Control Theory (RMS-1-612-s ; RIMA-1-612-s)

Laboratory no. 2 – State-Space Control (updated: 28th May 2018)

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Recommended reading: Chapter 8 of [1] and Chapters 3 & 11 of [2].

[1] Roland S. Burns, *Advanced Control Engineering*, Butterworth-Heinemann, 2001, ISBN 978-0-7506-5100-4.

[2] Richard C. Dorf and Robert H. Bishop, *Modern Control Systems*, 8th edition, Addison-Welley, Boston, MA, 1997, ISBN: 978-0-2013-0864-8.

[3] Phong B. Dao, *Control Theory (RMS-1-612-s)*, Lecture 07. State-space methods for control system design.

Remarks: (1) Short instructions are provided to support you in solving exercises. You should read the instructions carefully. Though some parts of the exercises are explained and guided in the instructions but they are general solutions and not yet complete. Therefore, based on the instructions provided you should present and discuss in the report your solutions and results obtained in detail; (2) Before using any Matlab commands you should first check the syntax, descriptions, input arguments, output arguments, and illustrated examples by typing help "command name"; (3) The report must be typed on a word-processor and submitted as a single PDF file via **UPeL platform**.

Exercise 1

Consider the state feedback controller and the state observer designed in *regulator.m* for the plant given by the transfer function $G(s) = \dfrac{1}{s(s+1)(0.2s+1)}$.

regulator.m

```
% First enter the transfer function G(s)
numG = 1;
denG = conv ( conv ( [1  0], [1  1] ), [0.2  1] );
% Convert to state-space model
[ Ag, Bg, Cg, Dg ] = tf2ss ( numG, denG );
% Check the controllability and observability of the system
system_order = length(Ag)    % equals "3"
M = ctrb(Ag, Bg);
rank_of_M = rank(M)    % equals "3"
N = obsv(Ag, Cg);
rank_of_N = rank(N)    % equals "3"
% Compute the poles of a second-order system
damping = 0.707;
wn = 3;
[ num2, den2 ] = ord2 (wn, damping);
% Select desired poles to include poles of the second-order system
dominant = roots(den2);
desiredpoles = [dominant'    10 * real( dominant(1) ) ];
% Compute the controller gain K
K = acker (Ag, Bg, desiredpoles);
```

```
% Compute the closed-loop state variable feedback system
Asf = Ag – Bg * K;  Bsf = Bg;  Csf = Cg;  Dsf = 0;
[numsf, densf] = ss2tf (Asf, Bsf, Csf, Dsf);
% Select observer poles to be 10 times faster than controller poles
observerpoles = 10 * desiredpoles;
% Compute the observer gain L
L = acker (Ag', Cg', observerpoles);
% Compute the closed-loop system with both controller and observer
Areg = [ (Ag – Bg * K)  Bg * K; zeros( size(Ag) )  (Ag - L' * Cg) ];
Breg = [ Bg; zeros( size(Bg) ) ];
Creg = [ Cg  zeros( size(Cg) ) ];
Dreg = 0;
[numreg, denreg] = ss2tf ( Areg, Breg, Creg, Dreg );
damp(denreg);
```

Questions:

(a) Suppose that we want the closed-loop system to have two dominant poles corresponding to a second-order system with a damping ratio of 0.707 and a natural frequency of 3. Now to ensure that these two poles are dominant we require the third closed-loop pole to be 20 times the real part of the dominant closed-loop poles. Compute the state feedback gain matrix (or the controller gain) **K**. How does this compare with the gain **K** calculated in *regulator.m*? How do the dynamics of the new closed-loop system compare with that of the closed-loop system in *regulator.m*? Explain in detail.

(b) Repeat part **(a)** assuming that we allow the third closed-loop pole to be only 4 times the real part of the dominant closed-loop poles. Plot step responses of the closed-loop system for three different cases and discuss the relevant dynamic characteristics obtained.

(c) Let the third closed-loop pole be 10 times the real part of the dominant closed-loop poles, as given in *regulator.m.* However, now suppose that we require the observer poles to be 20 times as great as the desired closed-loop poles. Compute the new observer gain matrix **L**. How does this compare to the **L** previously calculated in *regulator.m*? How do the dynamics of the new closed-loop system compare to the dynamics of the closed-loop system in *regulator.m*? Explain in detail.

(d) Repeat part **(c)** assuming that we allow the observer poles to be only 4 times as great as the desired closed-loop poles. Plot step responses of the closed-loop system for three different cases and discuss the relevant dynamic characteristics obtained.

(e) Return back to the parameter values given in *regulator.m* calculate the maximum natural frequency ω_n one can achieve, assuming that all elements of the state feedback gain matrix **K** will not be greater than 10?

Instructions:

1. Though most control systems of interest are of higher order, they often have a dominant mode that is represented by a complex pole pair of lower frequency than the other poles (or in other words, the system has a slow complex pole pair and some faster poles). The dominant (low frequency or slow) complex pole pair mainly decides the behaviour of the system and therefore it is commonly analysed to provide approximate performance and design insight. For that reason, the higher-order systems are often approximated by a second-order system that contains only the dominant mode of the actual system. It is important to note that the closer to the imaginary axis the complex pole pair is, the slower it is.

2. The Ackermann's formula (Matlab command: **acker**) or the pole placement design (Matlab command: **place**) can be used to determine the state feedback gain matrix K to place the system poles at the desired locations. However, the closed-loop system pole locations can be arbitrarily placed if and only if the system is controllable. *Therefore, the controllability of the system must be checked before designing the state variable feedback controller*. When the full-state is not available for feedback, we may employ an observer. The observer design process can also be performed by using the Ackermann's formula. The state variable compensator is obtained by connecting the full-state feedback law to the observer. *The observability of the system should be always checked before designing the state variable observer*.

The controllability and observability of a system can be determined (or verified) by using the Matlab command **ctrb** and **obsv**, respectively. Type "help ctrb" or "help obsv" in Matlab to understand how to use these commands.

3. It can be observed in the step responses that the dynamic characteristics of the new closed-loop system are almost the same in comparison with the ones of the closed-loop system presented in *regulator.m*. The reason is that the dominant complex pole pair is the one that primarily decides the dynamic performances of the system. The third closed-loop pole that equals 10 times or 20 times the real part of the dominant closed-loop poles does not cause much influence with regard to the closed-loop system dynamics. In summary, if other poles move far away from the dominant pole, they have less effect and thus sometimes can be neglected. For example: in this case, the dominant closed-loop poles are the complex conjugate pole pair located at $-2.121 \pm j2.1216$ and the third closed-loop pole is the real pole -42.42.

- The dominant closed-loop poles create an oscillatory response in terms of $Ae^{-2.121t}\sin(2.1216t+\phi)$ that exhibits some overshoot. The oscillation will decay approximately in $t = 4\tau = 4 * \frac{1}{2.121} = 1.89$ seconds because of the $e^{-2.121t}$ damping term, where τ is the time constant.
- The third closed-loop pole corresponds to a decaying exponential term $Ce^{-42.42t}$ with the time constant $\tau = \frac{1}{42.42} = 0.023$ seconds, decays rapidly and is significant only for $t = 4\tau = 0.09$ seconds.

