Control Theory (RMS-1-612-s ; RIMA-1-612-s)

Laboratory no. 3 – Robust Control (updated: 1st May 2020)

Issue date: 1st May 2021
Due date: 31st May 2021

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Keywords: feedback control, robust control, systems with uncertain parameters, system sensitivity, system stability, robust stability, robustness margins, disturbance rejection, Nyquist stability criterion, Routh-Hurwitz stability criterion, lead-lag compensator, robust PID controllers, prefilter design, Matlab/Simulink, Control Systems Toolbox.

Recommended reading: Chapter 9 of [1] and Chapter 12 of [2].

[1] Roland S. Burns, *Advanced Control Engineering*, Butterworth-Heinemann, 2001, ISBN 978-0-7506-5100-4.

[2] Richard C. Dorf and Robert H. Bishop, *Modern Control Systems*, 8th edition, Addison-Welley, Boston, MA, 1997, ISBN: 978-0-2013-0864-8.

[3] Phong B. Dao, *Control Theory (RMS-1-612-s)*, Lecture 04. Closed-loop control systems.

[4] Phong B. Dao, *Control Theory (RMS-1-612-s)*, Lecture 09. Robust control system design.

[5] Okko H. Bosgra, H. Kwakernaak and G. Meinsma, *Design Methods for Control Systems*, Notes for a course of the Dutch Institute of Systems and Control, Winter term 2006-2007.

Acknowledgements: Exercises used in this laboratory originate from Chapter 12 in [2].

Remarks:

(1) Short instructions and final results are provided to support you in solving these exercises. You should read the instructions carefully. Though some parts of the exercises were explained and guided in the instructions but they are general solutions and not yet complete. Therefore, based on the instructions provided you should elaborate and discuss in the report your solutions with Matlab code and results obtained in detail.

(2) Before using Matlab commands (or functions) you should first check the syntax, descriptions, input arguments, output arguments, and illustrated examples by typing "help command".

(3) The report must be typed on a word-processor and submitted as a single PDF file via **UPeL platform**.

Exercise 1

A new suspended, mobile, remote-controlled video-camera system to bring three-dimensional mobility to professional NFL football is shown in Figure 1a. The camera can be moved over the field as well as up and down. The motor control on each pulley is represented by the system in Figure 1b, where $\tau_1 = 20\,\text{ms}$ and $\tau_2 = 2\,\text{ms}$.

a) Let $D(s) = 0$. Select K such that the resonance peak $M_{P\omega} = 1.84$.

b) Plot $20\log|T|$ and $20\log|S|$ on one Bode diagram, where T is the complementary sensitivity function (i.e. the closed-loop transfer function) and S is the sensitivity function.

c) Evaluate $|S|$ at the bandwidth ω_B, $\frac{\omega_B}{2}$, and $\frac{\omega_B}{4}$. To maintain robustness in terms of sensitivity to parameter changes, what is the maximum frequency the system should be designed? Explain.

d) Let $R(s) = 0$ and determine the effect of disturbance $D(s) = 1/s$ for the gain K found in part (1a) by plotting $y(t)$.

For all questions, you should present your solutions with Matlab code and discuss the results obtained in detail.

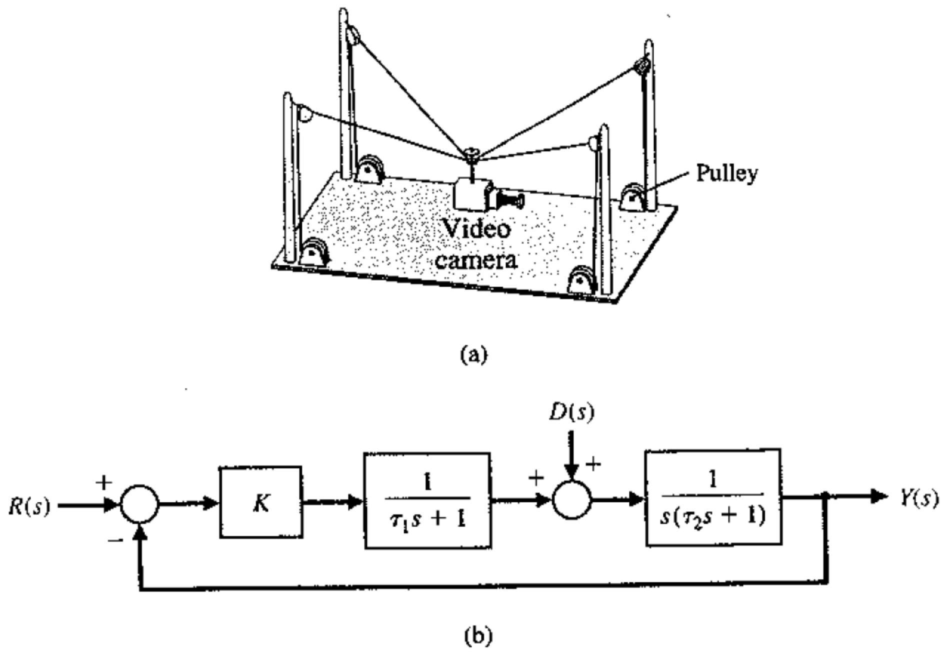
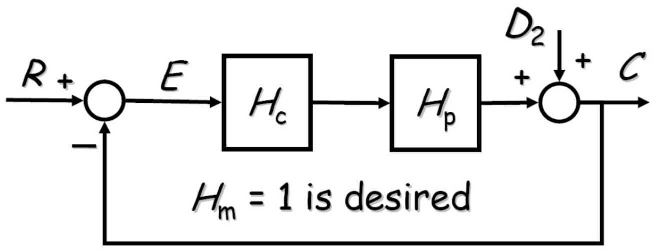


Figure 1.

Instructions:

1. The concepts of the complementary sensitivity function (T) and the sensitivity function (S) are shortly explained in the following. More details about closed-loop transfer functions can be found in [3].



The unity feedback control system

Sensitivity function

$$\frac{C}{D_2} = \frac{1}{1 + H_c H_p}$$

A system with zero sensitivity is ideal (i.e. changes in the system parameters have no effect on the transfer function)

$$\frac{C}{D_2} + \frac{C}{R} = 1$$

Complementary sensitivity function

$$\frac{C}{R} = \frac{H_c H_p}{1 + H_c H_p}$$

Closed-loop transfer function

$$\frac{1}{1 + H_c H_p} + \frac{H_c H_p}{1 + H_c H_p} = 1$$

To solve exercise (1a) you are advised to read pages (428, 429, 440, 441) in [2]. It is necessary to note that the resonance peak $M_{P\omega}$ can be obtained from the complementary sensitivity function plot (see Figure 8.25, page 440 in [2]). In general, the magnitude $M_{P\omega}$ indicates the relative stability of a system.

$$M_{P\omega} = |G(\omega_r)| = \left(2\xi\sqrt{1-\xi^2}\right)^{-1} \qquad (1)$$

The resonance peak $M_{P\omega}$ is the maximum value of the magnitude $|G(\omega)|$ of the frequency response at the resonant frequency

$$\omega_r = \omega_n \sqrt{1-2\xi^2} \tag{2}$$

where ω_n is the natural frequency and ξ is the damping ratio.

2. In order to find K such that the resonance peak $M_{P\omega} = 1.84$ you first need to solve Equation (1). The results obtained consist of four solutions of the damping ratio ($\xi = [0.9590 \ \ 0.2834 \ \ -0.2834 \ \ -0.9590]$). Obviously, only the two solutions with positive values (i.e. $\xi = [0.9590 \ \ 0.2834]$) are further considered. Next, these two damping ratio values are analysed using the root locus method. More specifically, the two Matlab commands `rlocus` and `rlocfind` should be employed for the *open-loop* system. This analysis will result in two corresponding open-loop gain values (i.e. $\xi = 0.9590 \Rightarrow \mathrm{K}_1 \cong 12.9$ and $\xi = 0.2834 \Rightarrow \mathrm{K}_2 \cong 100$). For each value of K found, create a complementary sensitivity function (or the closed-loop transfer function) and then check its step response and Bode diagram. The value of K that results in the step response and Bode diagram – as shown in Figures 3a and 3b, respectively – is the gain value that meets the requirement and thus being selected. It is because of the fact that this gain value causes the overshoot and oscillations in the step response and the resonance peak in the Bode diagram. The another gain value does not produce this behaviour. You should remember to set $K = 1$ and $D = 0$ in this analysis (before running `rlocus` and `rlocfind`).

Matlab functions:

rlocus(SYS) computes and plots the root locus of the single-input, single-output LTI model SYS.

[K,POLES] = rlocfind(SYS) is used for interactive gain selection from the root locus plot of the SISO system SYS generated by `rlocus`. The command `rlocfind` puts up a crosshair cursor in the graphics window that is used to select a pole location on an existing root locus. The root locus gain associated with this point is returned in K and all the system poles for this gain are returned in POLES.

MATLAB code examples for feedback control system design using `rlocus` and `rlocfind` are given below. The major idea of the technique used is that by selecting a point along the root locus that coincides with a desired damping ratio and/or a desired natural frequency, an open-loop gain K can be calculated and then implemented in the controller.

```
open_sys = R*(K*G1+D)*G2; ⇒ form the open-loop system.
figure;
rlocus(open_sys);
axis([-520 20 -400 400]);
...solve Equation (1) to find solutions of the damping ratio given the required value of MPw.
damping_ratio = 0.2834; % check this damping ratio value.
wn = 0:100:400;
sgrid(damping_ratio, wn);
[K, closed_loop_poles] = rlocfind(open_sys) ⇒ return open-loop gain K and closed-loop poles.
open_sys = R*(K*G1+D)*G2; ⇒ reform the open-loop system with the new value of gain K.
closed_sys = feedback(open_sys,1);
figure;
step(closed_sys);
figure;
bode(closed_sys);
```

It should be noted that when Matlab asks you to "select a point in the graphics window" the desired point should be selected as shown in Figure 2.

