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Portfolio Optimization with Trade Paring Constraints

A New Feature in the Barra Optimizer

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1 Introduction

In the context of portfolio optimization, paring constraints refer to a special class of constraints that control the number of assets or trades in a portfolio, or set a minimum requirement on the holding levels for assets or the transaction levels for trades. Broadly speaking, paring constraints can be classified into three categories—asset paring constraints, trade paring constraints, and level paring constraints.

Paring constraints play an important role in portfolio management. Quantitative portfolio managers frequently and widely employ paring constraints in their portfolio construction and rebalancing processes. However, these constraints also introduce a significant amount of complexity into the optimization procedure. Paring constraints often change the nature of an optimization problem from a well-behaved *convex* quadratic programming problem to a more complicated *discrete* quadratic programming problem.

This paper is devoted to portfolio optimization problems with trade paring constraints, or “trade paring (problems)” in short. It is organized into five sections. Section 2 provides the definitions, applications, and variations of trade paring. Section 3 discusses various methods for solving the trade paring problems. Section 4 presents computational results to compare the performance of several trade paring heuristics, and to show the advantages of the integrated trade paring approach in the Barra Optimizer. Concluding remarks follow in Section 5.

2 What Are Trade Paring Problems?

As mentioned before, trade paring constraints belong to one of the three broad categories of paring constraints. In this section, we will first briefly define these paring constraints, then provide a few examples of trade paring applications, and finally discuss variations of the trade paring constraints supported in the Barra Optimizer. Although asset paring and level paring are not the focus of this paper, we provide their definitions here for comparison and reference.

2.1 Definitions of Paring Constraints

The three classes of paring constraints are defined as follows.

Asset Paring: A constraint that specifies the maximum or minimum number of assets in an optimal portfolio. For example, a constraint asking for a maximum of 100 assets or a minimum of 10 assets in an optimal portfolio is an asset paring constraint.

Trade Paring: A constraint that specifies the maximum or minimum number of trades involved in transacting an initial portfolio into an optimal one. For example, a constraint requiring a maximum of 20 trades or a minimum of 5 trades is a trade paring constraint.

Level Paring: A constraint that establishes a minimum threshold level for assets or trades at the portfolio or individual asset/trade level. The following are two types of level paring.

(1) Minimum Holding Level Paring: A constraint that specifies how the holding level for any asset cannot be below a certain level. For example, a constraint requiring that the holding level for all assets in the optimal portfolio be at least 1% of the portfolio value is a minimum holding level paring constraint.

(2) **Minimum Transaction Size Level Paring:** A constraint that specifies how the transaction size for any trade cannot be below a certain level. For example, a constraint requiring that the transaction size for any trade be at least 0.5% of the portfolio value is a minimum transaction size level paring constraint.

Paring constraints are discrete in nature. A portfolio optimization problem with paring constraints can be formulated as a mixed-integer quadratic programming problem. By introducing binary variables, a simple mean-variance portfolio optimization problem with paring constraints has the following mathematic form:

$$\underset{\mathbf{h}}{\text{Maximize}} \quad \mathbf{r}^T\mathbf{h} - \lambda\, \mathbf{h}^T \boldsymbol{\Sigma}\, \mathbf{h} \tag{1}$$

Subject to

$$\mathbf{e}^{\mathrm{T}}\mathbf{h} = 1 \qquad \text{(Holding Constraint)} \tag{2}$$

$$\mathbf{L}_s \le \mathbf{A}\mathbf{h} \le \mathbf{U}_s \qquad \text{(General Linear Constraints)} \tag{3}$$

$$\mathbf{L}_h \le \mathbf{h} \le \mathbf{U}_h \qquad \text{(Asset Bounds)} \tag{4}$$

$$K_{\min} \le \sum_{i=1}^{N} \delta_i \le K_{\max}, \quad \delta_i = \begin{cases} 0, & \text{if } h_i = 0 \\ 1, & \text{otherwise} \end{cases} \qquad \text{(Asset Paring Constraints)} \tag{5}$$

$$T_{\min} \le \sum_{i=1}^{N} \sigma_i \le T_{\max}, \quad \sigma_i = \begin{cases} 0, & \text{if } h_i - h_{0i} = 0 \\ 1, & \text{otherwise} \end{cases} \qquad \text{(Trade Paring Constraints)} \tag{6}$$

