# ECE 105: Introduction to Electrical Engineering

Lecture 15

Short Linear Algebra Intro

Yasser Khan

Rehan Kapadia

# What is a vector?

- In everyday life, we often deal with quantities that have only magnitude, like temperature (e.g., 20°C) or mass (e.g., 5 kg). These are scalars

- A vector is a mathematical object that has both magnitude (size) and direction.
- For example, when you're driving a car, your velocity is a vector because it includes both your speed (magnitude) and the direction you're traveling.

- In linear algebra, we represent vectors as a list of numbers. Each number in the list corresponds to a component of the vector in a particular dimension.

# Representing Vectors

USC University of Southern California

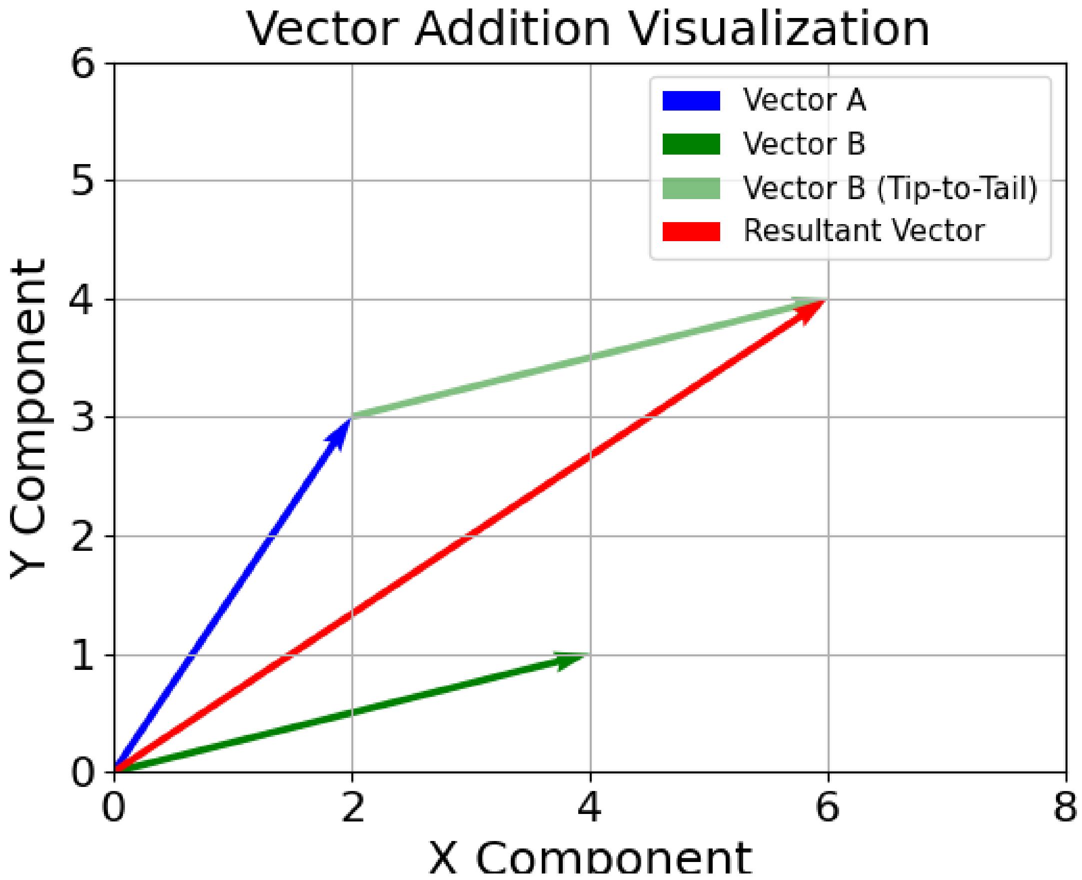
- Let's start with a simple 2-dimensional (2D) space, like a flat map. In this space, we can represent any point using two numbers: its horizontal position (x-coordinate) and its vertical position (y-coordinate).