In case when the third closed-loop pole is only 4 times the real part of the dominant closed-loop poles then this third closed-loop pole causes considerable influence on the dynamics of the closed-loop system because it is quite close to the dominant closed-loop poles.

Moving the non-dominant poles away from the imaginary axis causes increasing the value of all elements in the state feedback gain matrix **K** because these poles need high gain to be able to move further to the left from their initial positions.

4. The closed-loop dynamics of the observed state variable feedback control system is described by the equation

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - L^T C \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad (1)$$

So that the characteristic equation of the closed-loop system is

$$\det(sI - A + BK) \times \det(sI - A + L^T C) \qquad \text{or} \qquad |sI - A + BK|\,|sI - A + L^T C| = 0 \qquad (2)$$

where the matrix **K** is in the form $K = [k_1, k_2, ..., k_n]$ and **L** is in the form $L = [l_1, l_2, ..., l_n]^T$.

The above equation shows that the desired closed-loop poles are not changed by the introduction of the state variable observer. Therefore, the dynamics of the closed-loop system are the same in all cases (i.e. when the observer poles are 4, 10, or 20 times as great as the desired closed-loop poles). This behaviour can be observed in the step responses of these systems with different desired observer poles (*these step responses should be shown in the report*). This corresponds to the so-called **separation principle**, i.e. the observer and the controller can be designed separately; and the closed-loop system with the dynamic measurement feedback is stable, given that the control and observer systems are stable.

Because the observer is normally designed to have a more rapid response than the control system with full order observed state feedback therefore the observer poles are always designed to be several times as great as the desired closed-loop poles. Notice that higher speed of observer's poles results in higher gains of the observer matrix **L**.

5. To determine the maximum natural frequency ω_n one can achieve, assuming that all elements of the controller gain **K** are not greater than 10, one may consider implementing a procedure that performs the following algorithm:

```
for wn = 3:-0.01:0
    - determine the new desired poles based on the new value of wn;
    - compute the new controller gain K by using the Ackermann's formula;
    - stop if all elements of the controller gain K are smaller than 10;
end
```

Finally, one should obtain $\omega_n = 1.1225$ corresponding to K = [3.5233 8.8563 9.9995]. This task can be understood as finding the possible fastest system with all elements of the state feedback gain matrix **K** do not exceed a certain value.

Exercise 2

(a) Check the controllability and observability of the following systems. Are these systems stable?

$$(i) \quad \dot{x} = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 16 \\ 0 & -25 & -20 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} -1 & 3 & 0 \end{bmatrix} x$$

$$(ii) \quad \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -4 & -3 \end{bmatrix} x + \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

(b) For each system of part (a) – assuming that the state variables are available for feedback – determine whether or not a state variable feedback controller can be designed to stabilize the system.

Instructions:

1. The first system is uncontrollable, but observable. The second one is uncontrollable and unobservable.

2. This exercise involves the controller design method for ***uncontrollable systems***: The canonical decomposition theorem is first used to transform state equation of an uncontrollable system into the controllable-uncontrollable canonical form (or the controllability staircase form) and then the obtained factorized/transformed system is investigated:

- If the *unstable pole(s)* of the transformed system are *controllable* then a state variable feedback controller can be designed to stabilize the system.
- If the *unstable pole(s)* of the transformed system are *uncontrollable* then we cannot design a state variable feedback controller to stabilize the system.

One can obtain the controllability staircase form by using the Matlab command **ctrbf**. Type "help ctrbf" in Matlab to understand the command and how the controllability staircase is formed.

In principle, the function **ctrbf** decomposes the original state-space system represented by (A, B, C) into the controllability staircase form ($\overline{A}, \overline{B}, \overline{C}$) which satisfies

$$\overline{A} = TAT^T, \quad \overline{B} = TB, \quad \overline{C} = CT^T \tag{3}$$

$$\text{and } \overline{A} = \begin{bmatrix} A_{uc} & 0 \\ A_{21} & A_c \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} C_{uc} & C_c \end{bmatrix} \tag{4}$$

where (A_c, B_c) is a controllable subspace, and A_{uc} is uncontrollable states. The controllable subequation has the same transfer function as the original system, or in other words, the original system and the transformed system have the same poles, i.e.

$$C_c(sI - A_c)^{-1}B_c = C(sI - A)^{-1}B \tag{5}$$

In order to determine the controllable subspace (A_c, B_c) and the uncontrollable states (A_{uc}) we should understand how the command **ctrbf** transforms a state-space system to the controllability staircase form.

[A_bar, B_bar, C_bar, T, k] = ctrbf(A, B, C) $\Rightarrow$ transforms a state-space system represented by A, B, and C into the controllability staircase form represented by A_bar, B_bar, and C_bar.

Important output results:

T is the similarity transformation matrix and k is a vector of length n, where n is the order of the system represented by A.

sum(k) is the number of states in A_c, i.e. the controllable portion of A_bar.

For examples:

If sum(k) = 1 then A_c is an 1×1 matrix (i.e. a scalar or single number).

If sum(k) = 2 then A_c is an 2×2 matrix.

3. Now, we consider the first system:

A1 = [1 4 3; 0 2 16; 0 -25 -20];

B1 = [-1; 0; 0];

C1 = [-1 3 0];

D1 = [0];

poles1 = eig(A1) $\Rightarrow$ the first system has one unstable pole ($p = 1$).

[Af1, Bf1, Cf1, T1, k1] = ctrbf(A1, B1, C1) $\Rightarrow$ obtain the controllability staircase form.

poles1_new = eig(Af1) $\Rightarrow$ the poles of the transformed system are the same as that of the original one.

controllable_poles = eig(Af1(3,3))

uncontrollable_poles = eig(Af1(1:2, 1:2))

The transformed system has one controllable pole ($p = 1$) located at Af1(3,3), i.e. this unstable pole is controllable. Therefore, one can design a state variable feedback controller to stabilize this system.

4. Next, consider the second system. By using the same method presented you should achieve the results:

Both the original and the transformed systems have one unstable pole ($p = 1$), one stable pole ($p = -3$), and one at origin ($p = 0$). Investigating the controllability staircase form you can notice that the transformed system has two controllable poles ($p = 1$ and $p = -3$). It means that you can move the unstable pole $p = 1$, but the pole $p = 0$ is uncontrollable. Therefore, it is not possible to design a state variable feedback controller to stabilize this system, i.e. the system will be at the edge of stability.

Exercise 3

For the given state-space model:

A = [-5 -2; -3 0];

B = [1; -3];

C = [1 -1];

D = 0;

Design a state variable feedback controller, which will stabilize this unstable system, with assumption that the state variables are available for feedback.

Instructions:

1. First, the controllability and observability of the given system should be checked. As a result, you know that this system is uncontrollable, but observable. In addition, the system has two poles ($p_1 = -6$ and $p_2 = 1$) so that the system is unstable. A step response might be plotted to illustrate this.

2. Next, you can use the same approach presented in Exercise 2 to check whether or not a state variable feedback controller can be designed to stabilize this system.

[Af, Bf, Cf] = ctrbf(A, B, C) $\Rightarrow$ the original system is decomposed into the controllability staircase form

controllable_poles = eig(Af(2,2))

Investigating the controllability staircase form you can notice that the sytem is not fully controllable, but fortunately the unstable pole ($p_2 = 1$) is controllable. Hence, you can use the Ackerman's formula to design the state variable feedback controller that moves the unstable pole ($p_2 = 1$) to a desired stable pole, such as $p_2 = -10$. It should be noted that you do not need to move the stable pole ($p_1 = -6$).