$$\begin{cases} h_i \ge \xi \;\; or \;\; h_i \le -\xi \;\; or \;\; h_i = 0 \\ h_i - h_{0i} \ge \zeta \;\; or \;\; h_i - h_{0i} \le -\zeta \;\; or \;\; h_i - h_{0i} = 0 \end{cases} \qquad \text{(Level Paring Constraints)} \tag{7}$$

$$i = 1, 2, \cdots, n$$

where $\mathbf{h}$ is the vector of portfolio holdings, $\mathbf{r}$ is the vector of asset returns, $\boldsymbol{\Sigma}$ is the asset-by-asset covariance matrix, and λ is the risk aversion parameter. More specifically, the asset paring constraints (5) specify that the number of assets in the optimal portfolio cannot be greater than $K_{\max}$ or less than $K_{\min}$. The trade paring constraints (6) require that the number of trades in the trade list be no greater than $T_{\max}$ or less than $T_{\min}$. The level paring constraints (7) set the minimum holding level at ξ and the minimum trading level at ζ. More detailed descriptions of the portfolio optimization problem in general, and the paring constraints in particular, can be found in Stefek, Liu and Xu (2010).

2.2 Applications of Trade Paring

As discussed in Liu and Xu (2004), as well as in Kopman, et al (2009), paring problems arise frequently in financial applications. They often reflect a portfolio manager's desire to control administrative and transaction costs, maintain liquidity, seek risk diversification, as well as comply with legal or other practical restrictions.

Both asset paring constraints and trade paring constraints are valuable in portfolio construction and rebalancing. The following are three examples of practical trade paring problems.

Case 1: Rebalancing when there is a cash infusion or withdrawal

Imagine that a manager has a portfolio with 100 asset names. If there is cash infusion or cash withdrawal, the manager may need to rebalance the portfolio. For efficiency or other reasons, the manager could use a limited number of trades, say five, to invest the cash added or obtain cash for withdrawal.

Case 2: Rebalancing when new alpha information is available on one or more assets

An active money manager often receives new alpha information on one or more assets in the investment universe from time to time. When this happens, the manager may want to rebalance the portfolio to reflect new information. To prevent an excessive number of trades from occurring, the manager may want to limit the total number of trades; for example, a maximum of three trades.

Case 3: Monthly or quarterly rebalancing in the context of backtesting

When conducting regular, periodic backtesting, one may want to control both the number of assets in each period and the number of trades from one period to the next. For example, one could specify that the maximum number of assets allowed in the optimal portfolio at any period is 100, and that the maximum trades from one period to another is 20. If one only uses a trade paring constraint without an asset paring constraint, the number of assets in the optimal portfolio may grow from period to period, and in the worst case may eventually reach the total number of assets in the investment universe. On the other hand, employing only an asset paring constraint without a trade paring constraint may result in excessive trades in one or more periods. In the worst case scenario, the total number of trades could be twice the maximum number of assets specified, after all assets in the optimal portfolio of the previous period are sold and a new basket of assets are bought.

2.3 Variations of Trade Paring

In addition to constraining the total number of trades, trade paring constraints can also be used to control the number of buy trades or sell trades separately. When shorting is allowed, trade paring constraints also include those that limit the total number of long trades or short trades.

A trade occurs when an asset's final position differs from its initial position. If the final position is greater than the initial position, the trade is a buy. Conversely, the trade is a sell.

A long trade occurs when an asset has a positive initial position and its final position differs from this initial one, or when the initial position is zero and the final position is positive. Conversely, a short trade arises when an asset has a negative initial position with a different final position, or a zero initial position, but a negative final position. In the case where an asset has a negative initial position and a positive final position, the transaction actually involves two segments—a short trade (covering a short position) plus a crossover long trade (buying a long position). Similarly, the transaction going from a positive initial position to a negative final position consists of a long trade (selling a long position) plus a crossover short trade (initiating a short position). The Barra Optimizer gives the user a choice of treating a transaction with a crossover segment as either one trade or two trades.