- For example, let's say we have a vector v = [3, 4]. This means:- The vector extends 3 units in the x-direction (horizontally)- The vector extends 4 units in the y-direction (vertically)

# Vector Addition

- **Vector Addition**: Combines two or more vectors to produce a resultant vector.
- **Tip-to-Tail Method**: Place the tail of one vector at the tip of the other.
- **Resultant Vector**: Drawn from the starting point of the first vector to the endpoint of the last.
- **Component-wise Addition**: Add corresponding components of vectors.
- Example: If **A = [a1, a2]** and **B = [b1, b2]**, then **A + B = [a1 + b1, a2 + b2]**.
- **Geometric Interpretation**: Represents cumulative effect (e.g., forces, velocities).
- **Commutative Property**: **A + B = B + A**; order does not change the result.

# Vector Addition

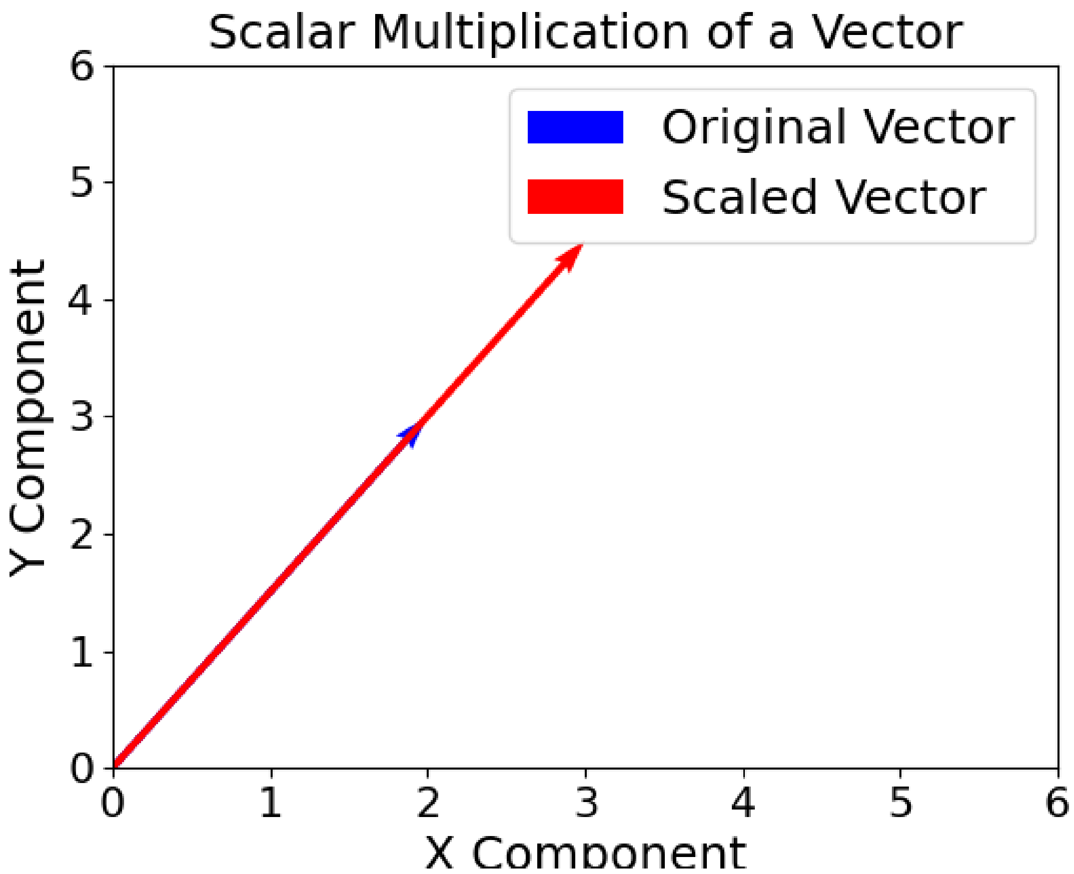


# Scalar Multiplication

- **Scalar Multiplication**: Scalar multiplication involves multiplying a vector by a scalar (a single real number).

- **Magnitude Adjustment**: The scalar changes the magnitude (length) of the vector but **not** its direction.
  - If the scalar is greater than 1, the vector lengthens.
  - If the scalar is between 0 and 1, the vector shortens.
  - If the scalar is negative, the vector reverses direction.
- **Mathematical Operation**: Each component of the vector is multiplied by the scalar.
  - For example, given a vector **v = [v1, v2, v3]** and scalar **k**, the result is **k * v = [k * v1, k * v2, k * v3]**.
- **Geometric Interpretation**: The new vector is in the same or opposite direction as the original, with its length scaled by the absolute value of the scalar.