3. Finally, you should check if you obtain these two desired stable poles, i.e. $p_1 = -6$ and $p_2 = -10$, and then plot a step response of the new system.

Exercise 4

In the example *regulator.m* the steady-state error could be cancelled by multiplication of the input signal by a constant gain. However, it is not a reliable method because it requires perfect knowledge of the system. A better solution is to apply an **integral control scheme**. Assuming that all state variables are available for feedback, *design a state variable feedback controller with the integral element*, which fulfills the design requirements: damping ratio $\xi = 0.707$, natural frequency $\omega_n = 3$, and provides zero steady-state error. Compare and discuss the simulation results obtained for the state variable feedback controller without and with the integral control element.

Instructions:

Because a state variable feedback controller might not reject the effects of disturbances, particularly in case of input disturbances, therefore the design of a state variable feedback controller with an integral control element can be used to compensate for the disturbances and to provide zero steady-state error.

First we will design and simulate a state variable feedback controller without integral control element.

Start from the example *regulator.m*.

Step 1. Consider a state-space system of the form

$$(I) \quad \begin{cases} \dot{x} = A_g x + B_g u \\ y = C_g x + D_g u \end{cases}$$

The controllability and observability of the system should be first checked.

numG = 1;

denG = conv(conv([1 0], [1 1]), [0.2 1]);

[Ag, Bg, Cg, Dg] = tf2ss(numG, denG);   % convert to state-space model

system_order = length(Ag) $\Rightarrow$ equals "3"

M = ctrb(Ag, Bg);

rank_of_M = rank(M) $\Rightarrow$ equals "3"

N = obsv(Ag, Cg);

rank_of_N = rank(N) $\Rightarrow$ equals "3"

Therefore, we know that the system (I) is controllable and observable.

Step 2. By designing a state variable feedback controller for the system (I) we obtain the system (II)

$$(II)\quad \begin{cases} \dot{x} = A_{sf}x + B_{sf}u \\ y = C_{sf}x + D_{sf}u \end{cases}$$

where $A_{sf} = A_g - B_g K$ with K is the state feedback gain matrix.

a) Compute the poles of a second-order system

```
damping = 0.707;
wn = 3;
[num2, den2] = ord2(wn, damping);
dominant = roots(den2);    % dominant complex pole pair
```

b) Select the desired poles that include two dominant poles of the second-order system and another one

```
desiredpoles = [dominant'    10 * real( dominant(1))];
```

c) Compute the state feedback gain (or the controller gain) K by using the Ackermann's formula

```
K = acker(Ag, Bg, desiredpoles)
```

$\Rightarrow$ As a result, you should obtain **K = [19.5 94 191]**.

d) Compute the closed-loop state variable feedback system

```
Asf = Ag - Bg * K;  Bsf = Bg;  Csf = Cg;  Dsf = 0;
[numsf, densf] = ss2tf(Asf, Bsf, Csf, Dsf);
```

Step 3. Create a step input signal and a step-like disturbance signal that will be used for simulation

```
t = 0:0.01:15;   % total time
t1 = 0:0.01:4.99;   % the interval (0÷4.99) [sec] in which the disturbance equals to 0
t2 = 5:0.01:15;   % the interval (5.0÷15.0) [sec] in which the disturbance equals to 1
input1 = 1.0 * ones(size(t));   % create a step signal
input2 = [0.4 * zeros(size(t1))  0.4 * ones(size(t2))];   % a step-like disturbance signal with amplitude 0.4
```

Step 4. Simulate the system (II) using the test signals created above

Notice that:

- The first way to inject a disturbance signal into a system is to add one more input. In this case, the matrix B_{sf} is modified to have two inputs, i.e.

$$B_{sf} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow B_{sf} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- The second way to add a disturbance signal into a system is to modify the matrix B_{sf}, for example:

$$B_{sf} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow B_{sf} = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}$$

```
figure(1);
subplot(1,3,1);
```

```
step(Asf, Bsf, Csf, Dsf, 1, t);
title('Step response without disturbance');
subplot(1,3,2);
lsim(Asf, [1 0; 0 1; 0 0], Csf, Dsf, [input1'  input2'], t);   % B_sf = [1 0; 0 1; 0 0]
title('Step response with disturbance');
legend('Input 1: step response', 'Input 2: disturbance signal')
subplot(1,3,3);
step(Asf, [0; 100; 0], Csf, Dsf, 1, t);   % B_sf = [0; 100; 0]
title('Disturbance response');
```

Consequently, we obtain the results shown in Figure 1 for the state variable feedback controller design without having the integral control term.

⇒ Based on these results, you should discuss and give some conclusions about the capability to reject input disturbance signal and to provide zero steady-state error of the designed system.

Next we will design and simulate the state variable feedback controller with integral control element.

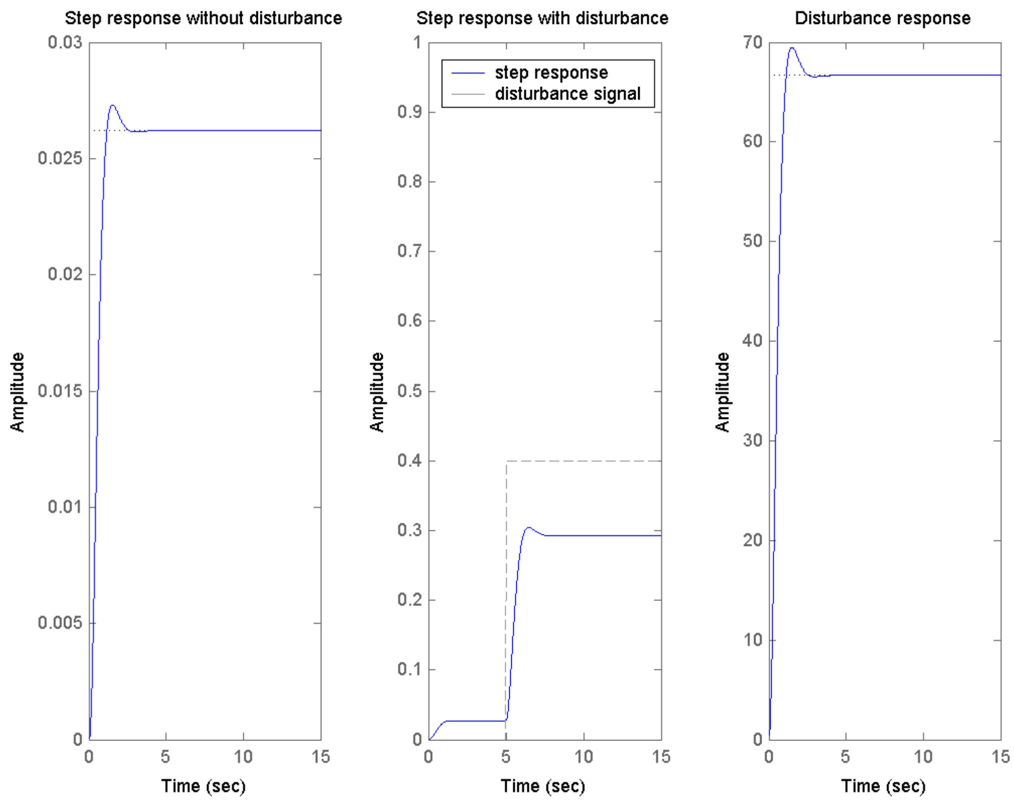


Figure 1. Simulation results for the state variable feedback controller without the integral control part.