In summary, the Barra Optimizer supports the following variations of trade paring constraints;

- Maximum number of trades
- Maximum number of buy trades
- Maximum number of sell trades
- Maximum number of long trades
- Maximum number of short trades
- Minimum number of trades
- Minimum number of buy trades
- Minimum number of sell trades
- Minimum number of long trades
- Minimum number of short trades

3 How to Solve Trade Paring Problems

In this section, we first discuss the difficulties of solving a trade paring problem, then, we consider several methods that can be used to solve trade paring problems. Lastly, we introduce the integrated approach that is used in the Barra Optimizer. The computational results for some of these methods will be presented in Section 4.

3.1 Difficulties in Solving Trade Paring Problems

Like asset paring constraints, trade paring constraints are cardinality constraints. In fact, if the initial portfolio consists of cash only, then a trade paring constraint that limits the total number of trades will be identical to an asset paring constraint that limits the total number of assets in the optimal portfolio.

Problems with cardinality constraints are particularly difficult to solve to optimality. They often change the nature of the optimization problem from a well-behaved convex problem to a complicated, ill-behaved non-convex one. A convex problem has the desirable property that any convex combinations of two feasible portfolios will also be a feasible portfolio. For example, if h1 and h2 are two feasible portfolios, then h3=0.5*h1 + 0.5*h2 is guaranteed to be a feasible portfolio, assuming all constraints are convex. However, in the presence of a cardinality constraint (for example, a constraint that limits the total number of trades to 20), then h3 may not be feasible. Just because h1 and h2 both have limits of no more than 20 trades, this does not automatically lead to h3 having a limit of no more than 20 trades. In other words, convex combinations of two feasible portfolios may not be feasible when a cardinality constraint is present. This makes the search for an optimal solution difficult.

Another example of the difficulties associated with the cardinality constraints in portfolio optimization is that the efficient frontier is no longer continuous (Chang, et al 2000), which further complicates the search for an optimal risk-return trade-off.

3.2 Methods for Solving Trade Paring Problems

In theory, a trade paring problem can be converted to an asset paring problem after certain mathematical transformations. Many methods for tackling asset paring can be applied to trade paring, too. However, transforming a trade paring problem into an asset paring problem may be messy, error-prone, inefficient, and sometimes impractical. Thus, trade paring deserves direct approaches of its own. In what follows, we introduce several methods that can be used to solve problems with trade paring constraints, or with mixed paring constraints, where trade paring is combined with other types of paring constraints.

3.2.1 Branch-and-Bound and Brand-and-Cut Methods

As mentioned in Section 2, a trade paring problem can be formulated as a mixed-integer quadratic programming (MIQP) problem. Naturally, one can employ the Branch-and-Bound (B&B) or Branch-and-Cut (B&C) methods to solve a trade paring problem. Many successful B&B and B&C methods are available for mixed-integer *linear* programming (MILP) problems (Nemhauser and Wolsey, 1998). However, most of the efficient cuts for MILP may not work well for MIQP because the optimal solution may lie in the interior feasible region for MIQP instead of the boundaries for MILP. Bienstock (1996) extended some MILP cuts to MIQP when the asset-by-asset covariance matrix has a low rank. Although all B&B and B&C methods will find the true optimal portfolio in theory, the main disadvantage of a generic B&B or B&C method is that it may take a long time to get the optimal portfolio. Given current computer power, it is impractical to use the B&B or B&C approaches to solve meaningful trade paring problems, especially if the problem size is large (that is, more than 200 assets in the investment universe).

3.2.2 Genetic Algorithm, Simulated Annealing, Tabu Search and Other Heuristics

Chang, et al (2000) applied Genetic Algorithm (GA), Simulated Annealing (SA), Tabu Search (TS), and other heuristics to cardinality constrained portfolio optimization problems. They tested and compared those methods for five data sets involving problems with small or middle size (up to 225 assets). Liu and Stefek (1995) also developed and investigated a special genetic algorithm for asset paring problems. Coleman, et al (2006) considered the problem that minimizes tracking error with an asset paring constraint. Similar to the B&B and B&C methods, these methods in general are found to be computationally inefficient and impractical.

