Control Theory Exercises 1

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1.1. DC motor

Consider a DC motor shown in Figure 1. The rotor and the shaft are assumed to be rigid. The input is the armature voltage V in [volts] (i.e. the motor is driven by a voltage source). The measured output variables are the shaft angle θ in [radians] and the angular velocity of the shaft ω in [radians per second].

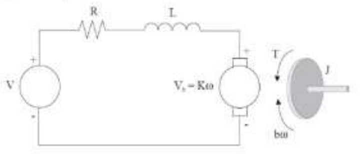


Figure 1. Schematic representation of the considered DC motor.

$J = 0.02\ kgm^2$

$b = 0.05\ Nms$

$K = 0.02\ Nm/A$

$R = 1\,\Omega$

$L = 0.03\ H$

T - torque e - emf $T = k\,i(t)$

J - inertia $e = k\,\dot{\theta}(t)$

b - motor viscous friction

mechanical power

$P_1 = T\,\dot{\theta}(t)$

electrical power

$P_2 = -e\,i(t)$

$P_1 + P_2 = 0$

$$\boxed{\begin{cases} J\,\ddot{\theta}(t) = T - b\,\dot{\theta}(t) \\ V(t) - R\,i(t) - L\dfrac{di(t)}{dt} - e(t) = 0 \end{cases}}$$

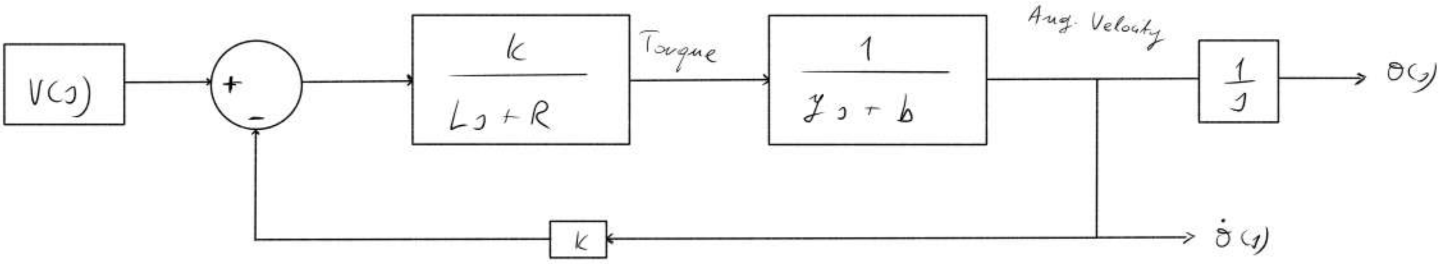
$$\begin{cases} J\,\ddot{\theta}(t) = k\,i(t) - b\,\dot{\theta} \\ V(t) = R\,i(t) - L\dfrac{di(t)}{dt} - k\,\dot{\theta}(t) \end{cases} \Rightarrow \begin{cases} Js^2\,\Theta(s) = k\,I(s) - bs\,\Theta(s) \\ V(s) = R\,I(s) + Ls\,I(s) + ks\,\Theta(s) \end{cases} \Rightarrow I(s) = \frac{Js^2 + bs}{k}\Theta(s) \Rightarrow$$

$$\Rightarrow V(s) = R\,\frac{Js^2+bs}{k}\Theta(s) + Ls\,\frac{Js^2+bs}{k}\Theta(s) + ks\,\Theta(s)$$

$$V(s) = \frac{s}{k}\left((R+Ls)(Js+b)+k^2\right)\Theta(s) \Rightarrow$$

$$\boxed{G_\theta(s) = \frac{\Theta(s)}{V(s)} = \frac{k}{s\left((R+Ls)(Js+b)+k^2\right)}}$$