# Scalar Multiplication



# Dot Product

•**Dot Product**: Scalar value representing the multiplication of two vectors.

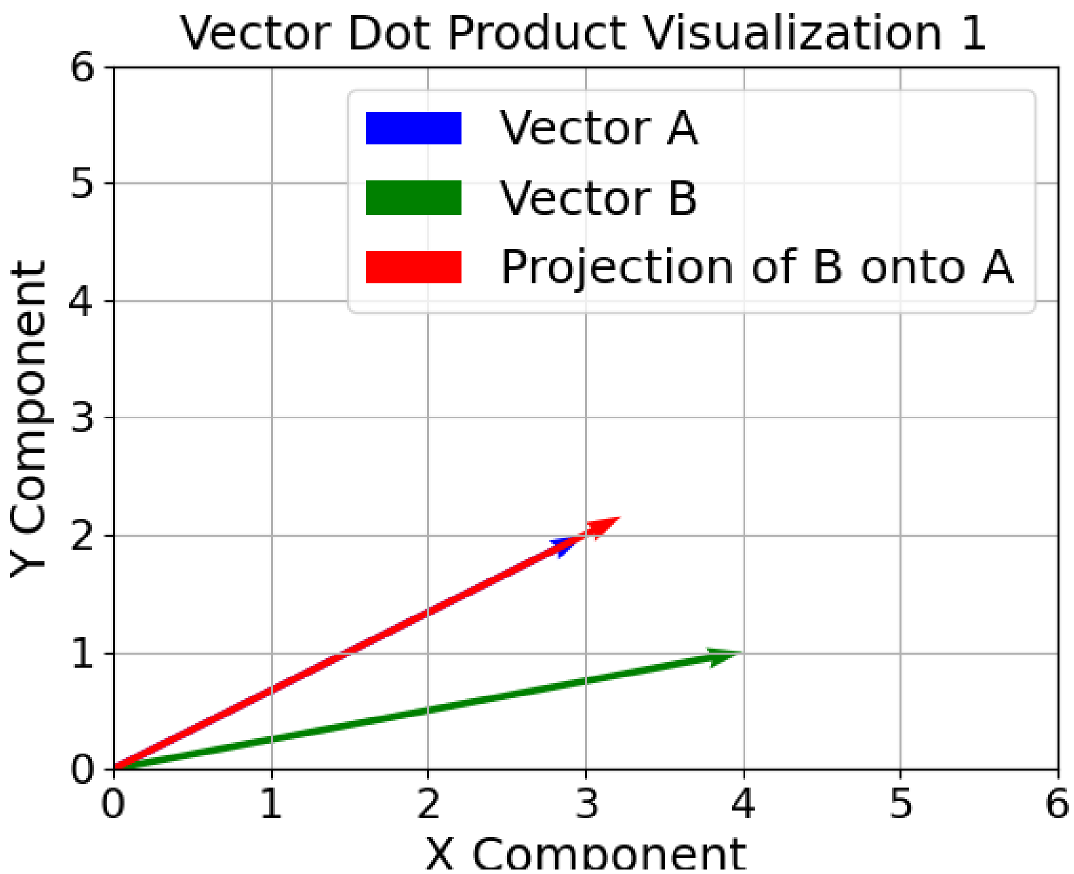
•**Mathematical Definition**: Given vectors **A** and **B**, dot product is **A · B = |A| |B| cos(θ)**, where **θ** is the angle between them.

•**Component-wise Calculation**: For vectors **A = [a1, a2]** and **B = [b1, b2]**, **A · B = a1 * b1 + a2 * b2**.