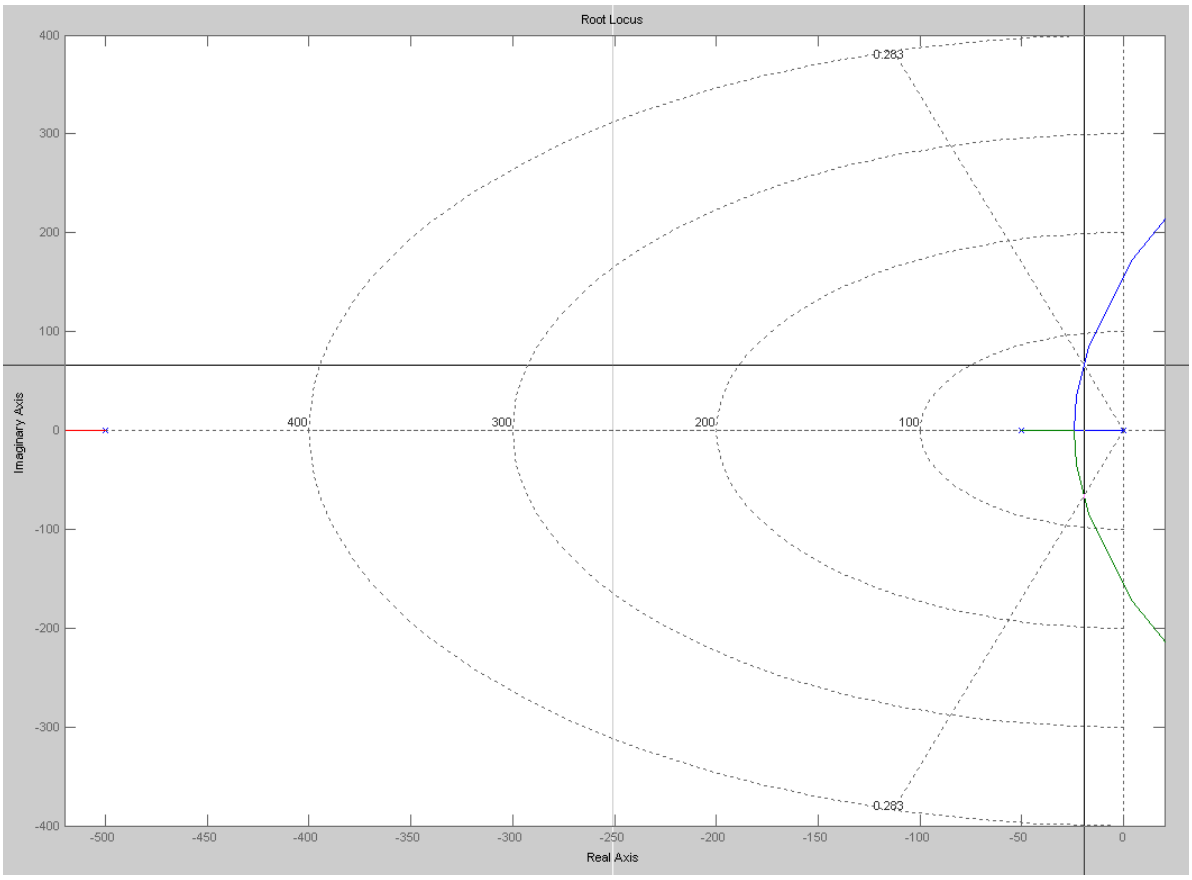


Figure 2. Select a point at the intersection between the root locus and the damping ratio.

After finding K you need to check if your design meets the requirement (i.e. $M_{P\omega} = 1.84$). It can be observed in Figure 3b that the magnitude characteristic $20\log|T|$ has the "peak" value that is approximately equal to 5.3 [dB] at the resonant frequency $\omega_r = 64.3$ [rad/s]. It is then easy to calculate the value of $|T|$ and compare it with the required resonance peak $M_{P\omega} = 1.84$.

3. It is well known that the bandwidth (ω_B) is the frequency obtained from the Bode diagram for the complementary sensitivity function (T) at which the frequency response declines *3dB* from its low-frequency value (see Figure 8.25, page 440 in [2] for details). This $-3dB$ bandwidth is a measure of a system's ability to accurately reproduce an input signal and it can be related to the speed of the transient response [2]. More specifically, the speed of response to step input is roughly proportional to ω_B and the settling time is inversely proportional to ω_B. Thus as the bandwidth ω_B increases, the rise time of the step response of the system will decrease.

In exercise (1c), the bandwidth $\omega_B \cong 102$ [rad/s] can be obtained from the Bode diagram for the complementary sensitivity function, as shown in Figure 3b. Next, the sensitivity function (S) can be evaluated at ω_B, $\frac{\omega_B}{2}$, and $\frac{\omega_B}{4}$. For example, at $\omega_B = 102$ [rad/s] the sensitivity function has the magnitude characteristic $20\log|S|$ that is approximately equal to 4.46 [dB] and thereby $|S| = 1.67$.

Finally, based on the Bode diagram for the sensitivity function (S) it can be found that the system is robust with respect to parameter variations for the frequency up to the maximum value $\omega \cong 48.4$ [rad/s], where $20\log|S| = 0$, i.e. $|S| = 1$. You need to show this result on the Bode diagram.

The values that you have obtained may be fairly different from the results given in the instruction. This divergence is probably due to the use of different Operating Systems (OS) and/or Matlab versions.

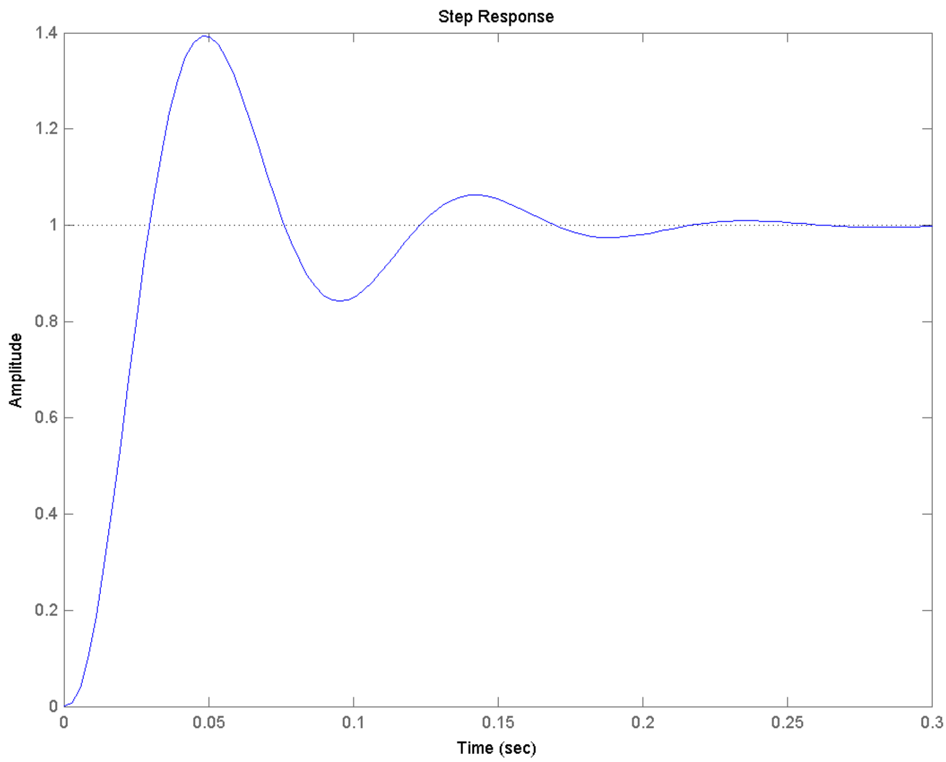


Figure 3a. Step response for the complementary sensitivity function (or the closed-loop system).

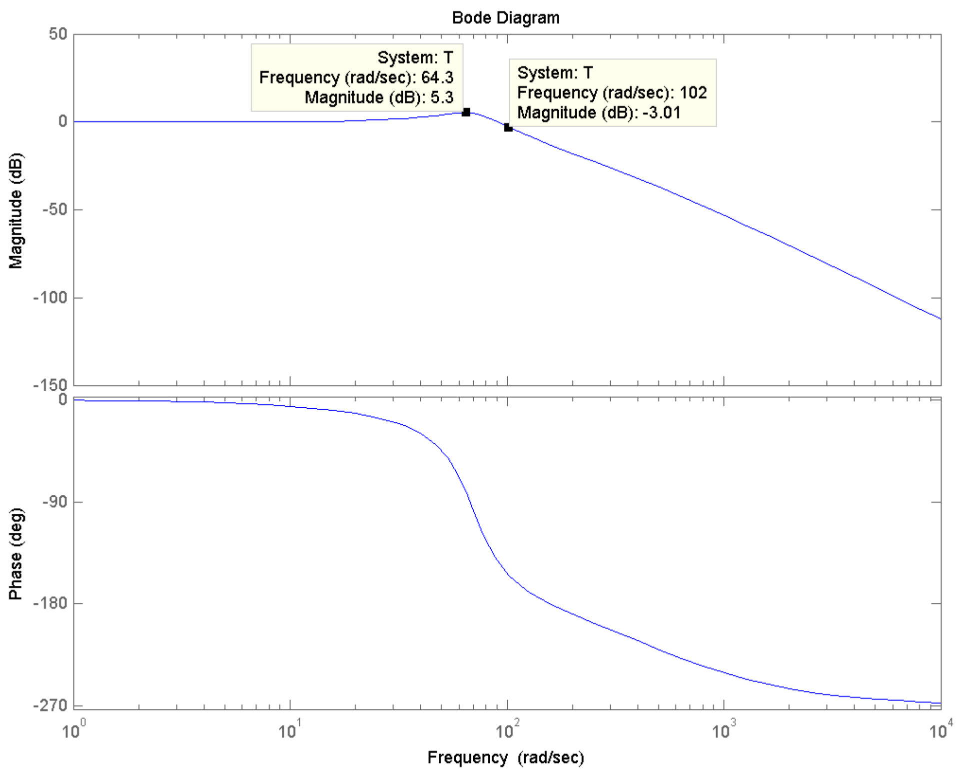


Figure 3b. Bode diagram for the complementary sensitivity function (or the closed-loop system).

4. For exercise (1d) you should perform the analysis for both $K_1 \cong 12.9$ and $K_2 \cong 100$ and obtain the results in Figure 4. Give some comments on the robustness of the system in terms of disturbance rejection for both cases with K_1 and K_2. It should be noted that the disturbance D(s) = 1/s means that the disturbance input is a unit step signal.

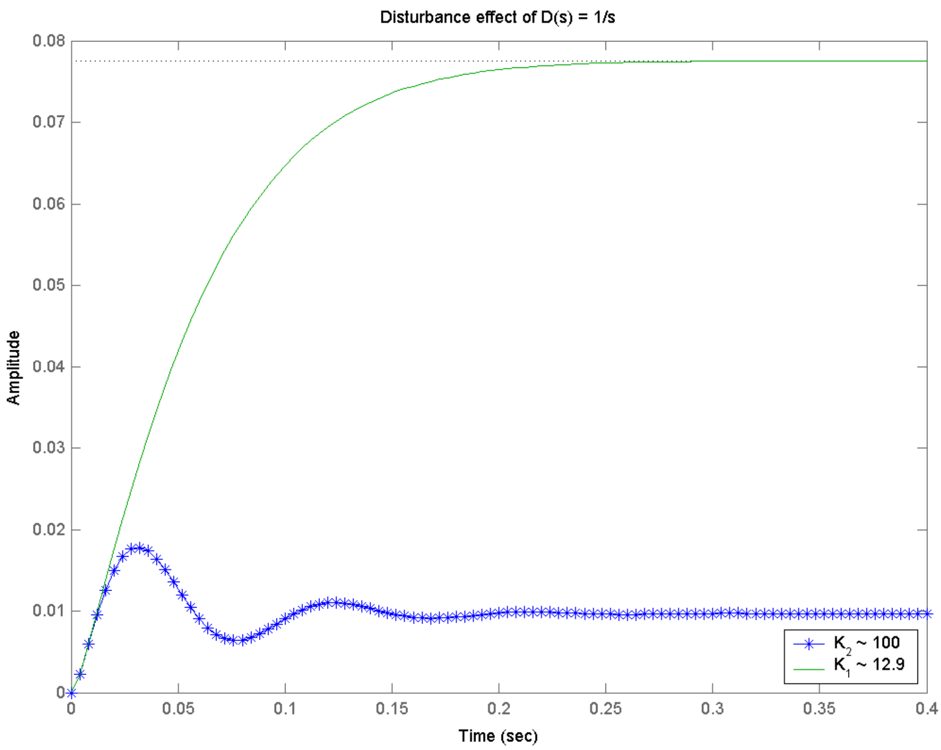


Figure 4. Disturbance effect of D(s) = 1/s for different values of the gain K.