$$\boxed{G_\omega(s) = \frac{\omega(s)}{V(s)} = \frac{k}{(R+Ls)(Js+b)+k^2}}$$



calculating partial derivatives for lagrangian:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left[(M+m)\dot{x} + ml\dot{\theta}\cos\theta\right] = (M+m)\ddot{x} + ml(\ddot{\theta}\cos\theta - \sin\theta\dot{\theta}^2) = (M+m)\ddot{x} - ml\sin\theta\dot{\theta}^2 +$$

$$+ ml\cos\theta\ddot{\theta}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left[ml\dot{x}\cos\theta + ml^2\dot{\theta}\right] = ml^2\ddot{\theta} + ml\frac{d}{dt}(\dot{x}\cos\theta) = ml^2\ddot{\theta} + ml(\ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta) =$$

$$= ml\left[l\ddot{\theta} + \ddot{x}\cos\theta - \dot{x}\dot{\theta}\sin\theta\right]$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta$$

substituting int Euler-Lagrange equations:

$$\begin{cases} ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta = 0 \\ (M+m)\ddot{x} - ml(\sin\theta)\dot{\theta}^2 + ml(\cos\theta)\ddot{\theta} = u \end{cases}$$

$$ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \quad /:ml$$

$$\ddot{x}\cos\theta + l\ddot{\theta} = g\sin\theta$$

we have obtained following state diff. equations

$$\begin{cases} (M+m)\ddot{x} - ml(\sin\theta)\dot{\theta}^2 + ml(\cos\theta)\ddot{\theta} = u \\ \ddot{x}\cos\theta + l\ddot{\theta} = g\sin\theta \end{cases}$$

the state eq. are the same as for method 1 (Newton)

so the final result will also be the same

1.3. One-link planar manipulator

Build a mathematical model in terms of *nonlinear state differential equations* for the one-link planar manipulator with load shown in Figure 5 by using the Euler-Lagrange method (or the Lagrangian approach).

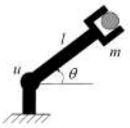


Figure 5. One-link planar manipulator.

M : mass of link [kg];
m : mass of load [kg];
l : link length [m];
l_c : distance from the centre of gravity of the manipulator to the rotation axis [m];
B : viscous friction coefficient at the rotation axis [Nms];
Input $u(t)$ is the external torque acting on the rotation axis of the manipulator [Nm];
Output $\theta(t)$ is the rotation angle between the link and the horizontal axis [rad];
Manipulator parameters: $M = 3.5kg$, $m = 0.6kg$, $l = 1.4m$, $l_c = 0.5m$, $B = 0.01Nms$ and $g = 9.81m/s^2$.

$$J = Ml_c^2 + ml^2 \qquad D = \frac{1}{2}B\dot{\theta}^2 \qquad P = (Ml_c + ml)\,g\sin\theta$$

$$J_k = \frac{1}{2}J\omega^2 = \frac{1}{2}(Ml_c^2 + ml^2)\dot{\theta}^2$$

Lagrangian:

$$L = J_k - P = \frac{1}{2}(Ml_c^2 + ml^2)\dot{\theta}^2 - (Ml_c + ml)\,g\sin(\theta)$$

$$\frac{\partial L}{\partial \theta} = -(Ml_c + ml)\,g\cos(\theta) \;;\quad \frac{\partial L}{\partial \dot\theta} = (Ml_c^2 + ml^2)\dot{\theta} \;;\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = (Ml_c^2 + ml^2)\ddot{\theta} \;;\quad \frac{\partial D}{\partial \dot\theta} = B\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot\theta} = u \Rightarrow \boxed{(Ml_c^2 + ml^2)\ddot{\theta} + (Ml_c + ml)\,g\cos(\theta) + B\dot{\theta} = u}$$

1.4. Linearization of the inverted pendulum on a cart

This exercise uses the results obtained from **"Exercise 1.2. Inverted pendulum on a cart"**.