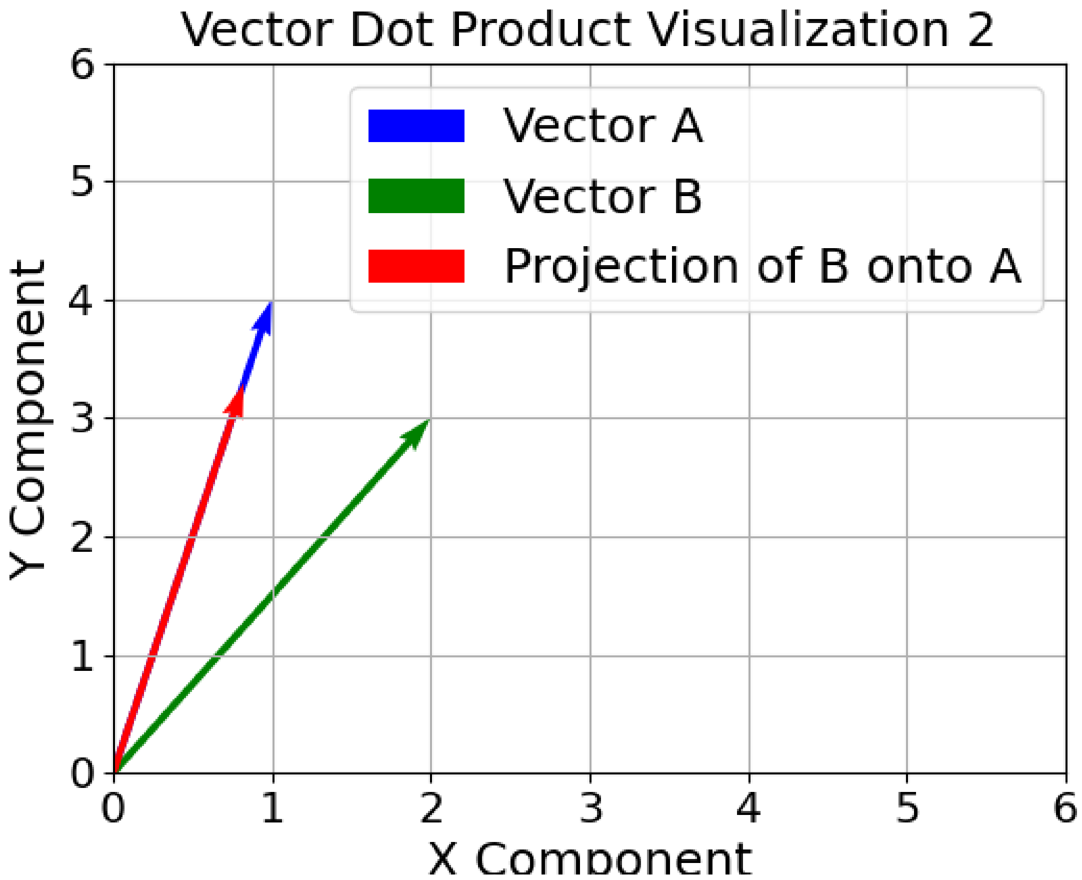
•**Geometric Interpretation**: Represents the magnitude of one vector projected onto another.

•**Orthogonality**: If **A · B = 0**, vectors are perpendicular.

# Dot Product



Vector Dot Product Visualization 1
Vector A
Vector B
Projection of B onto A
Y Component
X Component
0
1
2
3
4
5
6

Vector Dot Product Visualization 2
Vector A
Vector B
Projection of B onto A
Y Component
X Component
0
1
2
3
4
5
6

# Matrices

•**Matrix Definition**: A rectangular array of numbers arranged in rows and columns.

•**Dimensions**: Defined by the number of rows and columns (e.g., a 3x3 matrix).

•**Operations**: Can perform addition, subtraction, and multiplication with other matrices or scalars.

•**Uses**: Represent linear transformations, systems of equations, data storage, and more.

•**Notation**: Elements are accessed using indices, e.g., element $\mathbf{a}_{ij}$ is at row **i** and column **j**.

# Matrices



Matrix Representation

| | | | |
|---|---|---|---|
| 1 | 1 | 2 | 3 |
| 2 | 4 | 5 | 6 |
| 3 | 7 | 8 | 9 |

# Matrix Addition

- **Matrix Addition**: Adds corresponding elements of two matrices of the same dimension.

- **Element-wise Operation**: Given matrices **A** and **B**, each element $\mathbf{C}_{ij}$ of the resulting matrix **C** is calculated as $\mathbf{C}_{ij} = \mathbf{A}_{ij} + \mathbf{B}_{ij}$.

- **Conditions**: Matrices must have the same dimensions for addition.

- **Geometric Interpretation**: Represents combining transformations or cumulative effect.

# Matrix Addition

**Matrix A**

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

**Matrix B**

| | | |
|---|---|---|
| 9 | 8 | 7 |
| 6 | 5 | 4 |
| 3 | 2 | 1 |

**Matrix A + Matrix B**

| | | |
|---|---|---|
| 10 | 10 | 10 |
| 10 | 10 | 10 |
| 10 | 10 | 10 |

# Matrix Multiplication



•**Matrix Multiplication**: Combines two matrices to produce a third by taking the dot product of rows and columns.