Exercise 2

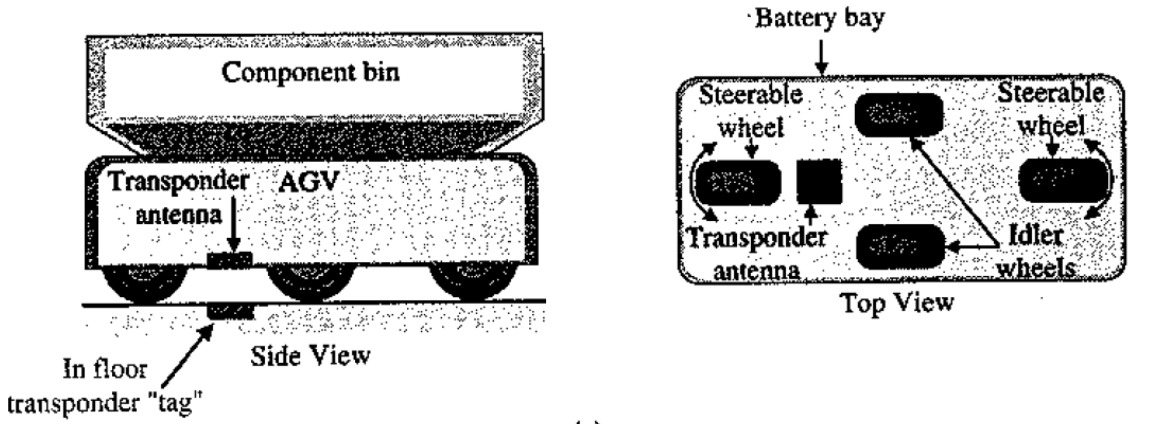
An automatically guided vehicle is shown in Figure 5a and its control system is shown in Figure 5b. The goal is to track the guided wire accurately, to be insensitive to changes in the gain K_1, and to reduce the effect of the disturbance $D(s)$. The gain K_1 is normally equal to 1 and $\tau = 1/25$ second.

a) Let $D(s) = 0$. Design a compensator (or controller) $G_c(s)$ so that the percent overshoot to a step input is less than or equal to 10%, the setting time (2% criterion) is less than 100 [ms], and the velocity error constant K_v for a ramp input is 100 (in other words, steady-state error $e_{ss} \leq 0.01$). Describe your design and present the results obtained in detail.

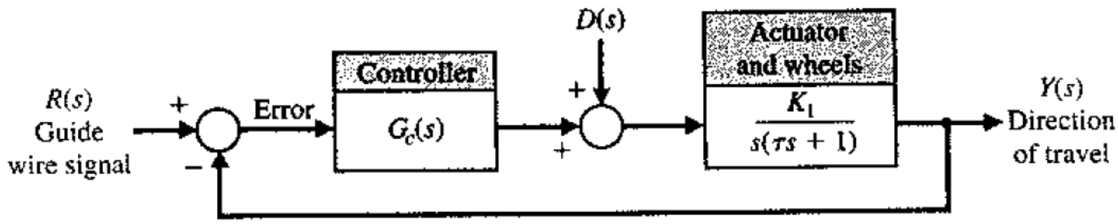
b) ~~For the compensator (or controller) designed in part (a), determine the sensitivity of the system to small changes in K_1 by determining $S_{K_1}^T$ or $S_{K_1}^r$.~~ (You do not need to solve this task)

c) If the gain K_1 changes to 2 while $G_c(s)$ of part (a) remains unchanged, find and plot the step response of the system with $K_1 = 2$ and compare the relevant performance specifications (i.e. the percent overshoot, setting time, and steady-state error) with those obtained in part (a) for $K_1 = 1$. Discuss the results obtained in detail. Do you think the compensator (or controller) that you have designed is robust to the change in the gain K_1? Explain.

d) Determine the effect of $D(s) = 1/s$ by plotting $y(t)$ when $R(s) = 0$ and $K_1 = 1$.



(a)



(b)

Figure 5.

Instructions:

1. The concept "velocity error constant K_v for a ramp input" is explained on **slides 20-22** in [3]. The required steady-state error (e_{ss}) obtained from the static velocity error constant $K_v = 100$ for a ramp input is given by

$$e_{ss} = \frac{1}{K_v} = 0.01 \tag{3}$$

More details about system types and steady-state errors can be found in [3].

The transfer function of the plant (actuator and wheels) is given by

$$G_p(s) = \frac{K_1}{s(\tau s+1)} = \frac{1}{s\left(\frac{1}{25}s+1\right)} = \frac{25}{s(s+25)} \tag{4}$$

So it has two poles: $p_1 = 0$ and $p_2 = -25$.

For exercise (2a) you are free to select any learned (or known) control structures (e.g. P/PI/PD/PID controllers, lead-lag compensators, or state variable feedback controllers, etc.) and a suitable design method (e.g. Ziegler-Nichols method, root locus technique, or pole placement method, etc.) to design a compensator (or a controller) that meets the requirements.

For example, Figures 6 and 7 show the results obtained for the lead-lag compensator $G_c(s)$ – described by Equations 5 and 6 – which was designed by using the Root-Locus Design GUI (`sisotool`) of Matlab Control System Toolbox. This compensator $G_c(s)$ was simply created by placing a zero at the same position of the pole $p_2 = -25$ and simultaneously introducing a new pole ($p_{new} = -100$) which is further from the origin in the complex s-plane.

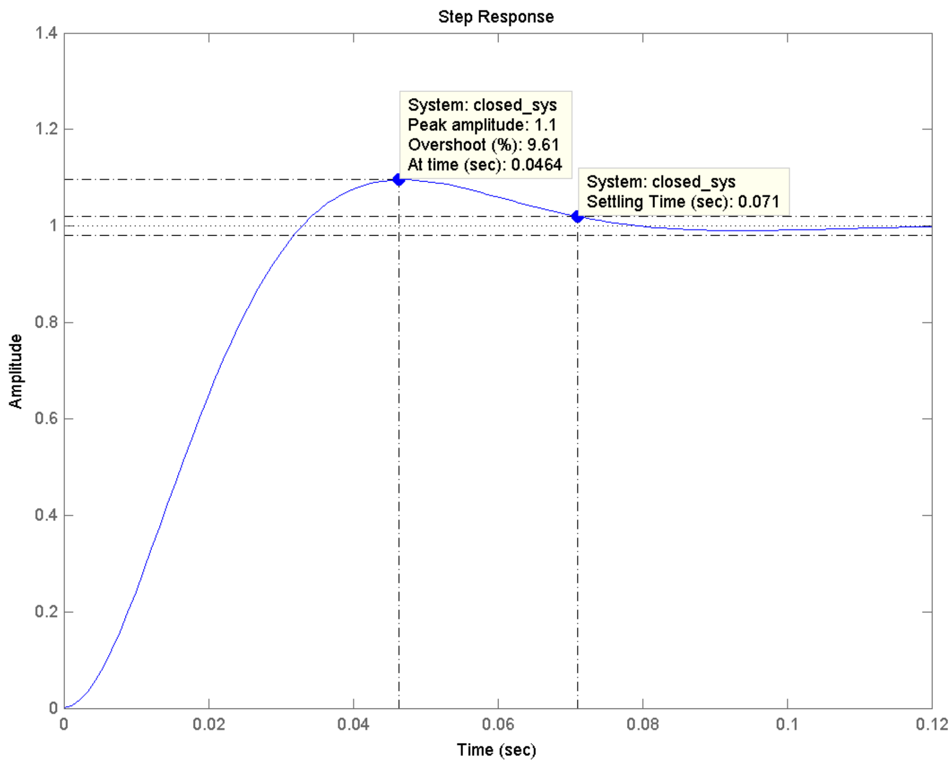


Figure 6. Step response using the lead-lag compensator designed (with $K_1 = 1$).

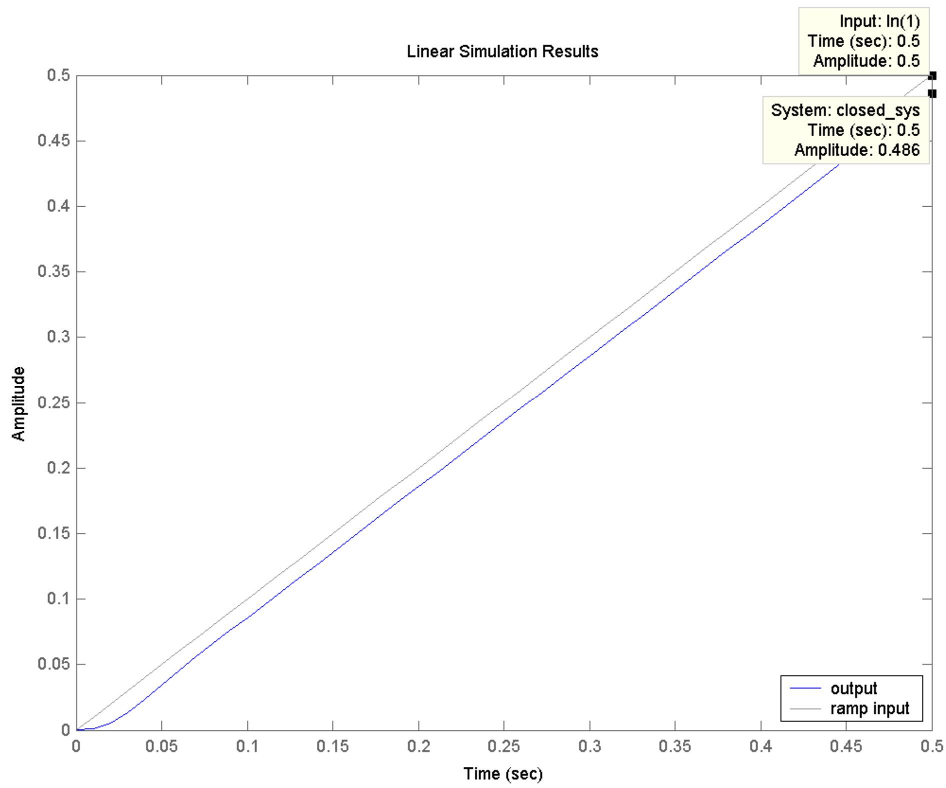


Figure 7. System response for a ramp input using the lead-lag compensator designed (with $K_1 = 1$).

The lead-lag compensator $G_c(s)$ has the basic form

$$G_c(s) = K_c \frac{\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)} \tag{5}$$

The design resulted in

$$G_c(s) = 70\frac{\left(\frac{1}{25}s+1\right)}{\left(\frac{1}{100}s+1\right)} = 70\frac{100}{25}\frac{s+25}{s+100} = \frac{2.8s+70}{0.01s+1} \tag{6}$$

From Equations (4) and (6) the open-loop transfer function can be calculated as

$$G(s) = G_c(s)G_p(s) = 70\frac{100}{25}\frac{s+25}{s+100}\frac{25}{s(s+25)} = 70\frac{100}{s(s+100)} \tag{7}$$

MATLAB code examples:

```
s = tf('s');
Kc = 70;
tau = 1/25;
K1 = 1;
Gp = ...
Gc = ...
closed_sys = feedback(Gc*Gp,1);
figure;
step(closed_sys);
figure;
t =0:0.01:0.5;
lsim(closed_sys,t,t); ⇒ check the steady-state error for a ramp input.
```

It can be noticed in Figures 6 and 7 that the lead-lag compensator designed meets the two requirements, i.e. the overshoot is equal to $9.61\% < 10\%$ and the setting time $t_s = 0.071\,[\text{sec}] = 71\,[\text{ms}] < 100\,[\text{ms}]$, but the steady-state error $e_{ss} = 0.5 - 0.486 = 0.014$ is larger than the required value $e_{ss} \le 0.01$.

