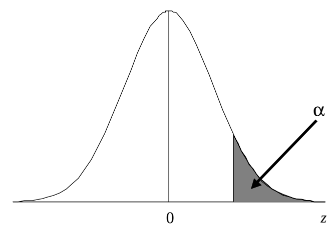
Hypothesis Testing

A ***one tailed*** Test



Acceptance Region

Critical Region

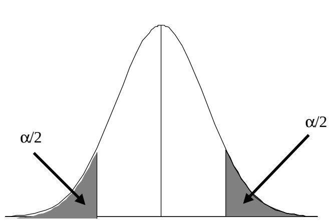
To define a statistical Test we

1. Choose a statistic (called the ***test statistic***)
2. Divide the range of possible values for the test statistic into two parts
   - The Acceptance Region
   - The Critical Region

To perform a statistical Test we

1. Collect the data.
2. Compute the value of the test statistic.
3. Make the Decision:
   - If the value of the test statistic is in the Acceptance Region we decide to ***accept*** H_0 .
   - If the value of the test statistic is in the Critical Region we decide to ***reject*** H_0 .

A ***two tailed*** Test



Critical Region

Acceptance Region

Critical Region

The z-test for Proportions

Testing the probability of success in a binomial experiment

The Test Statistic

$$z = \frac{\hat{p} - p_0}{s_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

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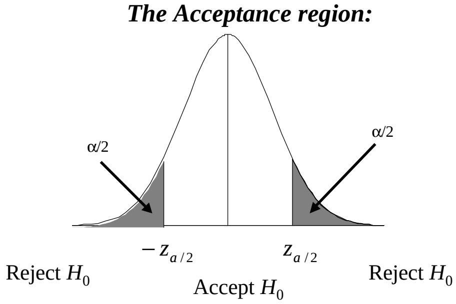
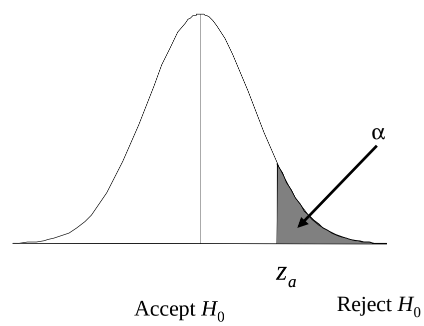
$$z = \frac{\hat{p} - p_0}{s_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

The Acceptance region:



The *Acceptance* and *Critical region:*



The one tailed z-test

$$H_0 : p \le p_0$$

$$H_A : p > p_0$$

The one tailed z-test

$$H_0 : p \ge p_0$$

$$H_A : p < p_0$$

The Test Statistic

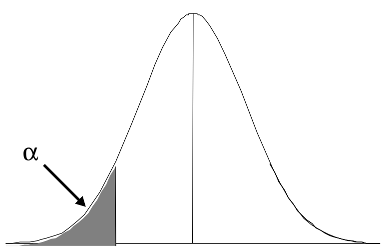
$$z = \frac{\hat{p} - p_0}{s_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

- If you are trying to prove a difference

$$H_A : p \neq p_0$$

- This is the alternative Hypothesis (H_A) - ***the Research Hypothesis***
- Use a ***two tailed test***

The Acceptance region:



- If you are trying to prove the true value p exceeds the hypothesized value p_0

$$H_A : p > p_0$$

- This is the alternative Hypothesis (H_A) - ***the Research Hypothesis***

Comments

- The alternative Hypothesis (H_A) is what the experiment is trying to prove - ***the Research Hypothesis***
- The alternative Hypothesis (H_A) will determine if you use a one-tailed test or a two tailed test

- If you are trying to prove the true value p does not exceed the hypothesized value p_0

$$H_A : p < p_0$$

- This is the alternative Hypothesis (H_A) - ***the Research Hypothesis***

- If you are trying to prove a difference

$$H_A : p \neq p_0$$

- This is the alternative Hypothesis (H_A) - ***the Research Hypothesis***

Example

- A new surgical procedure is developed for correcting heart defects infants before the age of one month.
- Previously the procedure was used on infants that were older than one month and the success rate was 91%
- A study is conducted to determine if the success rate of the new procedure is greater than 91% (n = 200)