Step 5. The integral control is taken into account in the state variable feedback controller design

By extending the system (I) with an integral control action we obtain a new state-space system (III) in terms of an extended state-space model shown in Figure 2. Notice that the matrix notations (A_g, B_g, C_g) are presented as (A, B, C) in Figure 2.

Figure 2 shows that we wish to design a feedback law of the form:

$u(t) = Fx(t) + F_I x_I(t) = -K_e x_e$ (i.e. the negative state feedback system)

where $x_I(t) = \int_0^t [y(\tau) - r(\tau)] d\tau$ or $\dot{x}_I(t) = y(t) - r(t)$, $K_e = [-F \quad -F_I]$, $x_e = \begin{bmatrix} x \\ x_I \end{bmatrix}$

and F is the state feedback gain matrix.

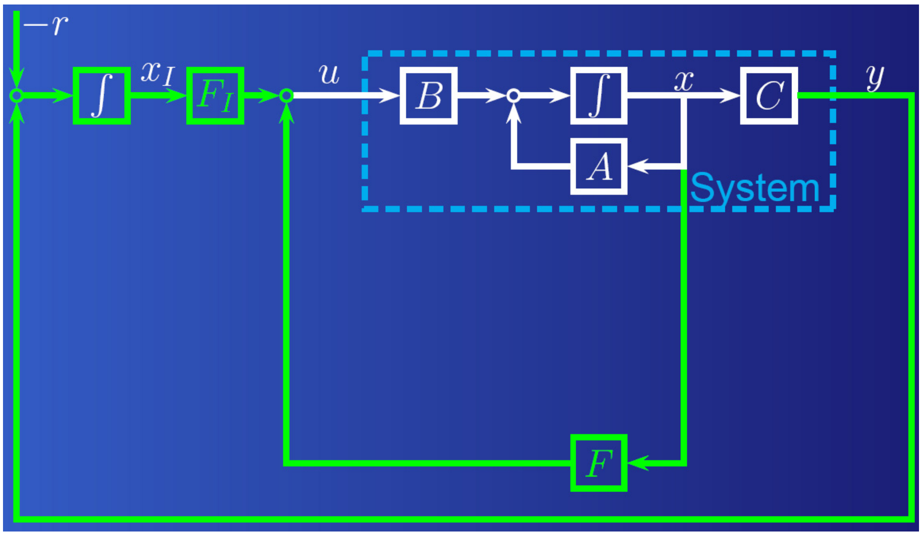


Figure 2. Structure of a state variable feedback control system augmented with integral control.

The state equations:

$$(III.1) \quad \begin{cases} \dot{x} = A_g x + B_g u \\ \dot{x}_I = y - r \\ y = C_g x \end{cases}$$

can be combined into an extended state-space model:

$$(III.2) \quad \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A_g & 0 \\ C_g & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} B_g \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r \\ y = [C_g \quad 0] \begin{bmatrix} x \\ x_I \end{bmatrix} \end{cases}$$

Hence, the integral control problem has been reduced to a conventional state feedback problem:

$$(III.3) \quad \begin{cases} \dot{x}_e = A_e x_e + B_e u + B_r r \\ y = C_e x_e \end{cases}$$

where $x_e = \begin{bmatrix} x \\ x_I \end{bmatrix}$, $A_e = \begin{bmatrix} A_g & 0 \\ C_g & 0 \end{bmatrix}$, $B_e = \begin{bmatrix} B_g \\ 0 \end{bmatrix}$, $C_e = [C_g \quad 0]$, and $B_r = \begin{bmatrix} 0 \\ -I \end{bmatrix} = [0 \quad 0 \quad 0 \quad -1]^T$.

Substitute $u = -K_e x_e$ into ($III.3$) then we obtain the closed-loop state feedback system augmented with the integral control term given in the form as

$$(III.4) \quad \begin{cases} \dot{x}_e = (A_e - B_e K_e) x_e + B_r r \\ y = C_e x_e \end{cases}$$

Notice that: the control signal $u = -K_e x_e$ is the input to the state-space system (I), but r is the input to the closed-loop system. Hence, the closed-loop equations will depend on both B_e and B_r.

Finally, we have

$$(III.5) \quad \begin{cases} \dot{x}_e = A_{sf\_ext} x_e + B_r r \\ y = C_e x_e \end{cases}$$

where $A_{sf\_ext} = A_e - B_e K_e$.

a) The extended state-space model (III) is implemented in Matlab as follows:

```
Ae = [Ag  zeros(size(Ag(:,1))); Cg  zeros(size(Cg(:,1)))];    % Ae = [Ag  0; Cg  0]
Be = [Bg; zeros(size(Bg(1,:)))];    % Be = [Bg; 0]
Ce = [Cg  zeros(size(Cg(:,1)))];    % Ce = [Cg  0]
De = [0];
```

b) The controllability and observability of the extended state-space system are checked:

system_order = length(Ae) $\Rightarrow$ equals "4"

M = ctrb(Ae, Be);

rank_of_M = rank(M) $\Rightarrow$ equals "4"

N = obsv(Ae, Ce);

rank_of_N = rank(N) $\Rightarrow$ equals "3"

As a result, the extended system (III) is controllable but not observable.

c) Adding the integral control part results in a 4th-order system so that you need to select the new desired poles that include two dominant poles of the second-order system and other two poles, for examples:

```
desiredpoles_ext = [dominant'   10*real(dominant(1))   20*real(dominant(1))];
```

d) Then, the state feedback gain matrix K_e of the extended system (III) can be computed by using the Ackermann's formula

```
K_e = acker(Ae, Be, desiredpoles_ext)
```

$\Rightarrow$ As a result, you should obtain **K_e = [62 1173 4389 1619]**.

e) Finally, the closed-loop equations of the extended system (III) is computed

```
Asf_ext = Ae - Be * K_e;
B_r = [0; 0; 0; -1];
[numsf_ext, densf_ext] = ss2tf(Asf_ext, B_r, Ce, De);
SYS_ext = tf(numsf_ext, densf_ext);
```

Step 6. Simulate the system (III) using the same test signals created above for the comparison

Notice that:

- The first way to inject a disturbance signal into a system is to add one more input. In this case, the matrix B_r is modified to have two inputs, i.e.

$$B_r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow B_r = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}$$

- The second way to add a disturbance signal into a system is to modify the matrix B_r, for example:

$$B_r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \Rightarrow B_r = \begin{bmatrix} 0 \\ 100 \\ 0 \\ 0 \end{bmatrix}$$

```
figure(2);
subplot(1,3,1);
step(Asf_ext, B_r, Ce, De, 1, t);   % Br = [0; 0; 0; -1]
title('Step response without disturbance');
axis([0 15 0 1.2]);
subplot(1,3,2);
lsim(Asf_ext, [0 0; 0 1; 0 0; -1 0], Ce, De, [input1'  input2'], t);   % Br = [0 0; 0 1; 0 0; -1 0]
title('Step response with disturbance');
legend('step response', 'disturbance signal')
axis([0 15 0 1.2]);
subplot(1,3,3);
step(Asf_ext, [0; 100; 0; 0], Ce, De, 1, t);   % Br = [0; 100; 0; 0]
title('Disturbance response');
```

Consequently, you should obtain the results as shown in Figure 3 for the state variable feedback controller design with the integral control term.