In order to fulfill all performance requirements, one should either design a "better" controller or improve this preliminary lead-lag compensator design. With regard to improving the lead-lag compensator, it is necessary to note that the parameter K_c of the compensator has much influence on the steady-state error. By increasing K_c the required steady-state error can be achieved; however this makes the overshoot increase. It is also advised that the position of the new pole strongly influences the dynamic characteristics (the overshoot and setting time) of the system so that one may consider selecting p_{new} to be more further from the origin in the complex s-plane to achieve better system performances. In summary, the lead-lag compensator $G_c(s)$ – described in Equations 5 and 6 – could be improved by concurrently modifying the parameter K_c and the position of p_{new}.

For example, the results using an improved lead-lag compensator are shown in Figures 8 and 9. Clearly, all required specifications are well met, i.e. the overshoot is equal to $0.433\% < 10\%$, the setting time $t_s = 0.0251\,[\text{sec}] = 25.1\,[\text{ms}] < 100\,[\text{ms}]$, and the steady-state error $e_{ss} = 0.5 - 0.49 = 0.01$.

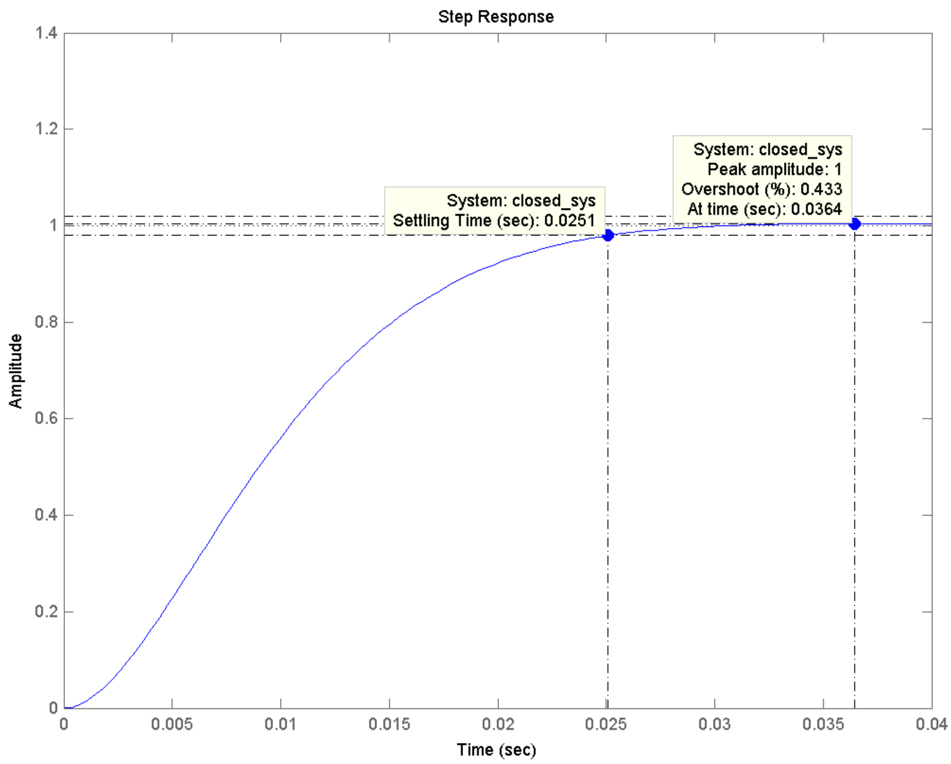


Figure 8. Step response using the lead-lag compensator improved (with $K_1 = 1$).

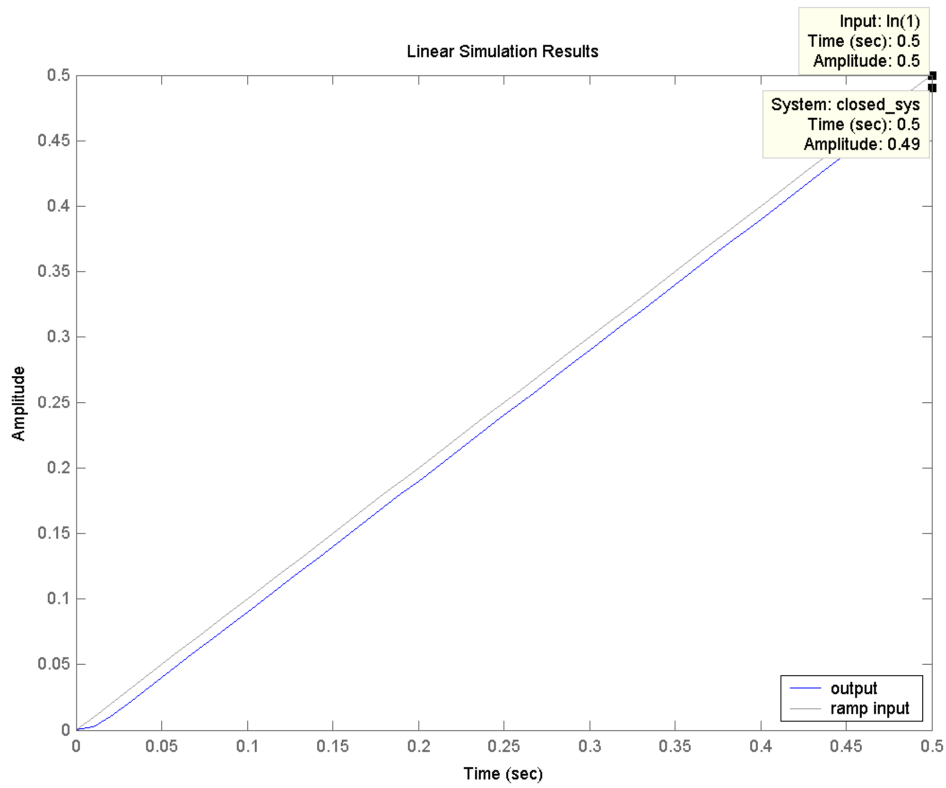


Figure 9. System response for a ramp input using the lead-lag compensator improved (with $K_1 = 1$).

You are now required to present in detail your own compensator (or controller) design that meets all performance requirements (i.e. the percent overshoot, setting time, and steady-state error). Showing only the resulting controller is not sufficient.

~~**2.** To solve exercise (2b) you are referred to Section 12.5 and Example 12.7 in [2]. This analysis involves the compensator (or controller) that you have designed in part (2a).~~

3. For exercise (2c) when the gain K_1 changes to 2 while the final solution of $G_c(s)$ – which is the improved lead-lag compensator in this case – remains unchanged. This compensator results in the step responses in Figure 10 for different values of K_1. In addition, Figure 11 shows the system response for a ramp input for $K_1 = 2$. One can notice that all system performances are still met the requirements for the case when $K_1 = 2$, which are: the overshoot is equal to $8.77\% < 10\%$, the setting time $t_s = 0.0244$ [sec] $= 24.4$ [ms] < 100 [ms], and the steady-state error $e_{ss} = 0.5 - 0.495 = 0.005 < 0.01$. The overshoot has increased due to higher gain, but steady-state error, also due to higher gain, decreased a bit. These results show that the improved lead-lag compensator is robust to the change in the gain K_1.

What happens with your design? Is the compensator (or controller) that you have designed robust when the gain K_1 changes to 2? In case if the designed compensator (or controller) is not robust to the change in the gain K_1 you need to improve/modify the compensator (or controller) designed in part (a). Give some comments on the robustness of the closed-loop system in terms of system disturbance rejection.

4. For exercise (2d) it is reminded that the disturbance D(s) = 1/s means that the disturbance input is a unit step signal. The unit step disturbance response when $R(s) = 0$ and $K_1 = 1$ with the use of the improved lead-lag compensator is presented in Figure 12. The results show that the disturbance response is reduced by a factor of 100.

In summary, this exercise investigates the design of a robust compensator (or controller) under system disturbances (i.e. the parameter change in the gain K_1 of the plant).

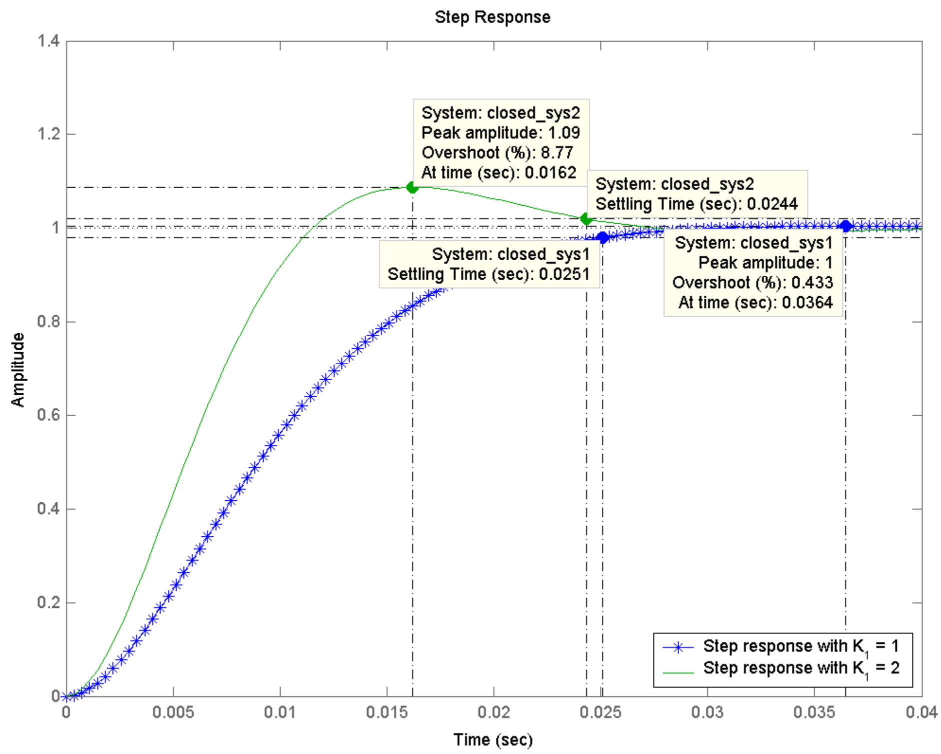


Figure 10. Step responses using the lead-lag compensator improved (with different values of K_1).