•**Dot Product**: Each element in the resultant matrix is the sum of products between corresponding elements in a row of the first matrix and a column of the second.

•**Conditions**: For matrices **A (m×n)** and **B (n×p)**, the resulting matrix **C** has dimensions **m×p**.

•**Not Commutative**: **A * B ≠ B * A** in general.

# Matrix Multiplication

- Given **Matrix A:** $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
- And **Matrix B:** $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$
- **Result:** $\begin{bmatrix} (1*2+2*1) & (1*0+2*3) \\ (3*2+4*1) & (3*0+4*3) \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 12 \end{bmatrix}$

# Matrix Transpose

- **Matrix Transpose**: Flips a matrix over its diagonal, swapping rows with columns.

- **Notation**: The transpose of a matrix **A** is denoted by $\mathbf{A}^T$.

- **Resulting Dimensions**: If **A** is an **m x n** matrix, $\mathbf{A}^T$ will be an **n x m** matrix.

- **Use Case**: Often used in linear algebra operations, such as dot products and matrix manipulation.

# Matrix Transpose

- Given **Matrix A:** $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$
- **Transpose of Matrix A:** $\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

# Matrix Determinant

- **Matrix Determinant**: A scalar value that can be calculated from a square matrix.

- **Notation**: The determinant of a matrix **A** is denoted by **|A|** or **det(A)**.

- **Conditions**: Only square matrices have determinants.

- **Use**: Helps determine matrix properties, like invertibility.

# Matrix Determinant

USC University of Southern California

- The determinant of a matrix represents a value that can indicate whether the matrix is invertible. A non-zero determinant means the matrix has an inverse, while a zero determinant means it is singular (non-invertible).

- **Geometric Interpretation**: For a 2x2 matrix, the determinant represents the area of the parallelogram formed by its row (or column) vectors.

- **Applications**: Determinants are used in solving systems of linear equations (using Cramer's Rule), in calculating eigenvalues, and in understanding linear transformations and their effects (like scaling and orientation).

# Matrix Determinant

- Given **Matrix A:** $\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix}$
- **Determinant:** $|A| = (3 * 6) - (8 * 4) = 18 - 32 = -14$

# Matrix Inverse

- **Matrix Inverse**: The inverse of a matrix **A** is another matrix $\mathbf{A}^{-1}$ such that $\mathbf{A} * \mathbf{A}^{-1} = \mathbf{I}$, where **I** is the identity matrix.

- **Conditions**: Only square matrices with a non-zero determinant have an inverse.

- **Use**: Helps solve matrix equations of the form $\mathbf{AX} = \mathbf{B}$.

# Matrix Inverse

- **Inverse Formula (2x2 Matrix)**:
  - For a 2x2 matrix **A = [ [a, b], [c, d] ]**, the inverse is given by:
  - **$A^{-1}$ = (1 / det(A)) * [ [d, -b], [-c, a] ]**
  - Where **det(A) = ad - bc**.
- **Condition**: **det(A)** must be non-zero for the inverse to exist.
- **General Method**: For larger matrices, use **Gaussian elimination** or **adjoint method** to find the inverse.

# Matrix Inverse

USC University of Southern California

- **Given Matrix A**:
  - **A = [ [4, 7], [2, 6] ]**
- **Step 1: Calculate Determinant**:
  - **det(A) = (4 * 6) - (7 * 2) = 24 - 14 = 10**
- **Step 2: Apply Inverse Formula**:
  - **$A^{-1}$ = (1 / 10) * [ [6, -7], [-2, 4] ]**
  - **$A^{-1}$ = [ [0.6, -0.7], [-0.2, 0.4] ]**
- **Verification**:
  - **A * $A^{-1}$ = I**, where **I** is the identity matrix **[ [1, 0], [0, 1] ]**.

# Matrix Inverse

USC University of Southern California

- A matrix must have a non-zero determinant to be invertible. If the determinant is zero, the matrix is termed **singular**, and it does not have an inverse.

- **Identity Matrix**: The result of multiplying a matrix by its inverse is always the identity matrix, which is a matrix with **1s** on the diagonal and **0s** elsewhere.

- **Applications**: Matrix inverses are crucial in solving systems of linear equations, particularly in situations involving multiple variables, and in finding solutions using matrix algebra.