⇒ Based on these results, you should discuss and give some conclusions about the capability to reject input disturbance signal and to provide zero steady-state error of the state variable feedback control system augmented with integral control.

Step 7. Build the state variable feedback controller model in Simulink

Use the state feedback gain matrix K and K_e found to build two cases of the state variable feedback controller (i.e. with and without the integral control part) in Simulink.

You should create a Simulink model as shown in Figure 4 and present the simulation results obtained.

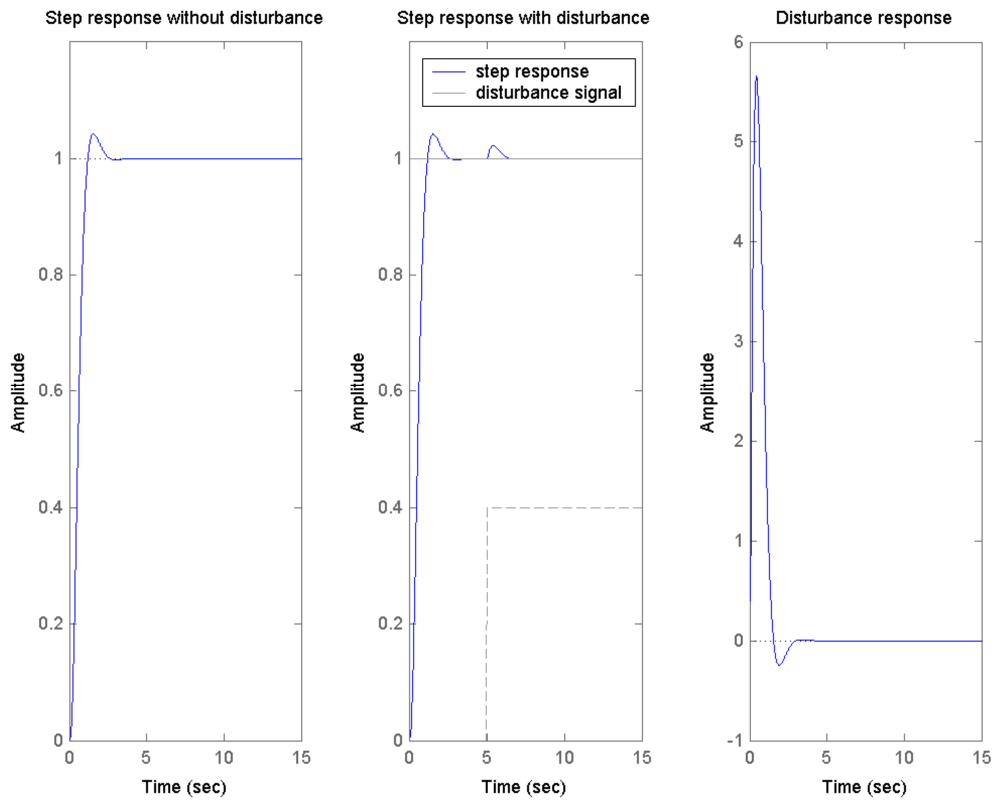


Figure 3. Simulation results for the state variable feedback controller with the integral control part.

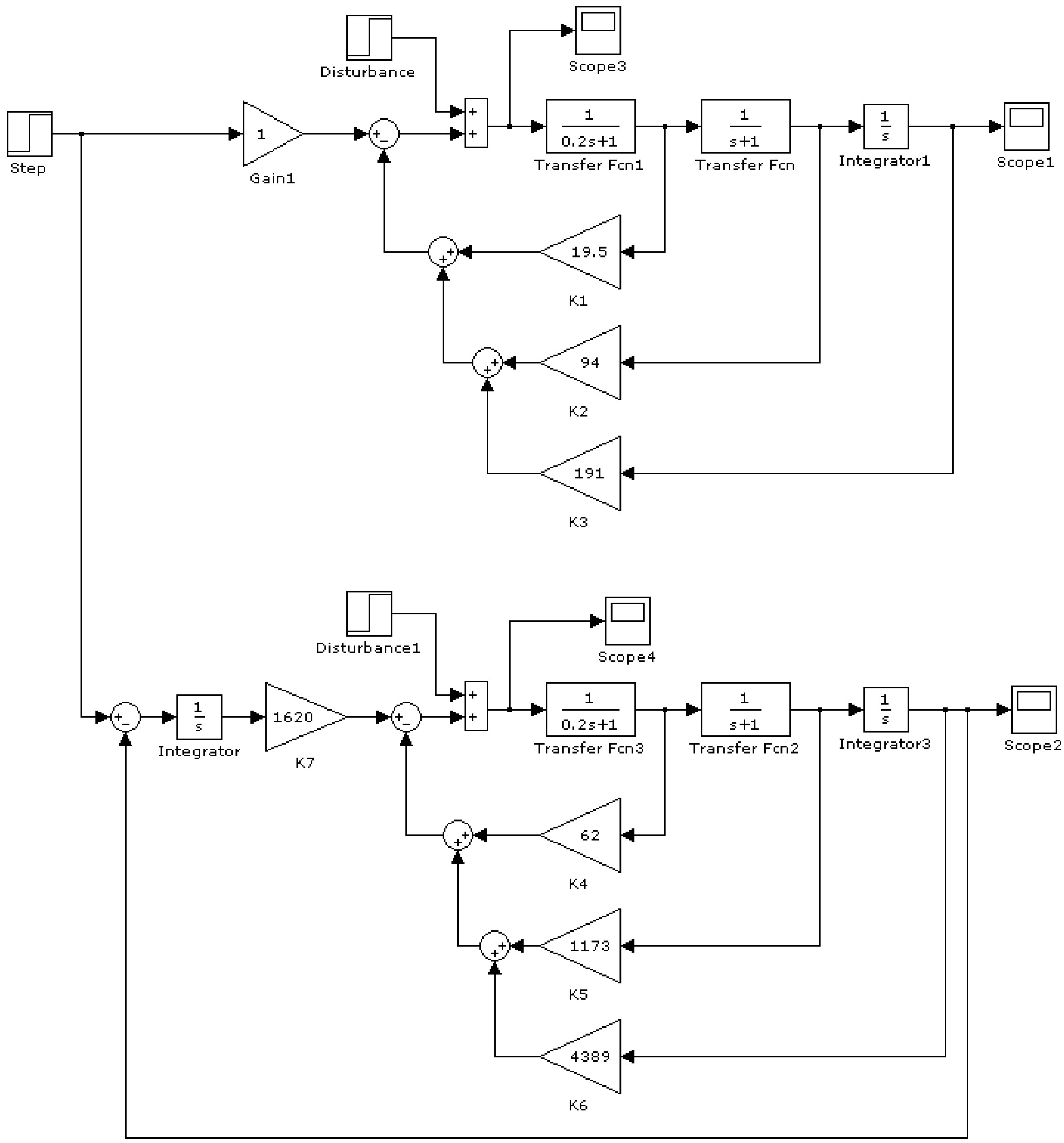


Figure 4. Simulink model of the state variable feedback controller with and without integral control.