3.2.3 Lagrangian Relaxation Methods

Taking advantage of the special structures of the covariance matrices of multiple-factor risk models, Kopman, et al (2009) have developed a Lagrangian relaxation method for solving asset paring problems. Based on their preliminary computational results, the authors found that this method was efficient for solving asset paring problems, especially when used to quickly find a feasible heuristic portfolio or a useful bound on the optimal utility. This Lagrangian relaxation method can also be applied to trade paring. We have investigated the Lagrangian relaxation methods for trade paring and will provide the testing results in the next section.

3.2.4 The “Pare-Down” Heuristic

The “Pare-Down” heuristic is the traditional Barra approach to asset paring. In fact, Barra invented this heuristic method in the early 90s and so-called “paring problems” are named after this approach.

Extended to trade paring, the “pare-down” heuristic begins by solving a standard optimization problem with all but the paring constraints. If the resulting portfolio satisfies the trade paring constraints, then the heuristic terminates processing, since the optimal portfolio is found. Otherwise, if the resulting portfolio has more trades than allowed, then the extra trades will be pared down from the current trading list sequentially, a portion at a time. This is where the term “paring” applies. With each iteration, a small number of trades are eliminated and all asset weights are re-optimized. The process is repeated until a feasible portfolio with the targeted number of trades is reached, or other stopping criteria are met. Many rules are available to select the trades to be pared down. The most intuitive and easy-to-implement is based on the trade size. For example, one may pick the five smallest trades from the current trading list as those to be pared-down next. Another rule is to rank the trades according to their impact to utility.

Theoretically, the “pare-down” heuristic emphasizes optimality since it starts the process with a standard (i.e., paring-relaxed) optimal portfolio that enjoys the highest utility possible. In some cases, particularly in active cases, the total number of trades in the standard optimal portfolio may be already very close to the target number of trades. Thus, only a few iterations may be needed to pare-down the extra trades. Intuitively, in such cases, the resulting heuristic portfolio should not differ much from the standard optimal portfolio, and its quality should be high. However, if the standard optimal portfolio has too many extra trades, the pare-down procedure may have difficulty in finding a good feasible solution, especially in the presence of many other complicated constraints, such as the leverage or risk constraints.

3.2.5 The “Build-Up” Heuristic

The “build-up” heuristic for asset paring was also developed in-house at Barra in 2004 to complement the “pare-down” heuristic. In contrast to the “pare-down” stress on optimality, the “build-up” heuristic emphasizes feasibility. The idea is to find a small-sized feasible portfolio quickly, and then to build upon it (i.e., to add more assets to it) to improve utility. Applied to trade paring, we start the “build-up” process from the initial portfolio, which has zero trades. Then we try to sequentially create more trades. With each iteration, we rank the assets outside the current trading list according to certain rules, e.g., the marginal contribution to utility. Then we pick a certain number of these assets and try to force them into the trading list by solving a modified standard optimization problem. This process is repeated until the number of trades reaches the maximum target or until every asset has been considered at least once.

3.3 The Integrated Heuristic Approach in the Barra Optimizer

Similar to the way it addresses the asset paring constraints (Liu and Xu (2004)), the Barra Optimizer employs a combination of innovative heuristic procedures to tackle the trade paring problem. The goal is to find a “near optimal” solution within a reasonable amount of time. Tradeoffs between utility and speed have to be made.

More specifically, the integrated approach begins with a preprocessor that detects two kinds of infeasible cases. Some cases are simply infeasible, even without the paring constraints. Others become infeasible when paring constraints are added. We refer to the former as the usual “infeasible” cases, but the latter as “paring too restrictive” cases. Identifying either type of infeasible case at the very beginning alleviates the time and work of going through the heuristics process when there is no feasible result possible.

3.3.1 The Underlying Heuristics

The “build-up” and “pare-down” heuristics described in Section 3.2 are the building blocks of the integrated approach to trade paring in the Barra Optimizer. Depending on the characteristics of a specific problem, the integrated approach may utilize one or both heuristics, and may alter certain components or parameters of the heuristics.

3.3.2 The Swap or Local Search Procedure

At the end of the “build-up” or “pare-down” heuristic, the integrated approach may pick a certain number of the non-trading assets, according to their marginal contribution to utility, add them into the trading list, and re-optimize based on the new trading list. If the resulting portfolio leads to more trades than the maximum target, it will re-optimize by forcing the smallest of the additional trades to zero. The two-step procedure can be viewed as one “swap” or local search procedure. Its goal here is to look for a portfolio with improved utility through a number of swaps between assets traded and those not traded.