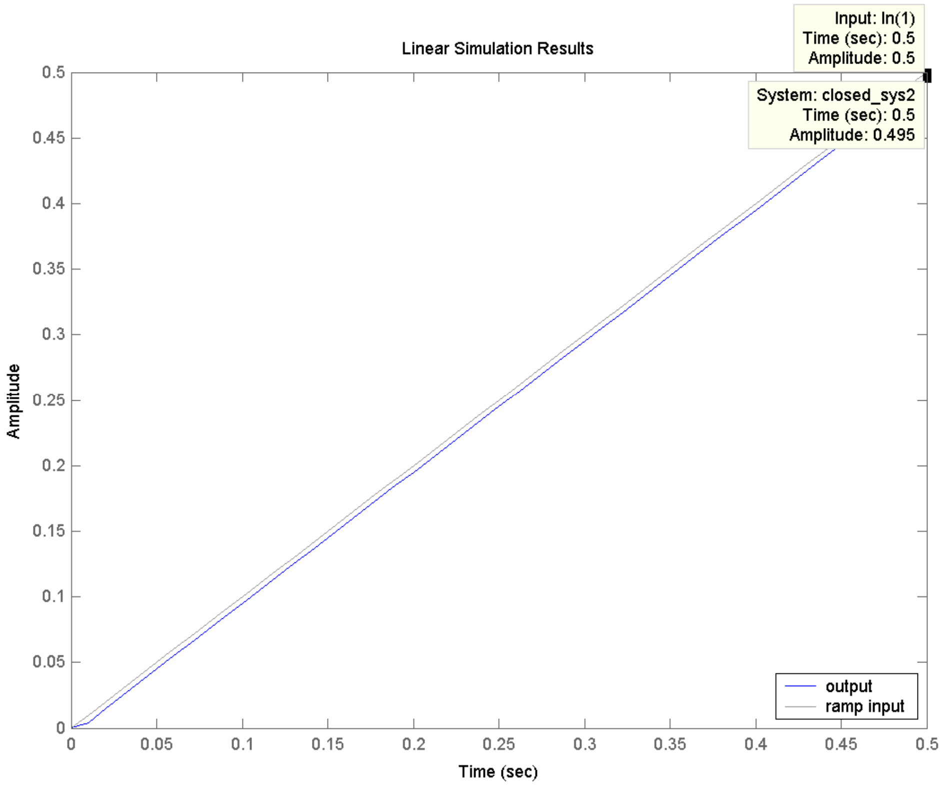


Figure 11. System response for a ramp input using the lead-lag compensator improved (with $K_1 = 2$).

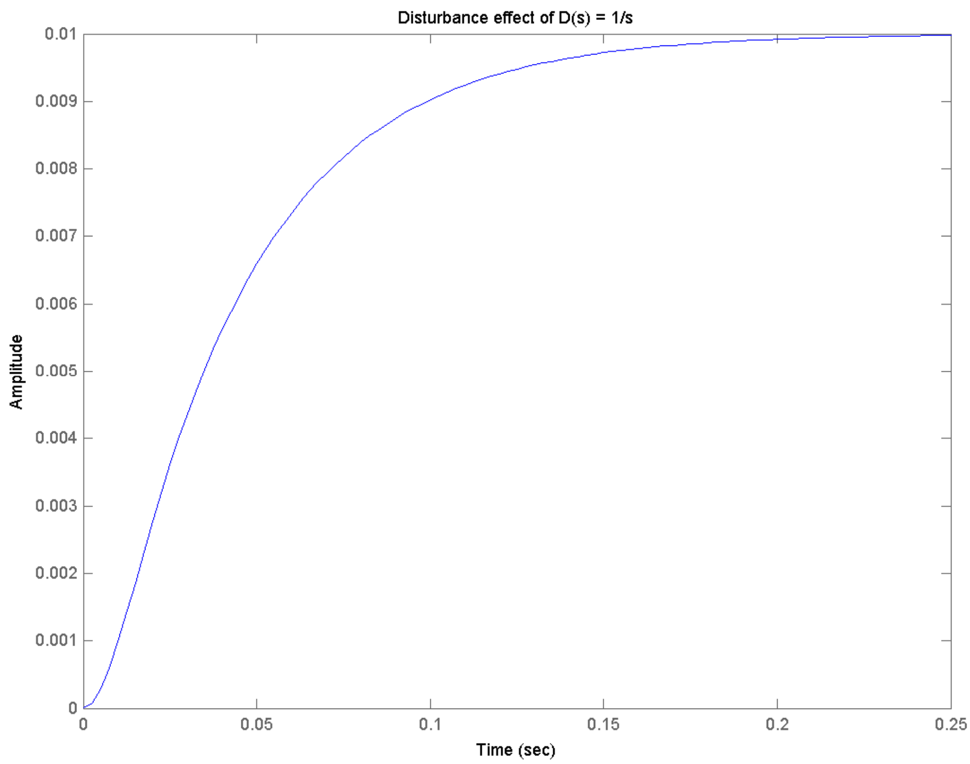


Figure 12. Disturbance effect of D(s) = 1/s when $R(s) = 0$ and $K_1 = 1$.

Exercise 3

A motor and load with negligible friction and a voltage-to-current amplifier K_a is used in the feedback control system in Figure 13. A designer has selected a PID controller $G_c(s) = K_1 + \frac{K_2}{s} + K_3 s$, where $K_1 = 5$, $K_2 = 500$, and $K_3 = 0.0475$.

a) Determine the appropriate value of K_a so that the phase margin φ_m of the system is 42^0. Create the Bode plot of the system with the found value of K_a.

b) For the gain value K_a obtained from part (a): (1) Plot the root locus of the system and find the roots of the closed-loop system; and (2) Plot the Nyquist plot of the system and determine if the closed-loop system stable or unstable? Explain.

c) Determine the maximum value of $y(t)$ when the disturbance $D(s) = 1/s$ and $R(s) = 0$ with the value of K_a found in part (a).

d) Determine the system response to a step input $r(t)$ in case with and without a prefilter.

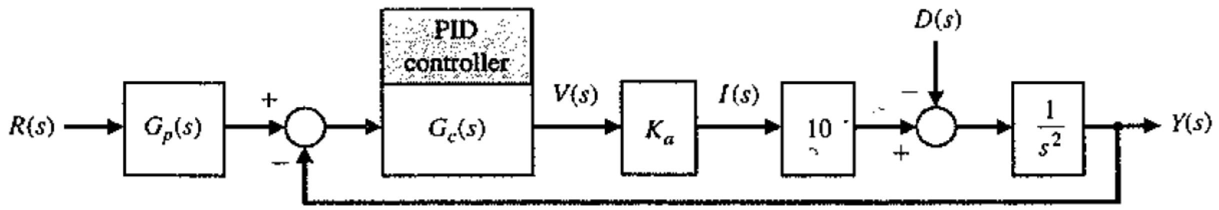


Figure 13.

Instructions:

First of all, it is acknowledged that some parts of the instructions given below derive from [5].

1. In this laboratory we have considered single-input single-output (SISO) feedback systems with the general configuration shown in Figure 14. We discuss their stability robustness, i.e. the property that the closed-loop system remains stable under changes of the plant and/or the compensator (or controller). This discussion focusses on the *loop gain* $L = PC$, with P is the plant transfer function and C is the compensator transfer function. For simplicity it is assumed that the system is *open-loop stable*, that is, both P and C represent the transfer function of a stable system.

It is well known that the closed-loop system in Figure 14 remains stable under perturbations of the loop gain L as long as the Nyquist plot of the perturbed open-loop system does not encircle the point -1. Intuitively, this may be interpreted as "keeping the Nyquist plot of the feedback system away from the point -1."

The classic gain margin and phase margin are well-known indicators for how closely the Nyquist plot approaches the point -1.

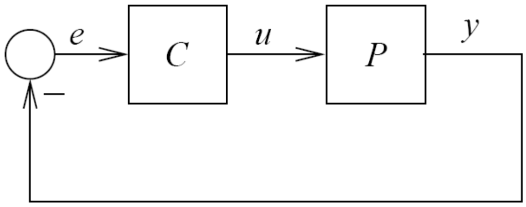


Figure 14. Feedback system configuration.

2. The concepts of gain and phase margins.

The *gain margin* is the smallest positive number k_m by which the Nyquist plot must be multiplied so that it passes through the point -1. We have

$$k_m = \frac{1}{b} = \frac{1}{|L(j\omega_r)|} \tag{8}$$

where ω_r is the angular frequency for which the Nyquist plot intersects the negative real axis furthest from the origin (see Figure 15).

The *phase margin* is the extra phase φ_m that must be added to make the Nyquist plot pass through the point -1. The phase margin φ_m is the angle between the negative real axis and $L(j\omega_m)$, where ω_m is the angular frequency at which the Nyquist plot intersects the unit circle closest to the point -1 (see Figure 15).

In classical feedback system design, robustness is often specified by establishing minimum values for the gain and phase margin. **Practical requirements are $k_m > 2$ for the gain margin and $30^0 < \varphi_m < 60^0$ for the phase margin.**

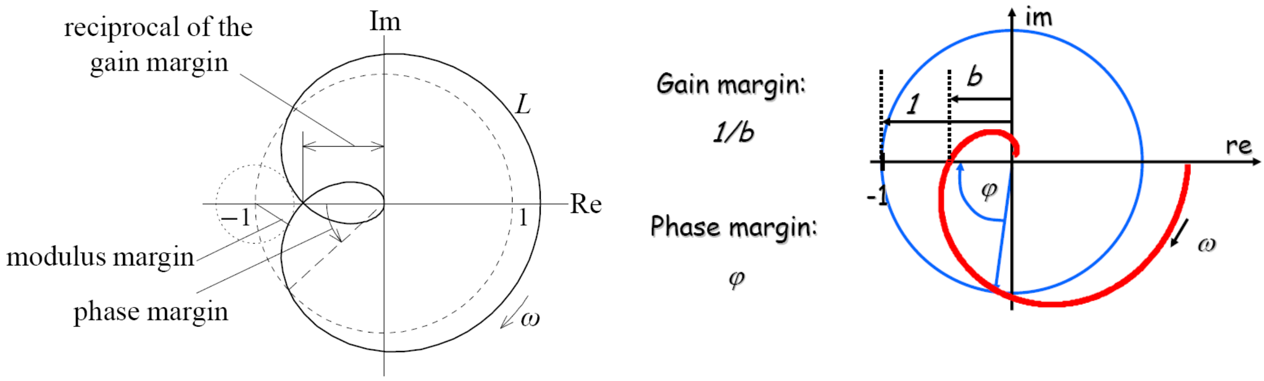


Figure 15. Robustness margins described on the Nyquist plot.

Gain and phase margins can also be achieved from the Bode plot. For example, one can estimate from Figure 16 that the phase margin $\varphi_m \approx -130^0 - (-180^0) = 50^0$.

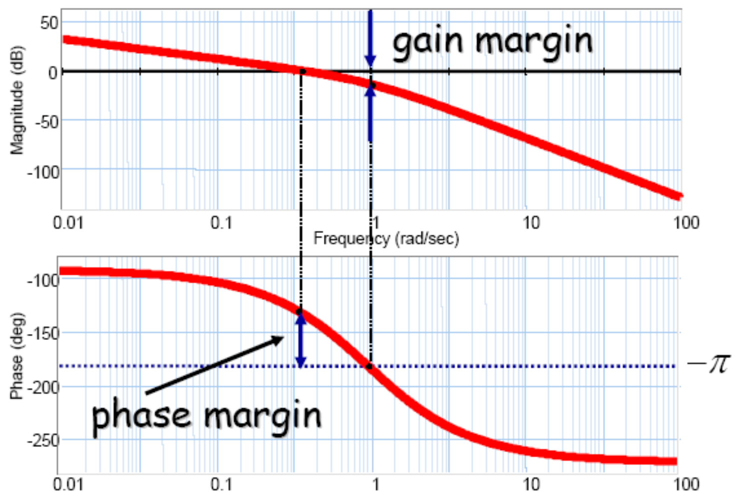


Figure 16. Robustness margins described on the Bode plot.

Gain and phase margins can also be obtained from the Nichols plot.