Exercise 5

The following model has been proposed to describe the motion of a constant velocity guided missile:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -0.1 & -0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} x$$

(a) Investigate the controllability and observability of the given system.

(b) Develop a controllable state variable model by computing the transfer function from the given state-space model and canceling the common factors in the numerator and denominator (i.e. pole-zero cancellation). With modified transfer function go back to the state-space model.

(c) Verify the controllability and observability of the modified system.

(d) Is the modified system stable? If it is unstable then you are required to design a state variable feedback controller to stabilize the system. The pole placement method using the Ackermann's formula should be used.

(e) Give comments on the relationship between the number of state variables and the controllability of the system. What happens in case if the system has pole-zero cancellation?

Instructions:

1. The original system is uncontrollable and unobservable.

In addition, the open-loop original system is unstable, as illustrated by the step response and Bode diagram in Figures 5 and 6, respectively.

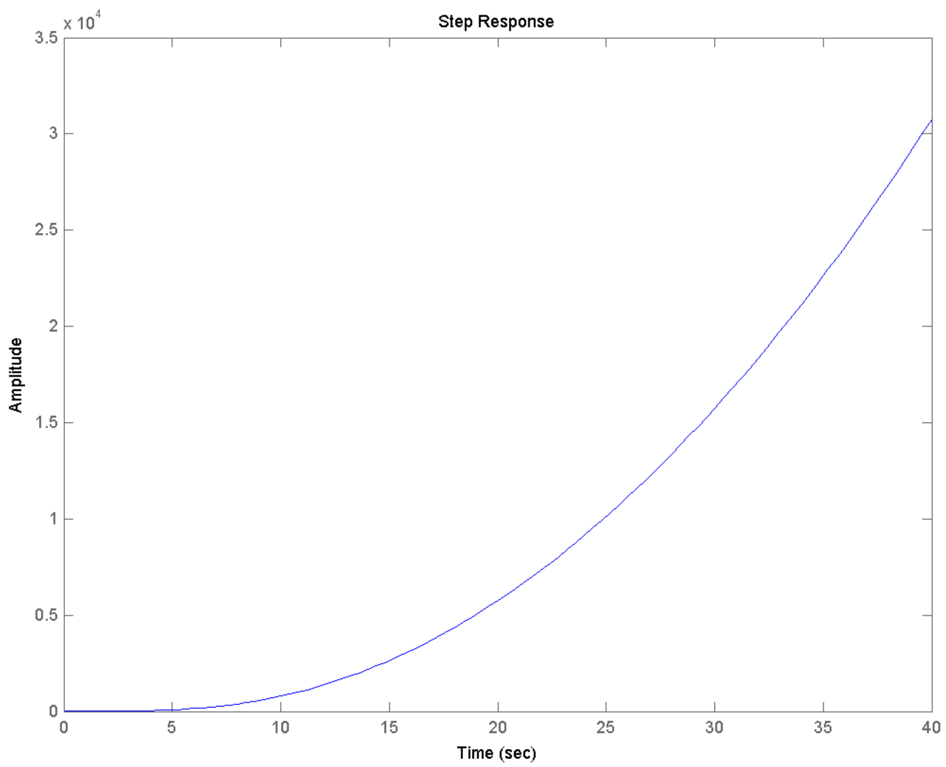


Figure 5. Step response of the original system.

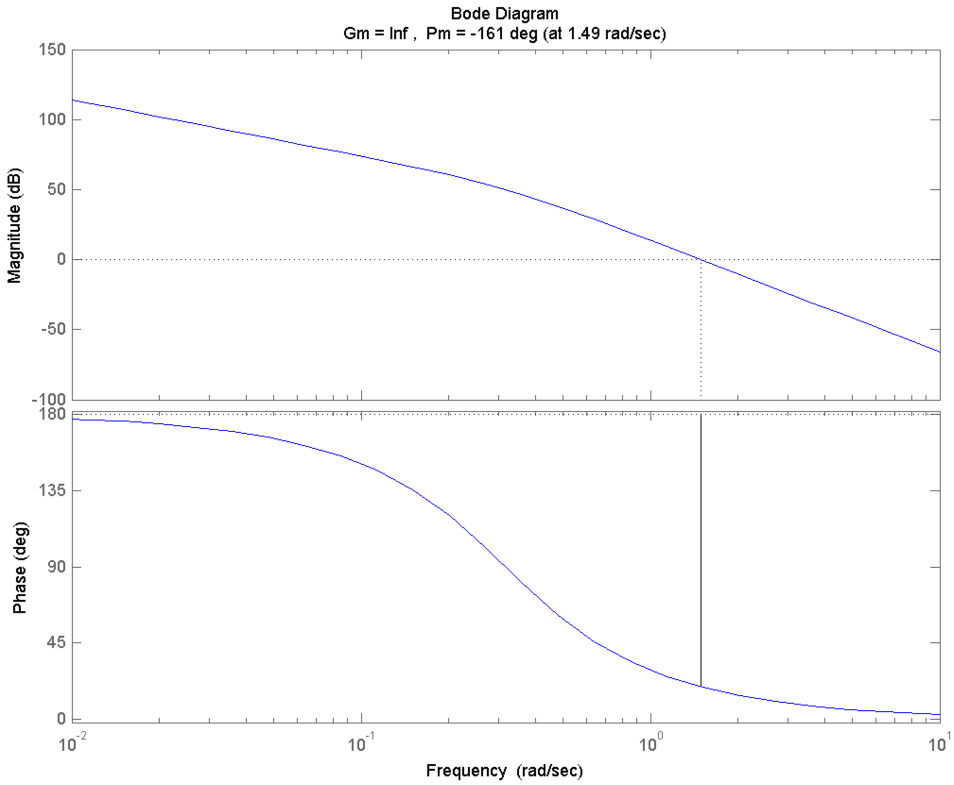


Figure 6. Bode diagram of the original system.

2. Pole-Zero simplification

As shown in this exercise, an uncontrollable and unobservable system can be turned into a controllable and observable system by means of pole-zero cancellation. However, one should be bear in mind the effect of pole-zero cancellation on the stability, controllability and observability of the system.

When an open-loop system has right-half-plane poles (i.e. the system is unstable), one idea to alleviate the problem is to add zeros at the same locations as the unstable poles, to in effect cancel the unstable poles. Unfortunately, this method is unreliable. The problem is that when an added zero does not exactly cancel the corresponding unstable pole (which is always the case in practice), a part of the root locus will be trapped in the right-half plane. This causes the closed-loop response to be unstable.

In case if pole-zero cancellation occurs, the cancelled modes are said to be unobservable. Of course, since we often start with a transfer function, any pole-zero cancellations should be dealt with at that point, so that the state-space realization will always be controllable and observable.