3.3.3 Handling Constraints on the Minimum Number of Trades

Portfolio managers sometimes impose constraints on the minimum number of assets or trades, addressing risk diversification and other legal or regulatory reasons. To handle a constraint on the minimum number of trades, the integrated approach will first try to detect whether this minimum constraint is binding. If it is not binding, then it will be ignored. Otherwise, only the "build-up" heuristic will be called upon. No "pare-down" heuristic is necessary.

4 Performance of Trade Paring Heuristics

Undoubtedly, heuristic approaches are not perfect. There is a trade-off between speed and optimality. How do we assess the "optimality" of heuristic solutions? The gap between the utility value of a heuristic solution and an appropriate bound on the true optimal utility value provides an intuitive and objective measure.

In this section, we will present computational results for trade paring problems. We will first examine eight representative cases to compare the performance of the "build-up," "pare-down," and "Integrated" heuristics, as well as the Lagrangian Relaxation and the Branch-and-Bound (B&B) methods. These results also show how optimal the heuristic solutions in the Barra Optimizer are. We then provide more simulation results to show the performance of the "build-up" and "pare-down" heuristics on more practical and complicated cases that may involve mixed paring or non-convex constraints.

4.1 Case Studies

In this subsection, we compare five heuristics on eight representative cases. The first four cases are passive index tracking cases. The remaining four are active portfolio management cases. In all these cases, asset covariance matrices come from the Barra US Equity Model, Long Horizon (USE3L) as of September 3, 2008. The MSCI US Mid Cap 450 Index (MSUSM450) for the same date is used as the benchmark. The investment universe is defined as the benchmark plus the cash asset, which happens to include 445 assets in total. Except for the holding constraint and the trade paring constraint on the maximum number of trades, no other constraints are present.

In the passive cases, the goal is to track the benchmark index with a small set of assets. Assume that a manager has an initial portfolio of 100 assets[1]. Now consider four scenarios. In the first three cases, the manager receives a 10%, 5%, and 3% cash infusion, respectively. In the fourth case, he needs to raise 5% cash for withdrawal. In all cases, the manager wants to rebalance his portfolio by making at most 10, 5, 3, and 10 trades, respectively.

In the active portfolio management cases, the objective is to beat the benchmark return on a risk-adjusted basis. The key difference between a passive investment strategy and an active one is the exploitation of expected returns, or asset alphas. Assume here that a manager starts with an initial portfolio of 97 assets that is optimal with respect to the current problem settings. Again, we consider four cases. The first two cases are for a 5% and 10% cash infusion, respectively. The third one is for a 3% cash withdrawal. The last one is for a 5% cash withdrawal, plus some alpha changes for three randomly selected assets.

[1] This "initial" portfolio is obtained by starting with an all-cash portfolio and solving an asset paring problem using the Barra Optimizer. The asset paring constraint should limit the maximum number of assets to 100.

Table 1: Passive Index Tracking Cases

Case Description	Tracking Error (Relative Gap = |B-H|/B*100%)						CPU Time (Seconds)				
	Best Bound [B]	Heuristic Solution [H]					Build-Up	Pare-Down	Integrated	Lagrangian	B&B (hours)
		Build-Up	Pare-Down	Integrated	Lagrangian	B & B					
Case 1: 10% Cash Inflow Max 10 Trades	1.7203	1.8066 (5.02%)	1.7658 (2.64%)	1.7658 (2.64%)	1.7552 (2.03%)	1.7485 (1.64%)	0.55	0.79	1.32	31.7	3
Case 2: 5% Cash Inflow Max 5 Trades	1.7835	1.8332 (2.79%)	1.8141 (1.72%)	1.8141 (1.72%)	1.8141 (1.72%)	1.8042 (1.16%)	0.61	0.62	1.16	34.51	3
Case 3: 3% Cash Inflow Max 3 Trades	1.8191	1.8864 (3.70%)	1.9033 (4.63%)	1.8864 (3.70%)	1.8863 (3.69%)	1.878 (1.56%)	0.62	0.56	1.15	28.96	3
Case 4: 5% Cash Outflow Max 10 Trades	1.8399	1.8812 (2.24%)	1.8701 (1.64%)	1.8701 (1.64%)	1.8601 (1.10%)	1.8497 (0.53%)	0.46	0.56	0.96	33.62	3