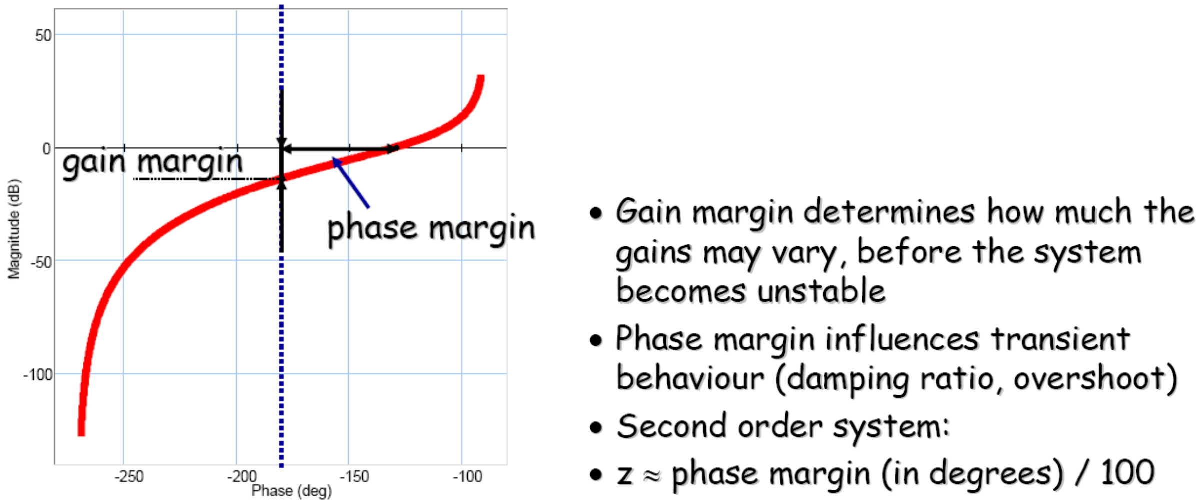


Figure 17. Robustness margins described on the Nichols plot.

3. The concepts of modulus and delay margins.

It is important to note that the gain and phase margin do not necessarily adequately characterize the robustness. Figure 18 shows an example of a Nyquist plot with excellent gain and phase margins but a relatively small joint perturbation of gain and phase suffices to destabilize the system. A typical example is the conditionally stable system which was investigated in the previous laboratory. For this reason, Landau et al. (1993) introduced two more margins.

The *modulus margin* s_m is the radius of the smallest circle with center -1 that is tangent to the Nyquist plot (see Figure 15). The modulus margin directly expresses how far the Nyquist plot stays away from the point -1.

The *delay margin* τ_m is the smallest extra delay that may be introduced in the loop that destabilizes the system. The delay margin is linked to the phase margin φ_m by the relation

$$\tau_m = \min_{\omega_*} \frac{\varphi_*}{\omega_*} \tag{9}$$

with ω_* ranges over all nonnegative frequencies at which the Nyquist plot intersects the unit circle, and φ_* denotes the corresponding phase $\varphi_* = \arg L(j\omega_*)$. In particular $\tau_m \le \frac{\varphi_m}{\omega_m}$.

A practical specification for the modulus margin is $s_m > 0.5$. The delay margin τ_m should be at least of the order of $\frac{1}{2B}$, where B is the bandwidth (in terms of angular frequency) of the closed-loop system.

4. Relation between robustness margins.

The gain margin k_m and the phase margin φ_m are related to the modulus margin s_m by the inequalities

$$k_m \ge \frac{1}{1-s_m} \quad \text{and} \quad \varphi_m \ge 2\arcsin\left(\frac{s_m}{2}\right) \tag{10}$$

This means that if $s_m \ge 0.5$ then $k_m \ge 2$ and $\varphi_m \ge 2\arcsin(0.25) \approx 28.96^0$.

Adequate margins are not only needed for robustness, but also to achieve a satisfactory time response of the closed-loop system. If the margins are small, the Nyquist plot approaches the point −1 closely. This means that the stability boundary is approached closely, manifesting itself by closed-loop poles that are very near to the imaginary axis. These closed-loop poles may cause an oscillatory response, which is called "ringing" if the resonance frequency is high and the damping is small.

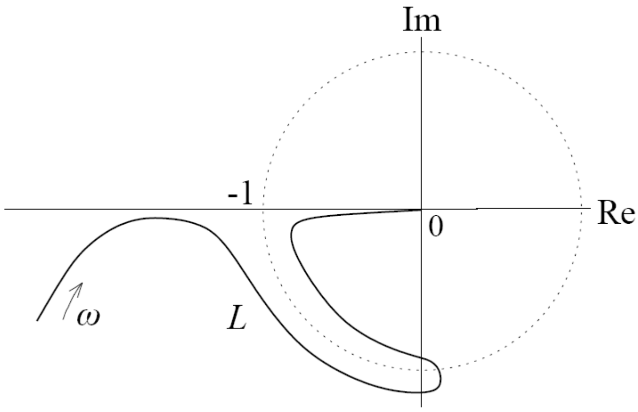


Figure 18. Nyquist plot has good gain and phase margins but a small simultaneous perturbation of gain and phase can destabilize the system [5].

5. For exercise (3a) there are several ways to determine the appropriate value of K_a to obtain the desired phase margin $\varphi_m = 42^0$.

First, one can implement a program that performs the following algorithm:

```
Ka = 1;
while (1==1)
    - calculate the (new) open-loop transfer function;
    - check the phase margin;
    - stop if the phase margin meets the requirement; otherwise increase Ka;
end
```

To compute the gain and phase margin for the SISO open-loop model SYS (continuous or discrete) one can use the following Matlab command:

```
[Gm, Pm] = margin(SYS); % Gm is the gain margin, Pm is the phase margin.
```

The gain margin in dB is derived by

```
Gm_dB = 20*log10(Gm);
```

The second way is to plot the phase margin of the open-loop system as a function of K_a and then utilize "look-up graph" method to determine the appropriate value of K_a.

```
Ka_test = linspace(350,400,100);
for i = 1:length(Ka_test)
    - calculate the (new) open-loop transfer function;
    - calculate the phase margin;
end
figure;
plot(Ka_test, Pm);
```

Using this method one should obtain the results plotted in Figure 19. Based on this relation it is easy to estimate the suitable value of K_a to obtain the desired phase margin.

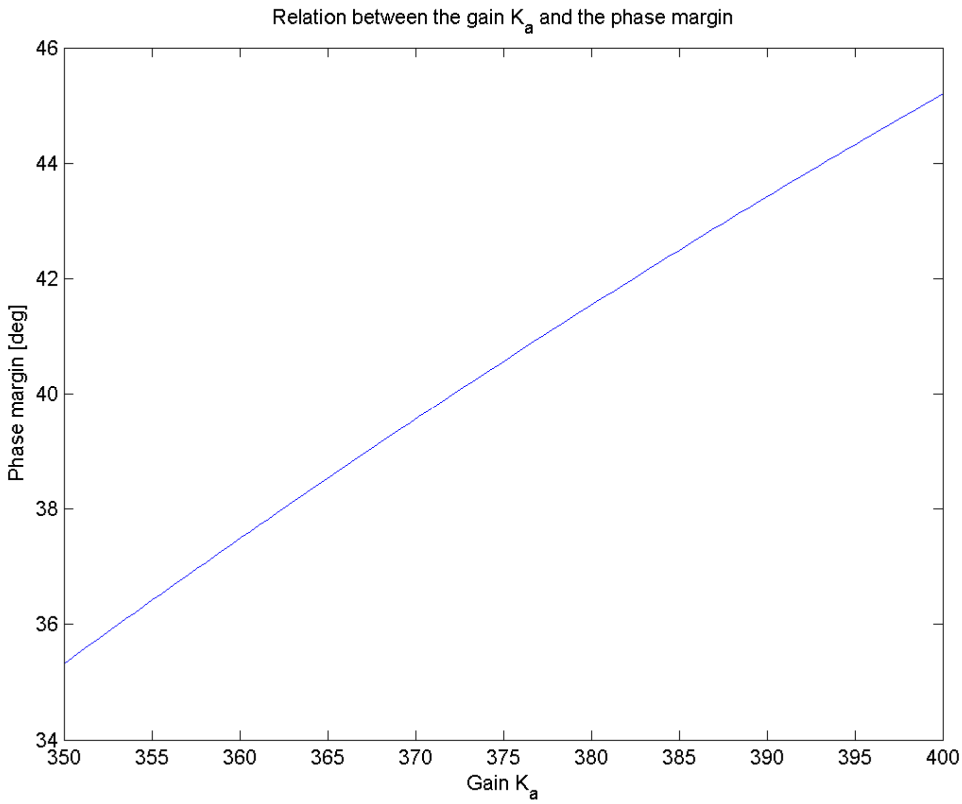


Figure 19. Relation between the gain K_a and the phase margin φ_m.

Another method is to use the root locus technique (`rlocus` and `rlocfind`) in combination with the open-loop Bode plot, as previously presented in the instruction for Exercise 1. The gain K_a is manually tuned until obtaining the desired phase margin. This way is thus time-consuming.

You might find a better solution. Let show it.

As a result, you should obtain the gain $K_a \cong 382.4$ together with the phase margin $\varphi_m = 42^0$ and the gain margin $k_m = 0.55$ (i.e. -5.18 dB). In comparison with the practical requirement $k_m > 2$ for the gain margin one can see that the obtained gain margin is quite small.

The open-loop Bode plot with $K_a = 382.4$ is shown in Figure 20. It looks like that we have a conditionally stable system.

6. For exercise (3b) the open-loop transfer function is

$$G(s) = G_c(s)G_p(s) = \frac{10K_a\left(5s + 500 + 0.0475s^2\right)}{s^3} \tag{11}$$

The open-loop root locus plot with $K_a = 382.4$ is shown in Figure 21. The Nyquist plot (being enlarged) is shown in Figure 22. These plots have confirmed that the system is conditionally stable.

Based on the Nyquist plot in Figure 22, is the closed-loop system stable or unstable? Explain.

The root locus plot is used to analyse the negative feedback loop and show the trajectory of the closed-loop poles when the feedback gain K_a varies from 0 to ∞. The main idea of root locus design is to predict the closed-loop response from the open-loop root locus plot. Then by adding zeros and/or poles via the compensator (or controller) design, the root locus can be modified in order to achieve a desired closed-loop response.

It should be noted that the Bode, Nyquist and root locus plots are created for the open-loop system.

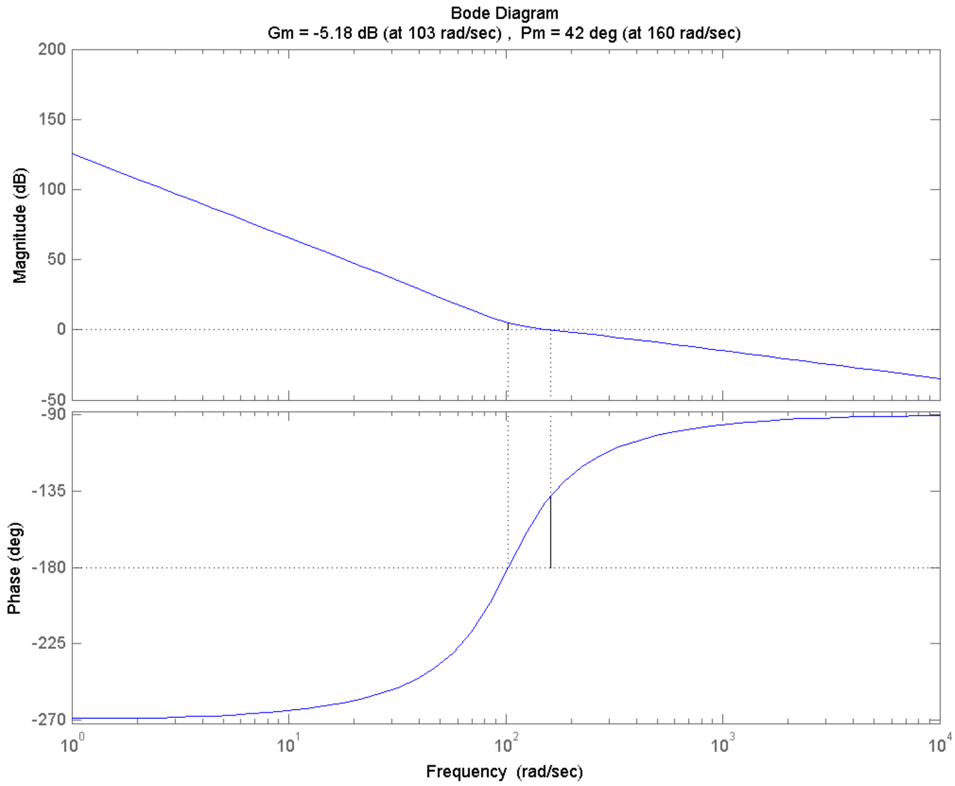


Figure 20. Bode plot of the open-loop system with $K_a = 382.4$.