If a mode is uncontrollable, the input cannot affect it. If a mode is unobservable, it has no effect on the output. Therefore, there is usually no reason to include unobservable or uncontrollable modes in a state-space model. An exception arises when the model is time varying. For example, a time varying output matrix C will cause time-varying zeros in the system. These zeros may momentarily cancel poles, rendering them unobservable for a short time. For this reason, one can consider using the minimum realization technique to create a transfer function via a state-space equation with elimination of its uncontrollable and unobservable parts.

3. Minimal realization or pole-zero cancellation can be done by using the following Matlab function.

`M_SYS = minreal(SYS,TOL);` eliminates uncontrollable or unobservable states in state-space models, or cancels pole-zero pairs in transfer functions or zero-pole-gain models. The output `M_SYS` has minimal order and the same response characteristics as the original model `SYS`. The parameter `TOL` is used to specify the tolerance used for state elimination or pole-zero cancellation.

4. After performing pole-zero cancellation you should obtain a modified system with the following transfer function

$$P(s) = \frac{5}{s^4 + 0.5s^3 + 0.1s^2}$$

The modified system is now controllable and observable; however it has two poles at the origin (i.e. two pure integrators). Hence, it is a 4th-order type 2 system. This kind of system is unstable for all values of open-loop gain K. This can be easily observed by plotting a step response. Also, more useful information about the system stability can be revealed using other tools such as Nyquist plot, Bode diagram, Nichols plot, or root-locus by using the Matlab command

```
ltiview({'step';'nichols';'pzmap';'bode'}, M_SYS);
```

For example: Step response, Nichols plot, Pole/Zero map, and Bode diagram are shown in Figure 7.

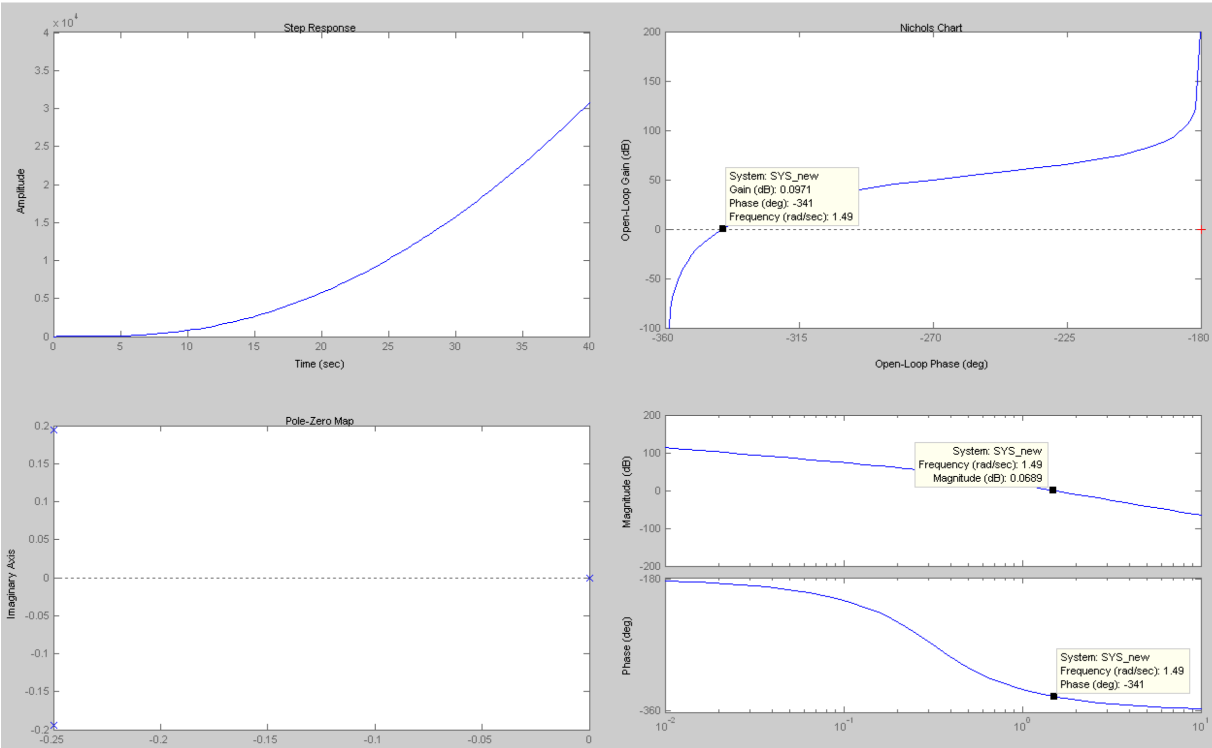


Figure 7. Multiple views on the system response.

It is shown in both Nichols plot and Bode diagram that the zero modulus crossover occurs at the frequency of *1.49 rad/s*, where a phase margin is equal to *-161 deg* (i.e. $\varphi_m = -341^0 - (-180^0) = -161^0$ so that the system is unstable. However, because the modified system is controllable the Ackermann's formula can be employed to design a state variable feedback controller with the desired stable closed-loop poles placed at, for examples:

new_desired_poles = [-0.25+0.25i -0.25-0.25i -0.5+0.5i -0.5-0.5i]   or

new_desired_poles = [-50 -100 -0.5+0.5i -0.5-0.5i]   % two dominant complex conjugate pole pair

5. One should give some comments and discussion on the relationship between the number of state variables, represented by the matrix *A* and the controllability of the system, represented by the controllability matrix *M*. What would one expect when the system has pole-zero cancellation?

Remarks:

- It is necessary to note that due to the loss of controllability and/or observability caused by pole-zero cancellation, for a system to be fully state controllable and completely observable its pulse transfer function must not have any pole-zero cancellations.

- A necessary and sufficient condition for controllability is that no single pole of the system is cancelled by a zero in all of the elements of the transfer function matrix $[sI - A]^{-1}B$. If such cancellation occurs, the system cannot be controlled in the direction of the cancelled mode.
- A necessary and sufficient condition for observability is that no single pole of the system is cancelled by a zero in all of the elements of the matrix $C[sI - A]^{-1}$. If such cancellation occurs, the cancelled mode cannot be observed in the output.

Exercise 6

The objective of the controller is to keep the satellite on the halo orbit trajectory that can be seen from the Earth so that the lines of communication are accessible at all times. The communication link is from the earth to the satellite and then to the far side of the moon. The linearized and normalized equations of motion of the satellite around the equilibrium point are:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 7.3809 & 0 & 0 & 0 & 2 & 0 \\ 0 & -2.1904 & 0 & -2 & 0 & 0 \\ 0 & 0 & -3.1904 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_3$$

The state vector x is the satellite position & velocity in the ξ, η, ζ directions; the inputs are the engine thrust accelerations in the ξ, η, ζ directions.

(a) Is the translunar equilibrium point a stable location?

(b) Is the system controllable from u_1 alone?

(c) Repeat the part (b) for u_2 and u_3.

(d) Suppose we want to observe the position in the η direction. Determine the transfer function from u_2 to η (hint: let $y = [0\ 1\ 0\ 0\ 0\ 0]x$).

(e) Verify if the system described by the transfer function from part (d) is controllable.

(f) Design a controller for the system from part (d) such that the closed loop poles of the system are at: $s_{1,2} = -1 \pm j$ *and* $s_{3,4} = -10$.