Comments

- Different objectives will result in different choices of the alternative hypothesis
- If you were interested in Proving that the new procedure is an ***improvement***:
- Then

$$H_A : p > p_0$$

- If you were interested in proving that the new procedure is not an improvement:
- Then

$$H_A : p < p_0$$

- If you were interested in proving only a difference between the new and the old:
- Then

$$H_A : p \neq p_0$$

We want to test

– H_0: $p \leq 0.91$ (91%)

Against

– H_A: $p > 0.91$ (91%)

p = the success rate of the new procedure

Performing the Test

1. Decide on a = P[Type I Error] = the significance level of the test

   Choose (a = 0.05)

2. Collect the data

- The number of successful operations in the sample of 200 cases is $x = 187$

$$\hat{p} = \frac{x}{n} = \frac{187}{200} = 0.935\,(93.5\%)$$

Comments

- When the decision is made to accept H_0 is made it should not be conclude that we have proven H_0.
- This is because when setting up the test we have not controlled b = P[type II error] = P[accepting H_0 when H_0 is FALSE]
- Whenever H_0 is accepted there is a possibility that a type II error has been made.

3. Compute the test statistic

$$z = \frac{\hat{p} - p_0}{s_{\hat{p}}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

$$= \frac{0.935 - 0.91}{\sqrt{\frac{0.91(1-0.91)}{200}}} = 1.235$$

4. Make the Decision $z_a = z_{0.05} = 1.645$

- Accept H_0 if: $z \leq 1.645$
- Reject H_0 if: $z > 1.645$

In the last example

The conclusion that there is a ***no*** significant (a = 5%) increase in the success rate of the new procedure over the older procedure should be interpreted:

We have been unable to proof that the new procedure is better than the old procedure

Since the test statistic is in the Acceptance region we decide to Accept H_0

Conclude that H_0: $p \leq 0.91\,(91\%)$ is true

There is a ***no*** significant (a = 5%) increase in the success rate of the new procedure over the older procedure

An analogy – a jury trial

The two possible decisions are

– Declare the accused innocent.
– Declare the accused guilty.

The null hypothesis (H_0) – the accused is innocent

The alternative hypothesis (H_A) – the accused is guilty

Hence: When decision of innocence is made:

- It is **not concluded** that innocence has been proven

but that

- we have been **unable to disprove** innocence

The two possible errors that can be made:

- Declaring an innocent person guilty. (type I error)
- Declaring a guilty person innocent. (type II error)

Note: in this case one type of error may be considered more serious

The z-test for the Mean of a Normal Population

We want to test, *m*, denote the mean of a normal population

Requiring all 12 jurors to support a guilty verdict :

- Ensures that the probability of a type I error (Declaring an innocent person guilty) is small.
- However the probability of a type II error (Declaring an guilty person innocent) could be large.

Situation

- A success-failure experiment has been repeated n times
- The probability of success p is unknown. We want to test
  - H_0: $p = p_0$ (some specified value of p)

  Against
  - H_A: $p \neq p_0$

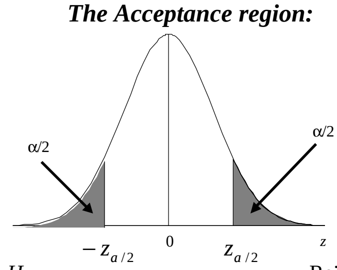
The Data

- Let x_1, x_2, x_3 , … , x_n denote a sample from a normal population with mean m and standard deviation s.
- Let

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} = \text{the sample mean}$$

- we want to test if the mean, m, is equal to some given value m_0.
- Obviously if the sample mean is close to m_0 the Null Hypothesis should be accepted otherwise the null Hypothesis should be rejected.

The Acceptance region:



Reject H_0 Accept H_0 Reject H_0

$$P[\text{Accept } H_0 \text{ when true}] = P[-z_{a/2} \le z \le z_{a/2}] = 1 - a$$

$$P[\text{Reject } H_0 \text{ when true}] = P[z < -z_{a/2} \text{ or } z > z_{a/2}] = a$$

The Test Statistic

- To decide to accept or reject the Null Hypothesis (H_0) we will use the test statistic

$$z = \frac{\bar{x} - m_0}{s_{\bar{x}}} = \frac{\bar{x} - m_0}{\frac{s}{\sqrt{n}}} = \sqrt{n}\,\frac{\bar{x} - m_0}{s} \approx \sqrt{n}\,\frac{\bar{x} - m_0}{s}$$

- If H_0 is true we should expect the test statistic z to be close to zero.
- If H_0 is true we should expect the test statistic z to have a standard normal distribution.
- If H_A is true we should expect the test statistic z to be different from zero.