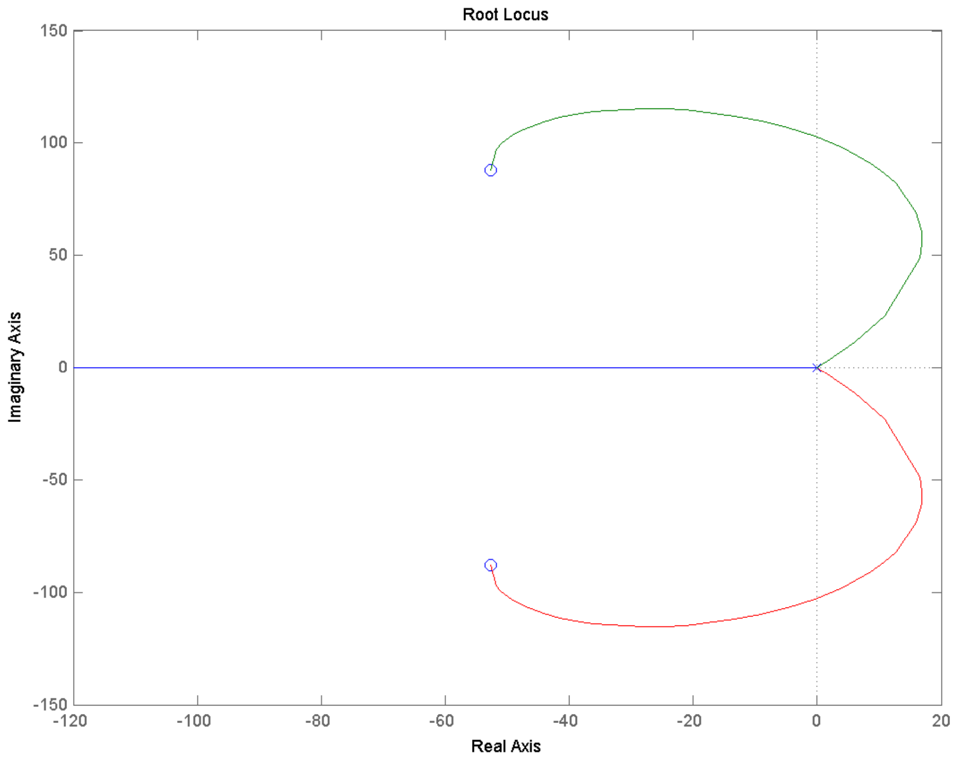


Figure 21. Root locus plot of the open-loop system with $K_a = 382.4$.

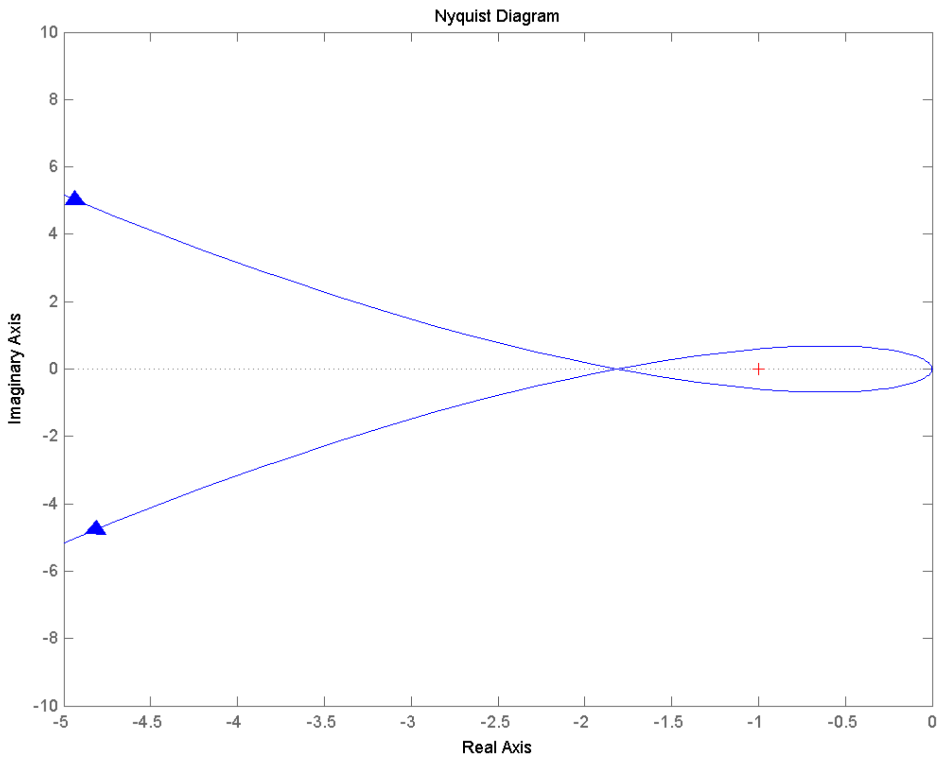


Figure 22. Nyquist plot of the open-loop system with $K_a = 382.4$.

The closed-loop system poles are

$$s_1 = -141.87 \ ; \ s_{2,3} = -19.89 \pm j114.38$$

These poles can be read from the root locus of the closed-loop system with $K_a = 382.4$ (see Figure 23). One could also use the Matlab command `eig` to obtain the results.

7. For exercise (3c) the transfer function from $D(s)$ to $Y(s)$ is

$$\frac{Y(s)}{D(s)} = \frac{-s}{s^3 + 181.6s^2 + 19120s + 1912000} \tag{12}$$

Notice that the disturbance $D(s)$ has a negative sign input to the system.

The maximum value of $y(t)$ can be obtained from the unit step disturbance response in Figure 24.

$$\max|y(t)| \cong 3.79 * 10^{-5} = 0.0000379$$

8. For exercise (3d), to understand about the prefilter design, you are referred to Section 10.10 in [2]. In general, the response on reference changes can be improved by means of a prefilter.

Following the design procedure you will obtain the transfer function of the prefilter:

$$G_p(s) = \frac{1912000}{181.6s^2 + 19120s + 1912000} \tag{13}$$

Step responses of the closed-loop system (with $K_a = 382.4$) in case with and without a prefilter is plotted in Figure 25.

Based on the instructions provided you are now required to elaborate and discuss in the report your solutions with Matlab code and results obtained in detail.

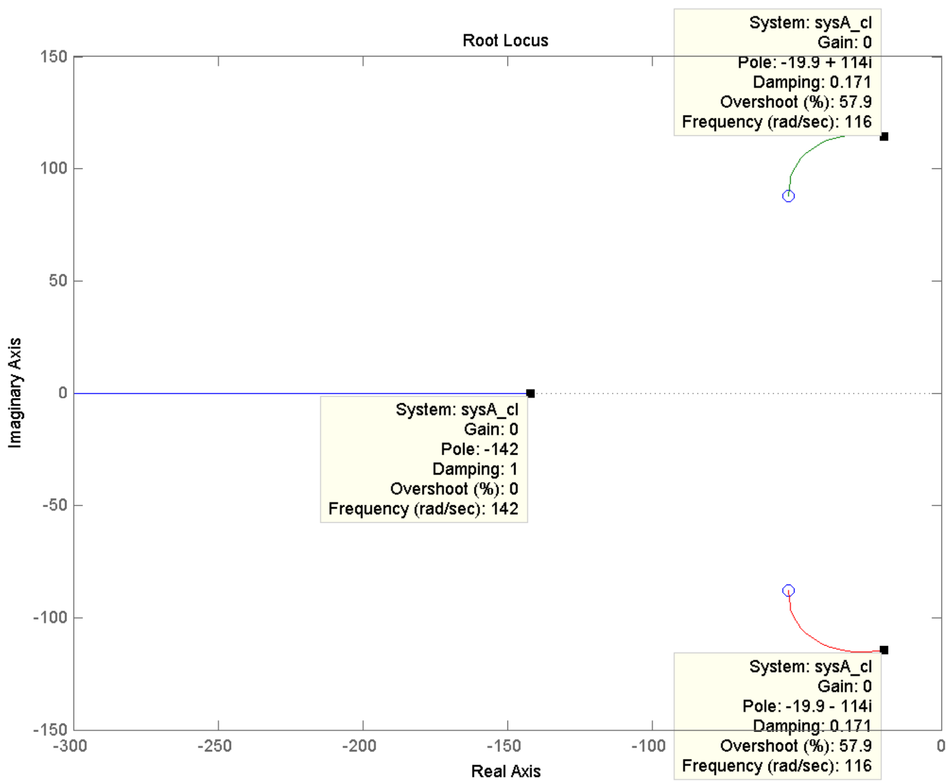


Figure 23. Root locus of the closed-loop system with $K_a = 382.4$.

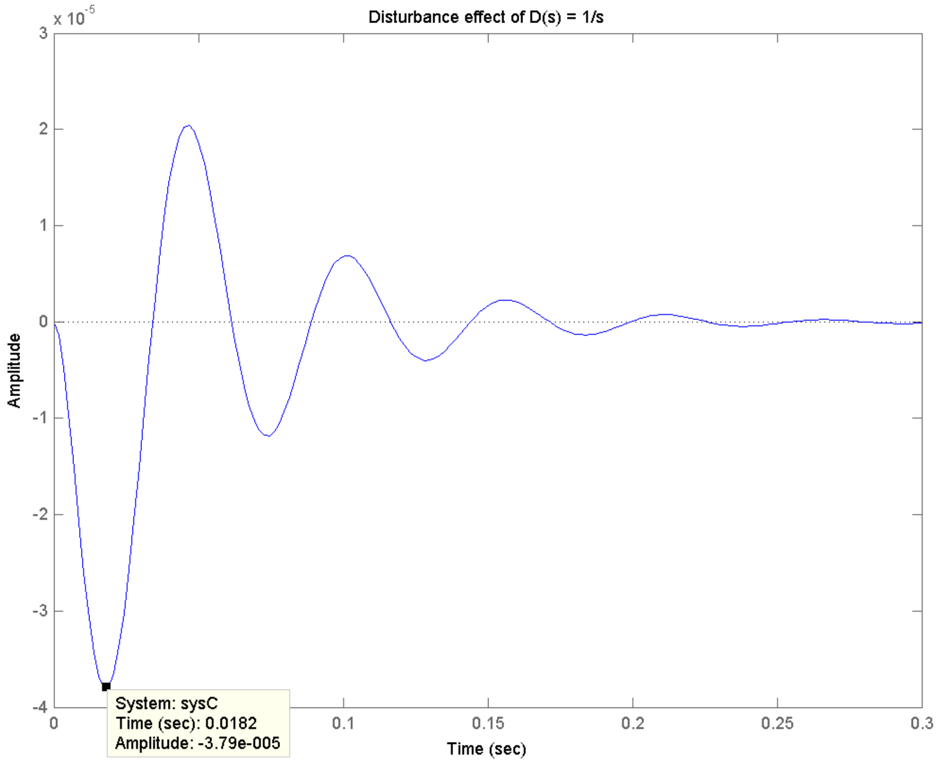


Figure 24. Disturbance effect of D(s) = 1/s when $R(s) = 0$ and $K_a = 382.4$.