# Solving Systems of Equations

- **Definition**: A system of equations is a set of two or more equations with the same variables.
- **Objective**: Find values for the variables that satisfy all the equations simultaneously.
- **Example**:
  - Equation 1: $2x + y = 8$
  - Equation 2: $x + 3y = 18$

# Solving Systems of Equations

- **Matrix Representation**: Any system of linear equations can be represented in matrix form.
  - **Coefficient Matrix (A)**: Contains the coefficients of the variables.
  - **Constant Vector (B)**: Contains the constants from each equation.
  - **Variable Vector (X)**: Contains the unknowns.
- **Example**:
  - Equations: 2x + y = 8, x + 3y = 18
  - Matrix Form: **A * X = B**
    - **A**: ⩽ [ [2, 1], [1, 3] ]
    - **X**: ⩽ [ [x], [y] ]
    - **B**: ⩽ [ [8], [18] ]

# Solving Systems of Equations

- **Inverse Method**: If **A** is invertible, we can solve for **X** using the formula: $\mathbf{X} = \mathbf{A}^{-1} * \mathbf{B}$.
- **Gauss Elimination**: Alternatively, use row operations to reduce the system to upper triangular form and solve via back-substitution.
- **LU Decomposition**: Decompose **A** into **L** (lower triangular) and **U** (upper triangular) for more computational efficiency.
- **Determinant Requirement**: **A** must have a non-zero determinant to be invertible and solve for **X**.

# Solving a System of Equations

- **Given System**:
  - $2x + y = 8$
  - $x + 3y = 18$
- **Step 1**: Write the matrix representation:
  - **A**: [ [2, 1], [1, 3] ]
  - **B**: [ [8], [18] ]
- **Step 2**: Find $\mathbf{A}^{-1}$ (Inverse of A):
  - $\mathbf{A}^{-1}$ = [ [0.6, -0.2], [-0.2, 0.4] ]
- **Step 3**: Solve for **X**:
  - $\mathbf{X} = \mathbf{A}^{-1} * \mathbf{B}$
  - **X =** [ [3.6], [4.8] ]
- **Solution**: $x = 3.6$, $y = 4.8$

# Eigenvalues and Eigenvectors

- **Definition**: An eigenvector of a matrix is a non-zero vector that only changes by a scalar factor when that matrix is applied to it.

- **Eigenvalue**: The scalar factor associated with an eigenvector is called an eigenvalue.

- **Basic Concept**: For a given square matrix **A**, if $\mathbf{A} * \mathbf{v} = \boldsymbol{\lambda} * \mathbf{v}$, then **v** is an eigenvector and $\boldsymbol{\lambda}$ is the corresponding eigenvalue.

# Eigenvalues and Eigenvectors

USC University of Southern California

- **Diagonalization**: Matrices can be simplified via diagonalization using their eigenvalues and eigenvectors.
- **Stability Analysis**: Used in control systems to determine system stability.
- **Principal Component Analysis (PCA)**: In machine learning, PCA uses eigenvectors to reduce the dimensionality of data.
- **Physical Interpretation**: Represent the axes along which a transformation acts by stretching or compressing.

# Eigenvalues and Eigenvectors

- **Eigenvalue Equation**: **A * v = λ * v**, where **A** is an **n x n** matrix, **v** is a non-zero vector, and **λ** is a scalar.
- **Characteristic Equation**: To find eigenvalues, solve **det(A - λI) = 0**, where **I** is the identity matrix.
- **Finding Eigenvectors**: Once eigenvalues are known, substitute **λ** back into **(A - λI)v = 0** to find eigenvectors.

# Eigenvalues and Eigenvectors

- **Given Matrix A**:
  - **A = [ [4, 1], [2, 3] ]**
- **Step 1: Find the Characteristic Polynomial**:
  - **det(A - λI) = det([ [4-λ, 1], [2, 3-λ] ]) = (λ^2 - 7λ + 10)**
- **Step 2: Solve for Eigenvalues**:
  - Solve **λ^2 - 7λ + 10 = 0**
  - Eigenvalues are **λ = 5** and **λ = 2**
- **Step 3: Find Eigenvectors**:
  - For **λ = 5**, solve **(A - 5I)v = 0**
  - Eigenvector for **λ = 5**: **v = [1, 2]**
  - For **λ = 2**, solve **(A - 2I)v = 0**
  - Eigenvector for **λ = 2**: **v = [-1, 1]**