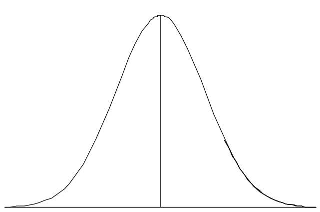
- Acceptance Region
  - Accept H_0 if: $-z_{a/2} \le z \le z_{a/2}$
- Critical Region
  - Reject H_0 if: $z < -z_{a/2}$ or $z > z_{a/2}$
- With this Choice

$$P[\text{Type I Error}] = P[\text{Reject } H_0 \text{ when true}]$$

$$= P[z < -z_{a/2} \text{ or } z > z_{a/2}] = a$$

The sampling distribution of z when H_0 is true:

The Standard Normal distribution



Reject H_0 Accept H_0 Reject H_0

Summary

To test mean of a Normal population

H_0: $m = m_0$ (some specified value of m)

Against

H_A: $m \ne m_0$

1. Decide on a = P[Type I Error] = the significance level of the test (usual choices 0.05 or 0.01)

2. Collect the data

3. Compute the test statistic

$$z = \sqrt{n}\,\frac{\bar{x} - m_0}{s} \approx \sqrt{n}\,\frac{\bar{x} - m_0}{s}$$

4. Make the Decision
   - Accept H_0 if: $-z_{a/2} \le z \le z_{a/2}$
   - Reject H_0 if: $z < -z_{a/2}$ or $z > z_{a/2}$

The one tailed test

To test

H_0: $m \le m_0$ (some specified value of m)

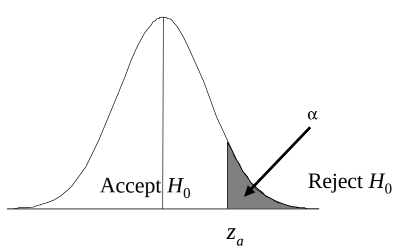
against

H_A: $m > m_0$

1. Use the test statistic

$$z = \sqrt{n}\,\frac{\bar{x} - m_0}{s} \approx \sqrt{n}\,\frac{\bar{x} - m_0}{s}$$

2. Use as the Acceptance and Critical Region
   - Accept H_0 if: $z \le z_{a/2}$
   - Reject H_0 if: $z > z_{a/2}$



The one tailed test – other direction

To test

H_0: $m \ge m_0$ (some specified value of m)

against

H_A: $m < m_0$

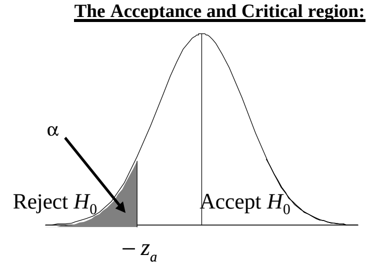
Acceptance and Critical Region

- Accept H_0 if: $z \ge -z_{a/2}$
- Reject H_0 if: $z < -z_{a/2}$

Test Statistic $z = \sqrt{n}\,\frac{\bar{x} - m_0}{s} \approx \sqrt{n}\,\frac{\bar{x} - m_0}{s}$

Expect z to be negative if H_0 is false

The Acceptance and Critical region:



Example:

We are interested in measuring the concentration of lead in water and we want to know if it exceeds the threshold level m_0 = 10.0

We take n = 40 one-litre samples measuring the concentration of lead.
Statistical results:

$$\bar{x} = 12.1 \quad \text{and} \quad s = 1.2$$

Test Statistic

$$z = \sqrt{n}\,\frac{\bar{x} - m_0}{s} \approx \sqrt{n}\,\frac{\bar{x} - m_0}{s}$$

$$= \sqrt{40}\,\frac{12.1 - 10.0}{1.2} = 11.07$$

Since z is greater than $z_{0.05} = 1.645$ we conclude that the average lead level is significantly higher than 10.0