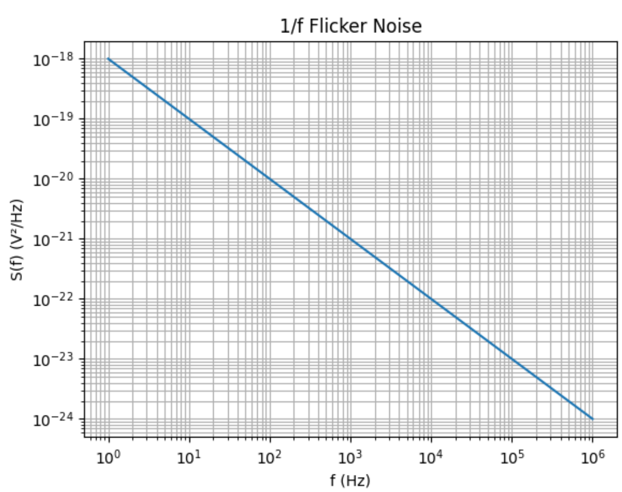

Figure 5: Solution for problem 9. See script `set1_prob9.py` for solution details.

4.1.10 Solution 1.10: ADC Resolution for Target SNR

$$6.02\,n + 1.76 \geq 60 \implies n \geq \frac{60 - 1.76}{6.02} \approx 9.64, \text{ so } n = 10 \text{ bits minimum.}$$

4.2 Noise: Solution Set 2 (Analytic/Conceptual)

4.2.1 Solution 2.1: Thermal Noise Derivation

1. In thermal equilibrium, each electrical degree of freedom has average energy $\frac{1}{2}k_{\mathrm{B}}T$.
2. A resistor can be modeled as a transmission line terminated in its own resistance. Over a small bandwidth Δf, the available noise power is $k_{\mathrm{B}}T\,\Delta f$ into a matched load.
3. Equating that to voltage noise $v_n^2/(4R)$ yields

$$\frac{v_n^2}{4R} = k_{\mathrm{B}}T\Delta f \quad \Longrightarrow \quad v_n^2 = 4k_{\mathrm{B}}TR\Delta f,$$

so $S_v(f) = v_n^2/\Delta f = 4k_{\mathrm{B}}TR$.

4.2.2 Solution 2.2: Shot Noise Formula

1. If charges arrive as a Poisson process at rate $\lambda = I/q$, then in time Δt the mean $\mu = \lambda\Delta t$ and variance $\sigma^2 = \lambda\Delta t$.
2. The variance of collected charge is $\sigma^2 q^2 = qI\,\Delta t$.
3. Over bandwidth $\Delta f = 1/(2\Delta t)$, noise current squared is $i_n^2 = 2\,q\,I\,\Delta f$.
4. Thus $S_i = i_n^2/\Delta f = 2qI$.

4.2.3 Solution 2.3: Flicker Noise Origins

- Charge trapping/detrapping at random defect sites produces random telegraph signals with a broad spectrum of time constants τ.
- A superposition of many such Lorentzian PSDs with a $1/\tau$ distribution yields $S(f) \propto 1/f^{\alpha}$.
- Empirically $\alpha \approx 0.8$–1.2 in MOSFETs, BJTs, thin-film resistors.

4.2.4 Solution 2.4: Uncorrelated Noise Addition

- By definition of variance, $\mathrm{Var}(v_{n1} + v_{n2}) = \mathrm{Var}(v_{n1}) + \mathrm{Var}(v_{n2}) + 2\,\mathrm{Cov}(v_{n1}, v_{n2})$.
- Uncorrelated $\Rightarrow \mathrm{Cov} = 0$.
- RMS voltage $= \sqrt{\mathrm{Var}}$.

4.2.5 Solution 2.5: Bandwidth and Integrated Noise

$$v_n^2 = \int_0^{\Delta f} S_v\,df = S_v\,\Delta f.$$

4.2.6 Solution 2.6: Quantization Noise vs. Jitter

- Quantization noise is spectrally flat (white) within Nyquist band.
- Jitter noise grows f_{in}^2; at higher input frequencies aperture jitter dominates.

4.2.7 Solution 2.7: Noise Figure Interpretation

$\mathrm{NF} = 10\log_{10}(\mathrm{SNR_{in}}/\mathrm{SNR_{out}})$. It quantifies how much an amplifier degrades SNR; low NF front-ends are crucial in weak-signal receivers.

4.2.8 Solution 2.8: Impedance Noise Matching

Noise-match optimizes the source impedance to achieve minimum noise figure, not maximum gain: often requires a complex source impedance that minimizes the real and imaginary noise-parameter trade-off.

4.2.9 Solution 2.9: Cross-Spectrum Noise Measurement

Simultaneously sample the same DUT output on two uncorrelated analyzers, compute cross-spectrum; uncorrelated instrument noise averages to zero, leaving DUT noise.

4.2.10 Solution 2.10: Noise Shaping in Modulators

Feedback loop with oversampling shapes quantization error transfer function: it is high-pass, thus attenuating in-band noise while pushing most quantization noise to higher frequencies, where digital filters remove it.