Suppose that the state variables are selected as follows:

$$\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \\ x_3(t) = x(t) \\ x_4(t) = \dot{x}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = \dot{\theta}(t) = x_2(t) \\ \dot{x}_2(t) = \ddot{\theta}(t) \\ \dot{x}_3(t) = \dot{x}(t) = x_4(t) \\ \dot{x}_4(t) = \ddot{x}(t) \end{cases} \tag{34}$$

and the system has one input $u(t)$ and two outputs $y(t) = \begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}$ (35)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ \dfrac{u\cos(x_1) - (M+m)g\sin(x_1) + ml\sin(x_1)\cos(x_1)x_2^2}{ml\cos^2(x_1) - (M+m)l} \\ x_4(t) \\ \dfrac{u + ml\sin(x_1)x_2^2 - mg\cos(x_1)\sin(x_1)}{M + m - m\cos^2(x_1)} \end{bmatrix} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}$$

linearised system of state-space equations:

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t) \end{cases}$$

Equilibrium point:

$$\begin{cases} \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \bar{u} = 0 \end{cases}$$

As a result, we obtain the state matrix A:

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}_{(0,0)} = \boxed{\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix}}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix} \qquad \frac{\partial f_1}{\partial u} = 0 \;;\quad \frac{\partial f_3}{\partial u} = 0 \quad [\text{no „u" in eq}]$$

→ Rest of eq f_2 is irrelevant because it has no „u"

$$\frac{\partial f_2}{\partial u} = \frac{\partial}{\partial u}\left(\frac{u\cos(x_1)}{ml\cos^2(x_1) - (M+m)l}\right) = \frac{\cos(x_1)}{ml\cos^2(x_1) - (M+m)l}$$

assumptions: $\begin{bmatrix} \cos(\theta) = \cos(x_1) \approx 1 \\ \sin(\theta) = \sin(x_1) \approx x_1 \end{bmatrix} \Rightarrow \dfrac{\partial f_2}{\partial u} \approx \dfrac{1}{-Ml}$

$$\frac{\partial f_4}{\partial u} = \frac{\partial}{\partial u}\left(\frac{u}{M + m - m\cos^2(x_1)}\right) = \frac{1}{M + m - m\cos^2(x_1)} \approx \frac{1}{M}$$

$$\boxed{B = \begin{bmatrix} 0 \\ \frac{-1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix}}$$

C

$$h_1(t) = x_1(t) = \theta(t)$$

$$h_2(t) = x_3(t) = x(t)$$

$$\frac{\partial h_1}{\partial x_1} = \frac{\partial}{\partial x_1}(x_1) = 1$$

$$\frac{\partial h_2}{\partial x_3} = \frac{\partial}{\partial x_3}(x_3) = 1$$

other = 0

$$\boxed{C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$$

D

$$\frac{\partial h_1}{\partial u} = \frac{\partial}{\partial u}(x_1) = 0$$

$$\frac{\partial h_2}{\partial u} = \frac{\partial}{\partial u}(x_3) = 0$$

$$\boxed{D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

$$\frac{\theta(t)}{u(t)} = G_1(s) = C_1(sI - A)^{-1}B + D_1 \qquad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ D_1 = 0$$

$$\frac{X(t)}{u(t)} = G_2(s) = C_2(sI - A)^{-1}B + D_2 \qquad C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},\ D_2 = 0$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{m}{M}g & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} \qquad (sI - A)^{-1} = Z = \begin{bmatrix} \frac{100s}{G_1} & \frac{100}{G_1} & 0 & 0 \\ \frac{4251}{G_1} & \frac{100}{G_1} & 0 & 0 \\ -\frac{2943}{10sG_1} & -\frac{2943}{10s^2G_1} & \frac{1}{s} & \frac{1}{s} \\ \frac{-2943}{10sG_1} & \frac{-2943}{10sG_1} & 0 & \frac{1}{s} \end{bmatrix} \qquad \mid\ G_1 = 100s^2 - 425$$