Instructions:

Given the state-space equations we can understand that the state variables are selected as follows:

x_1, x_2, x_3 are the satellite positions in the ξ, η, ζ directions, respectively.

x_4, x_5, x_6 are the satellite velocities in the ξ, η, ζ directions, respectively.

$$\Rightarrow \begin{cases} \dot{x}_1 = x_4 \\ \dot{x}_2 = x_5 \\ \dot{x}_3 = x_6 \\ \dot{x}_4 = 7.3809x_1 + 2x_5 + u_1 \\ \dot{x}_5 = -2.1904x_2 - 2x_4 + u_2 \\ \dot{x}_6 = -3.1904x_3 + u_3 \end{cases}$$

It should be noted that the system has three control inputs – which are the engine thrust accelerations in the ξ, η, ζ directions respectively – therefore we have a *multi-input single-output (MISO)* system.

By checking **eig(A)** you will know that the translunar point is not a stable location because there is one pole being on the right side of the complex plane.

Through checking the controllability independently for each direction you will find that the system is not fully controllable from any u_1, u_2, or u_3 alone.

The transfer function from u_2 to η:

$$P(s) = \frac{-1.943e^{-16}s^5 + s^4 - 3.325e^{-15}s^3 - 4.191s^2 - 8.629e^{-15}s - 23.55}{s^6 + 1.943e^{-16}s^5 + 2s^4 + 3.325e^{-15}s^3 - 19.97s^2 + 8.629e^{-15}s - 51.58} \qquad (e^{-16} = 10^{-16})$$

It should be noted that the above result can be slightly different given the Matlab versions and running environment of computers (such as operating systems and processors) that we are using. However, after eliminating the elements with very small coefficient we should obtain the same result as following

$$P(s) = \frac{s^4 - 4.191s^2 - 23.55}{s^6 + 2s^4 - 19.97s^2 - 51.58}$$

Now, by checking the pole-zero map of the transfer function $P(s)$ we can notice that the system has pole-zero cancellation (see Figure 8).

zeros = -2.7168 2.7168 0.0000 + 1.7862i 0.0000 - 1.7862i	poles = -2.1587 2.1587 0.0000 + 1.8626i 0.0000 - 1.8626i 0 + 1.7862i 0 - 1.7862i

As we can see, the roots 0 + 1.7862i and 0 - 1.7862i appear in both numerator and denominator therefore they can be cancelled. After this reduction, go back to the state-space model and obtain the system of order 4. This system is both controllable and observable. Now you can use the Ackermann's formula to design a state feedback controller with the closed-loop poles placed at desired locations.

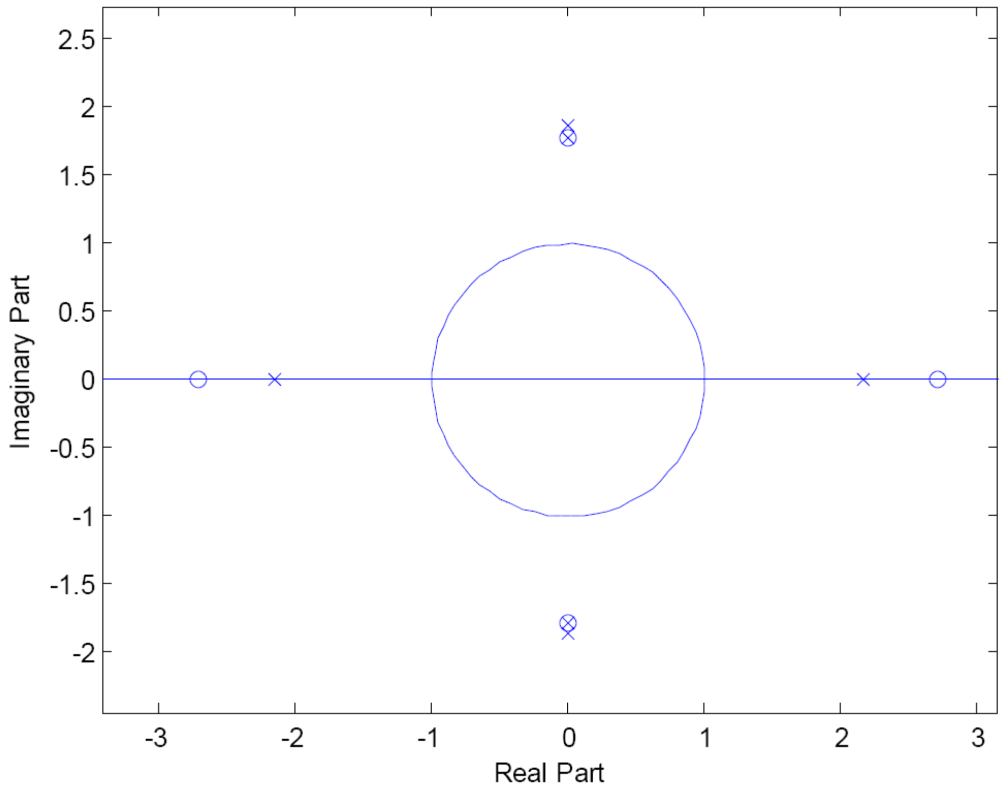


Figure 8. Pole-zero map.

Exercise 7

In order to make a comparison between classical control system design methods and the modern method, which is based on the state-space approach, consider again Exercise 4 from Laboratory no. 1 - Classical Feedback Control.

a) Assuming that the state variables are available for feedback, design a state variable feedback controller for the system *G(s) = 8/(s+0.01)(s+0.2)(s+1)* by using the Ackermann's formula (Matlab command: **acker**) or the pole placement design (Matlab command: **place**) to obtain the same required specifications (ξ, ω_n, t_s).

b) Discuss the advantages and disadvantages of the state-space design method in comparison with the Root-Locus Design GUI (sisotool) tool and the Ziegler-Nichols method.

c) Compare the results (i.e. dynamic characteristics) obtained by using these methods.

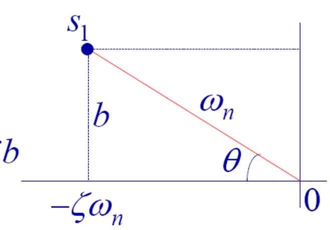
Instructions:

First, the controllability and observability of the system should be checked before designing the state variable feedback controller.

From the required design specifications (ξ, ω_n, t_s) you should determine the desired locations of the dominant closed-loop poles by using the guide below.

$$\zeta\omega_n = \frac{1}{\tau} \quad \text{and} \quad \theta = \cos^{-1}\zeta$$

$$b = \zeta\omega_n \tan\theta \quad \text{and} \quad s_1 = -\zeta\omega_n + jb$$



As a result, you will obtain the two dominant closed-loop poles, i.e. $s_{1,2} = -0.2 \pm j0.2$. The 3rd closed-loop pole should be selected as a non-dominant one, thereby it could be placed far away from the imaginary, e.g. $s_3 = -10$.

Appendix

Where to place the desired closed-loop poles?

The role of dominant poles (or modes) leads to the following design procedure:

- First, choose a second-order system with desired dynamics;
- Next, place two desired poles at the two poles of this second-order system;
- Then, choose all other desired poles to be faster (to render them less dominant) but not too fast (to avoid too large control action);
- Finally, assign the desired poles or modes and then evaluate the system by dynamic simulation.

Typically this process has to be iterated to achieve a good design.