4.2.11 Solution 2.11: Noise Through a First-Order RC Filter

(a) Setup the integral The output PSD is

$$S_{v,\text{out}}(f) = S_{v,\text{in}}(f)\,\big|H(j2\pi f)\big|^2 = 4k_{\text{B}}TR\,\frac{1}{1+(2\pi fRC)^2}.$$

Thus the total noise variance is

$$v^2_{\text{n,out}} = \int_0^\infty S_{v,\text{out}}(f)\,df = \int_0^\infty \frac{4k_{\text{B}}TR}{1+(2\pi fRC)^2}\,df.$$

Next, the variables are changed. Let $\omega = 2\pi f$, so $df = d\omega/(2\pi)$. Then

$$v^2_{\text{n,out}} = \int_0^\infty \frac{4k_{\text{B}}TR}{1+(\omega RC)^2}\,\frac{d\omega}{2\pi} = \frac{4k_{\text{B}}TR}{2\pi}\int_0^\infty \frac{d\omega}{1+(\omega RC)^2}.$$

Finally, the standard integral is evaluated. Recall the standard result:

$$\int_0^\infty \frac{d\omega}{1+(\omega\tau)^2} = \frac{\pi}{2\tau},$$

with $\tau = RC$. Therefore

$$v^2_{\text{n,out}} = \frac{4k_{\text{B}}TR}{2\pi}\cdot\frac{\pi}{2RC} = \frac{4k_{\text{B}}TR}{4RC} = \frac{k_{\text{B}}T}{C}.$$

(b) Since $v^2_{\text{n,out}} = k_{\text{B}}T/C$ depends only on C (and T), the RMS noise $\sqrt{k_{\text{B}}T/C}$ is independent of R. Physically, increasing R narrows the bandwidth but raises the source noise density, yielding a constant noise-charge on C.

4.3 Noise: Solution Set 3 (Challenge Problems)

4.3.1 Solution 3.1: Stochastic Multiplication by AND

By independence,

$$P\{y[n]=1\} = P\{a[n]=1 \,\wedge\, s[n]=1\} = P\{a[n]=1\}\,P\{s[n]=1\} = p(a)\,p(s).$$

Hence the average of the AND output over many bits converges to the product $p(a)\,p(s)$, implementing multiplication.

4.3.2 Solution 3.2: Stochastic Addition by Multiplexing

Condition on the select bit:

$$P\{x=1\} = P\{\text{SEL}=1\}\,P\{u=1\} + P\{\text{SEL}=0\}\,P\{v=1\} = \tfrac{1}{2}\,p_u + \tfrac{1}{2}\,p_v.$$

Thus the MUX performs an average (scaled sum) of the two inputs, implementing $(p_u+p_v)/2$.

4.3.3 Solution 3.3: Simulation of Stochastic Multiplication

Solution plot is shown in Fig. 6. As N increases, the random error $|\widehat{pq} - pq|$ scales roughly like $1/\sqrt{N}$, confirming the law of large numbers.

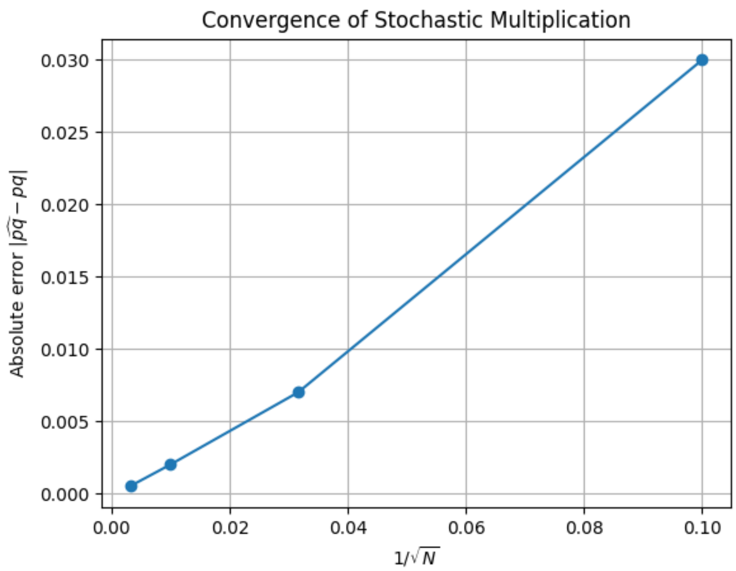

Figure 6: Solution for problem 3. See script `set3_prob3.py` for solution details.

4.3.4 Solution 3.4: Stochastic Multiply-Accumulate (MAC)

Solution plot is shown in Fig. 7. The printed estimate will be close to 0.34 (within a few 10^{-3}), and the histogram shows approximately 34,000 ones out of 100,000 samples, confirming the MAC behavior.

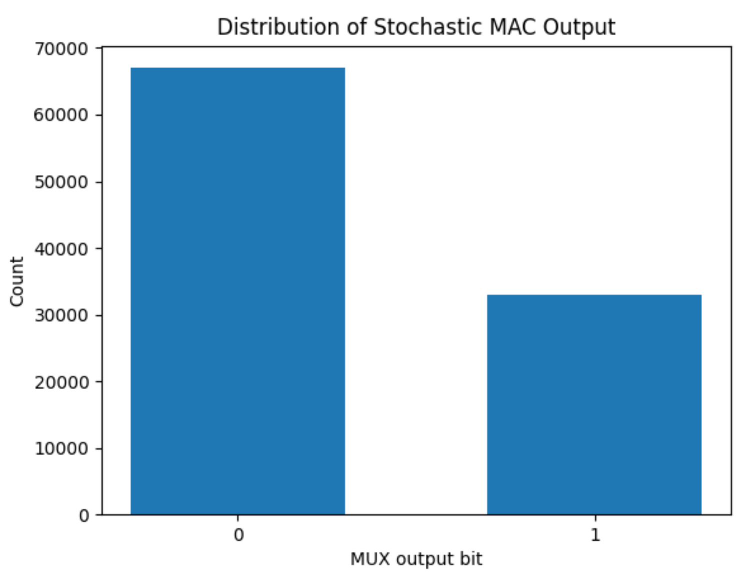

Figure 7: Solution for problem 4. See script `set3_prob4.py` for solution details.