$$K_1 = C_1 Z = \begin{bmatrix} \frac{100s}{G_1} & \frac{100}{G_1} & 0 & 0 \end{bmatrix} \qquad K_2 = C_2 \cdot Z = \begin{bmatrix} -\frac{2943}{10sG_1} & -\frac{2943}{10s^2G_1} & \frac{1}{s} & \frac{1}{s^2} \end{bmatrix}$$

$$\boxed{G_1(s) = K_1 B = -3.333 \cdot \frac{100}{G_1} = \frac{-333.3}{100s^2 - 425}} \qquad \boxed{G_2(s) = K_2 B = \frac{3.33 \cdot 2943}{10s^2(100s^2 - 425)} + \frac{1}{s^2}}$$

Determining static operation point at the equilibrium:

from the eq. we get:

$$\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \end{cases} \quad \text{therefore} \quad \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \ddot{\theta}(t) \end{cases} \qquad \begin{array}{l} u(t) - \text{input} \\ y(t) = \theta(t) - \text{output} \end{array}$$

$\bar{\theta} = \frac{\pi}{4}$

finding $\bar{x}$ & $\bar{u}$

$$x_1(t) = \theta(t) \,\big|_{x=\bar{x},\, u=\bar{u}} \Rightarrow \bar{x}_1 = \bar{\theta} = \frac{\pi}{4}$$

$$x_2(t) = \dot{\theta}(t) \,\big|_{x=\bar{x},\, u=\bar{u}} \Rightarrow \bar{x}_2 = 0$$

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi}{4} \\ 0 \end{bmatrix}$$

finding $u(t)$: from eq. (59)

$$(Ml_c^2 + ml^2)\ddot{\theta} + (Ml_c + ml)g\cos\theta + B\dot{\theta} = u$$

~~factoring out $\ddot{\theta}$~~ $\quad \ddot{\theta} = \dot{x}_2(t) = 0$

$\cancel{\ddot{\theta} = -(Ml_c}$ therefore:

$$u = (Ml_c + ml)g\cos\theta + B\dot{\theta} = (Ml_c + ml)g\cos x_1 + Bx_2 \,\big|_{x=\bar{x},\, u=\bar{u}} \Rightarrow$$

$$\Rightarrow \bar{u} = (Ml_c + ml)g\cos\bar{x}_1 = 17{,}9661$$

calculating matrices A, B, C, D: using eq (63) to (66)

$$A = \begin{bmatrix} 0 & 1 \\ \frac{(Ml_c + ml)}{(Ml_c^2 + ml^2)}\sin x_1 & \frac{-B}{Ml_c^2 + ml^2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{Ml_c^2 + ml^2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \frac{\partial h}{\partial u}\Big|_{(\bar{x},\bar{u})} = 0$$

substituting values:

$$A = \begin{bmatrix} 0 & 1 \\ 8.7595 & -0{,}0049 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0{,}4876 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

Matrix B has to be modified to accunt for the relation with $\dot{x}_2$ & u.

$$B = \begin{bmatrix} 0 \\ b_2 - \bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ -17{,}4785 \end{bmatrix}$$

calculating transfer function

$G_p(s) = C(sI - A)^{-1}B + D$

calculated using matlab for simplicity

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s+0,0049}{s(s+0,0049)-8,7597} & \frac{1}{s(s+0,0049)-8,7597} \\ \frac{8,7597}{s(s+0,0049)-8,7597} & \frac{s}{s(s+0,0049)-8,7597} \end{bmatrix}$$

$$C(sI-A)^{-1} = \begin{bmatrix} \frac{s+0,0049}{s(s+0,0049)-8,7597} & \frac{1}{s(s+0,0049)-8,7597} \end{bmatrix}$$

$$C(sI-A)^{-1}B = \frac{-17,4875}{s(s+0,0049)-8,7597} = G_p(s) \text{ because } D = 0$$

checking stability of the system:

characteristic eq: $s^2 + 0,0049s - 8,7597$

$\Delta = 0,0049^2 + 4 \cdot 8,7597 > 0$

ie. linearized model is unstable.