Table 2: Active Portfolio Management Cases

Case Description	Utility (%) (Relative Gap = |B-H|/B*100%)						CPU Time (Seconds)				
	Best Bound [B]	Heuristic Solution [H]					Build-Up	Pare-Down	Integrated	Lagrangian	B&B (hours)
		Build-Up	Pare-Down	Integrated	Lagrangian	B & B					
Case 5: 10% Cash Inflow Max 10 Trades	3.4971	3.4740 (0.66%)	3.4876 (0.27%)	3.4876 (0.27%)	3.4921 (0.14%)	3.4933 (0.11%)	0.55	0.79	0.93	31.7	3
Case 6: 5% Cash Inflow Max 5 Trades	3.5127	3.4967 (0.46%)	3.5047 (0.22%)	3.5047 (0.22%)	3.5084 (0.12%)	3.5084 (0.12%)	0.61	0.62	0.91	34.51	3
Case 7: 3% Cash Outflow Max 3 Trades	3.4968	3.4493 (1.36%)	3.4800 (0.48%)	3.4800 (0.48%)	3.4811 (optimal)	3.4811 (optimal)	0.52	0.37	0.92	16.4	3*
Case 8: 5% Cash Outflow Some Alpha Changes Max 10 Trades	3.4612	3.4309 (0.59%)	3.4481 (0.38%)	3.4481 (0.38%)	3.4514 (optimal)	3.4514 (optimal)	0.46	0.56	0.93	33.62	3*

* The Branch-and-Bound heuristic gets optimal portfolio for these two cases.

Table 1 and Table 2 present the results for these eight cases. The column titled "Best Bound" are the best bounds obtained using the Lagrangian relaxation procedure, with respect to the true optimal portfolio. For the passive cases in Table 1, these bounds are for tracking errors. That means the tracking errors of all feasible portfolios, including the optimal one, will be equal or larger than these bounds. For the active cases in Table 2, the best bounds are for utilities[2]. Thus, the utilities of all feasible portfolios, including the optimal one, will be equal or smaller than these bounds. The five columns under the common heading "Heuristic Solution (H)" provide heuristic tracking errors for passive cases, or the heuristic utilities for active cases. The value in brackets below each tracking error or utility is the "Relative Gap" that evaluates the performance of the heuristic. The columns under the common heading "CPU Time" demonstrate the speed for each heuristic. Except for the Branch-and-Bound method, all time measures provided are in seconds.

The results in Table 1 and Table 2 show that:

- The heuristic solutions for these representative cases are near optimal. The relative gaps for active cases are within 1.36%. The relative gaps for passive cases are within 5.02%. It is not surprising to see that the heuristics performed better in the active cases, since the optimal portfolio that ignores the paring constraints is typically closer to the true optimal portfolio with the paring constraints.
- Both the "build-up" and "pare-down" heuristics are much faster than the Lagrangian Relaxation or Branch-and-Bound Method. So is the "Integrated" heuristic, which essentially consists of both the "build-up" and "pare-down" heuristics in these cases.
- Both the Lagrangian relaxation and Branch-and-Bound procedures have produced optimal portfolios for the last two cases. However, it has taken the Lagrangian relaxation procedure much less time. We have also tried to let the Branch-and-Bound procedure run 10 hours for all the 8 cases, and found that the extra time did not improve the quality of the solutions.
- The Lagrangian relaxation procedure produces better solutions than the integrated approach in terms of the utility or tracking error. However, it takes more running time, and has other feature limitations, many of which are shared by asset paring.

The "integrated' heuristic solutions generated by the Barra Optimizer are very close to optimal for the active cases, and are of decent quality for the passive cases. Keep in mind that the gap is calculated relative to the bound of the optimal solution, and the true optimal solution must fall between this bound and the heuristic solution. Thus, the gaps between the heuristic and true optimal solutions are even smaller than the ones reported here.

4.2 Simulation Results

In this section, we report more testing results generated by our simulation tools. The cases here are randomly generated by varying the input parameters in the optimization problem, such as the number of factor constraints, the number of general linear constraints, and so forth. The parameters are well designed to produce cases of great practical relevance.

We group the cases by their basic profiles. For example, all cases in the first row of Table 3 are standard optimization problems with only a trade paring constraint. All cases in the third row, on the other hand, are long-short optimization cases with only a trade paring constraint. In general, "Mixed" in column 4 indicates that all the cases in the corresponding row have more than one type of paring constraints, and "L/S" means that the cases are long-short optimization problems. As the first column shows, the number of assets in the investment universe for the cases reported here ranges from 300 to 2,500.

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[2] In this paper, utility is defined as "return(%) – 0.075*risk(%)*risk(%)".

Table 3: Performance of Heuristics on More Complicated Cases

Case Description				# of Paring Failures		Average Utility		Average CPU Time (in seconds)	
Universe (# of Assets)	# of Cases	Paring Constraints		Build-Up	Pare-Down	Build-Up	Pare-Down	Build-Up	Pare-Down
		Max # of Trades	Other						
300	576	10		0	0	-4.768e-02	-4.724e-02	0.53	0.47
350	576	10	Mixed	0	1	-8.546e-02	-8.672e-02	0.58	0.83
300	576	30	L/S	0	0	-1.605e-02	-2.602e-02	1.06	2.25
750	576	20		0	0	-3.645e-02	-3.665e-02	0.84	0.84
750	576	20	Mixed	0	0	-8.778e-02	-8.750e-02	0.87	1.25
2500	576	30	Mixed	1	0	-3.398e-02	-2.652e-02	3.94	5.39
2500	288	30	L/S	0	0	-1.620e-02	-1.176e-02	8.85	19.30
550	2394	20	Mixed	0	0	-2.163e-01	-2.164e-01	0.42	0.40
550	1152	20	L/S	0	1	-5.123e-02	-5.127e-02	1.43	1.45

It is reasonable to expect that, as the cases get more complicated, the heuristics are more likely to fail. A failed case refers to one of the following two situations—(1) the case has no feasible solution at all, yet the heuristics cannot confirm infeasibility; (2) the case has at least one feasible solution, but the heuristics cannot find it within the allowed computing time frame. Both situations indicate that the paring constraints may be too restrictive. Relaxing one or all of these constraints is likely to yield favorable results. The two columns under the common heading "# of Paring Failures" show how many cases the "build-up" and "pare-down" heuristics have failed.

We see from Table 3 that, in general, the "build-up" and "pare-down" trade paring heuristics are quite comparable in terms of the solution quality and performance. Neither one is dominating the other. On many occasions, they are actually complementing each other.

Our other testing experiences also reveal that the overall success rate of the integrated approach is much higher than either one of the component heuristics. Its average utility is also better than either one of the components. In addition, the CPU time is usually less than the combination of the two components.

5 Summary

Trade paring constraints enable portfolio managers to control the number of trades when constructing and rebalancing their portfolios. For a standard optimization problem, users of the Barra Optimizer can set a maximum or minimum limit not only on the number of total trades, but also on the number of separate buy trades or sell trades. For a long-short optimization, users can set a maximum or minimum limit on the number of long trades or short trades separately, in addition to any constraints on the number of total trades, total buy trades, or total sell trades.

Portfolio optimization problems involving trade paring constraints are difficult to solve. As with other paring problems, resorting to heuristic procedures for a solution is often the norm rather than the exception. The Barra Optimizer employs two innovative, intuitive, and practical heuristics to complement each other. In this paper, we have demonstrated the performance of the heuristics, and provided some evidence of the quality of the heuristic solutions. We have shown that the integrated trade paring approach in the Barra Optimizer is viable and effective.

The trade paring capability, along with other paring capabilities in the Barra Optimizer, can be of great value to portfolio managers. However, a manager may wish to avoid setting a too-tight paring constraint in an optimization problem, unless it is absolutely necessary. Overly restricted paring constraints may not only limit the optimal risk-adjusted return, but also complicate the already-difficult solution process even further. In particular, a tight paring constraint combined with other non-convex constraints, such as the leverage or risk constraints, often make the problem infeasible, causing the optimizer to fail to find a feasible portfolio based on the user's constraints.

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