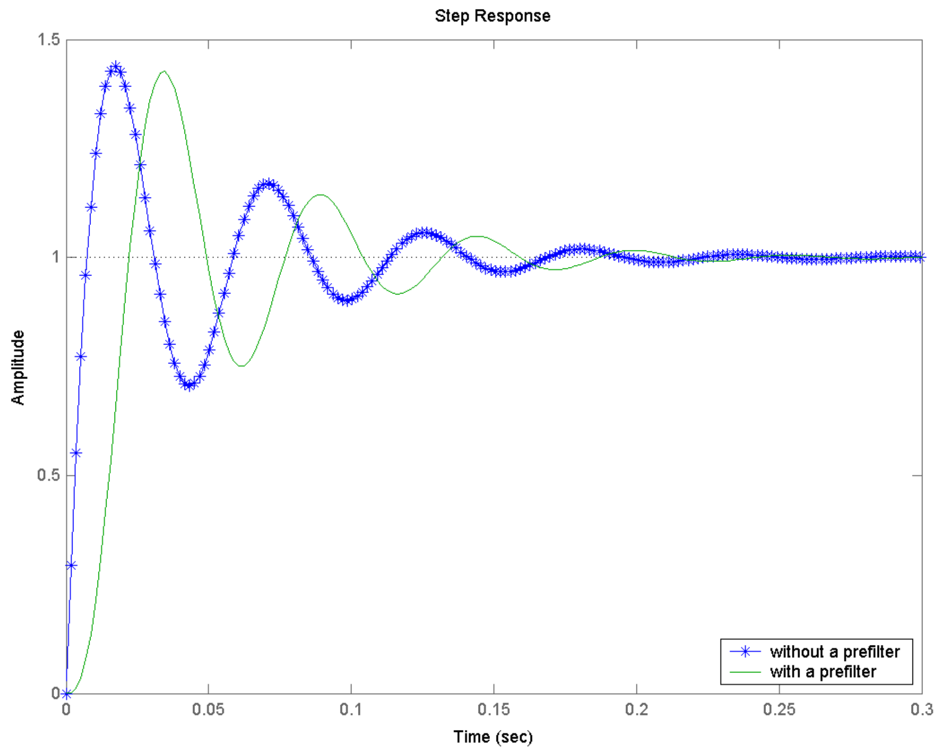


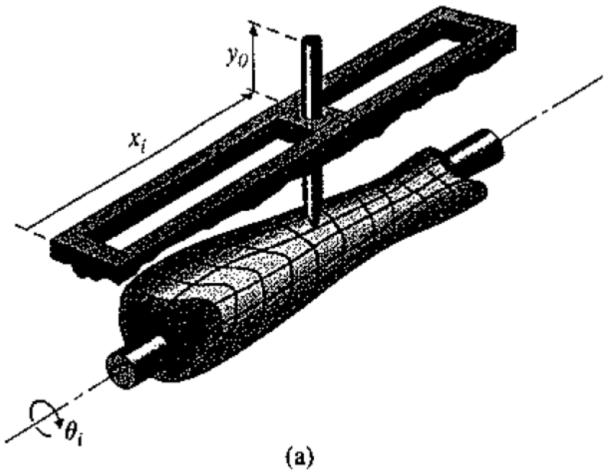
Figure 25. Step responses of the closed-loop system in case with and without a prefilter (with $K_a = 382.4$).

Exercise 4

A three-dimensional cam for generating a function of two variables is shown in Figure 26a. Both x and θ may be controlled using a position control system. The control of x can be achieved with a DC motor and position feedback of the form in Figure 26b, with the DC motor and load represented by:

$$G(s) = \frac{K}{s(s+p)(s+4)}, \quad \text{where } 1 \le K \le 3 \text{ and } 1 \le p \le 3\text{; normally } K = 2 \text{ and } p = 2.$$

1. Design a robust PID control system with a prefilter based on the Integral Time-weighted Absolute Error (ITAE) performance index (or criterion) so that the peak time response to a step input is less than 2.5 seconds and the maximum peak overshoot is less than 2% for the worst-case performance.
2. To evaluate if the designed ITAE-based control system achieves robust behaviour (i.e. still attaining the peak time response to a step input is less than 2.5 seconds and the maximum peak overshoot is less than 2%) while the plant $G(s)$ varies freely in the range indicated above, it is advised to test the designed robust PID controller with the prefilter $G_p(s)$ for the four special cases: $K=1, p=1$; $K=1, p=3$; $K=3, p=1$; $K=3, p=3$). Is the control system that you have designed robust in all cases. Discuss the results obtained in detail.



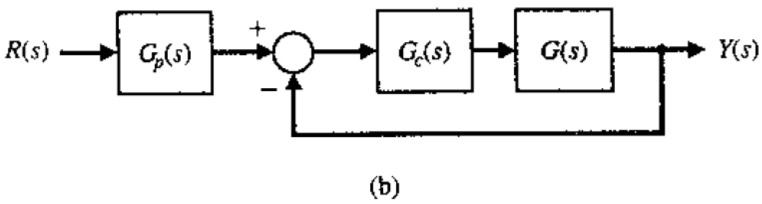


Figure 26.

Instructions:

First of all, one should use the Routh-Hurwitz criterion to check the stability of the plant under nominal conditions (i.e. when $K = 2$ and $p = 2$) and obtain

$$G_{no}(s) = \frac{2}{s(s+2)(s+4)} \Rightarrow \text{the nominal characteristic equation } q_{no}(s) = s^3 + 6s^2 + 8s + 2 = 0.$$

The Routh array is

$$\begin{array}{lll} s^3 & 1 & 8 \\ s^2 & 6 & 2 \\ s^1 & 46/6 & \\ s^0 & 2 & \end{array}$$

Therefore we know that this system is nominally stable.

Next, we can perform the design procedure in the sequence of four steps as follows:

Step 1: Follow Section 12.4 in [2] to determine the four polynomials $q_{1\ldots4}(s)$ of the system with known coefficients/parameters within bounds (i.e. $1 \le K \le 3$ and $1 \le p \le 3$).

We first consider the closed-loop system without prefiltering (i.e. $G_p(s) = 1$). Let $G_c(s) = 1$ then one will obtain the characteristic equation

$$q(s) = s^3 + (4+p)s^2 + 4ps + K = 0 \tag{14}$$

This has the form of a third-order system with the characteristic polynomial $s^3 + a_2 s^2 + a_1 s + a_0 = 0$.

Equating coefficients then uncertain coefficients can be found:

$$\begin{array}{lll} 1 \le a_0 = K \le 3 & \Rightarrow & \alpha_0 = 1, \beta_0 = 3 \\ 4 \le a_1 = 4p \le 12 & \Rightarrow & \alpha_1 = 4, \beta_1 = 12 \\ 5 \le a_2 = 4 + p \le 7 & \Rightarrow & \alpha_2 = 5, \beta_2 = 7 \end{array} \tag{15}$$

According to Section 12.4 in [2] we need to determine the four polynomials of the form:

$$\begin{aligned} q_1(s) &= s^3 + \alpha_2 s^2 + \beta_1 s + \beta_0 \\ q_2(s) &= s^3 + \beta_2 s^2 + \alpha_1 s + \alpha_0 \\ q_3(s) &= s^3 + \beta_2 s^2 + \beta_1 s + \alpha_0 \\ q_4(s) &= s^3 + \alpha_2 s^2 + \alpha_1 s + \beta_0 \end{aligned} \tag{16}$$

Substitute uncertain coefficients from Equation (15) to Equation (16) then we obtain the following four polynomials:

$$\begin{aligned} q_1(s) &= s^3 + 5s^2 + 12s + 3 \\ q_2(s) &= s^3 + 7s^2 + 4s + 1 \\ q_3(s) &= s^3 + 7s^2 + 12s + 1 \\ q_4(s) &= s^3 + 5s^2 + 4s + 3 \end{aligned} \tag{17}$$

We can also create transfer functions $G_{1\ldots4}(s)$ from polynomials $q_{1\ldots4}(s)$.

$$\begin{aligned} q_1(s) &= s^3 + 5s^2 + 12s + 3 \Rightarrow G_1(s) = \frac{3}{s^3 + 5s^2 + 12s} \\ q_2(s) &= s^3 + 7s^2 + 4s + 1 \Rightarrow G_2(s) = \frac{1}{s^3 + 7s^2 + 4s} \\ q_3(s) &= s^3 + 7s^2 + 12s + 1 \Rightarrow G_3(s) = \frac{1}{s^3 + 7s^2 + 12s} \\ q_4(s) &= s^3 + 5s^2 + 4s + 3 \Rightarrow G_4(s) = \frac{3}{s^3 + 5s^2 + 4s} \end{aligned} \tag{18}$$

Step 2: Determine the polynomial $q_w(s)$ from $q_{1...4}(s)$ (or the transfer function $G_w(s)$ from $G_{1...4}(s)$) that represents the worst case (e.g. exhibiting unstable behaviour or presenting the worst performance such as the greatest overshoot, the longest settling time, or the slowest peak time response).

The analysis of the four polynomials given in Equation (17) is sufficient to ascertain the stability of the given system. At the first attempt, it is advised to examine the stability of these four polynomials using the Routh-Hurwitz criterion to find the worst-case polynomial that may indicate unstable performance.

The Routh array for $q_{1...4}(s)$ from left to right has the form:

$$\begin{array}{lcc} s^3 & 1 & 12 \\ s^2 & 5 & 3 \\ s^1 & 57/5 & \\ s^0 & 3 & \end{array} \qquad \begin{array}{lcc} s^3 & 1 & 4 \\ s^2 & 7 & 1 \\ s^1 & 27/7 & \\ s^0 & 1 & \end{array} \qquad \begin{array}{lcc} s^3 & 1 & 12 \\ s^2 & 7 & 1 \\ s^1 & 83/7 & \\ s^0 & 1 & \end{array} \qquad \begin{array}{lcc} s^3 & 1 & 4 \\ s^2 & 5 & 3 \\ s^1 & 17/5 & \\ s^0 & 3 & \end{array}$$

Therefore we know that the system is stable for all the range of uncertain parameters.

Since all four polynomials are stable so that we need to find the polynomial (or transfer function) that represents the worst-case performance for the given system. A good approach is to plot and investigate step responses of the four closed-loop systems created with transfer functions $G_{1...4}(s)$. The results are shown in Figures 27 and 28. Table 1 presents a summary of system performances obtained for four cases for comparison. One can easily notice that the closed-loop system using the transfer function $G_3(s)$ represents the worst performance, because of its longest peak time, rise time and setting time. Therefore we have found that $G_w(s) = G_3(s)$.

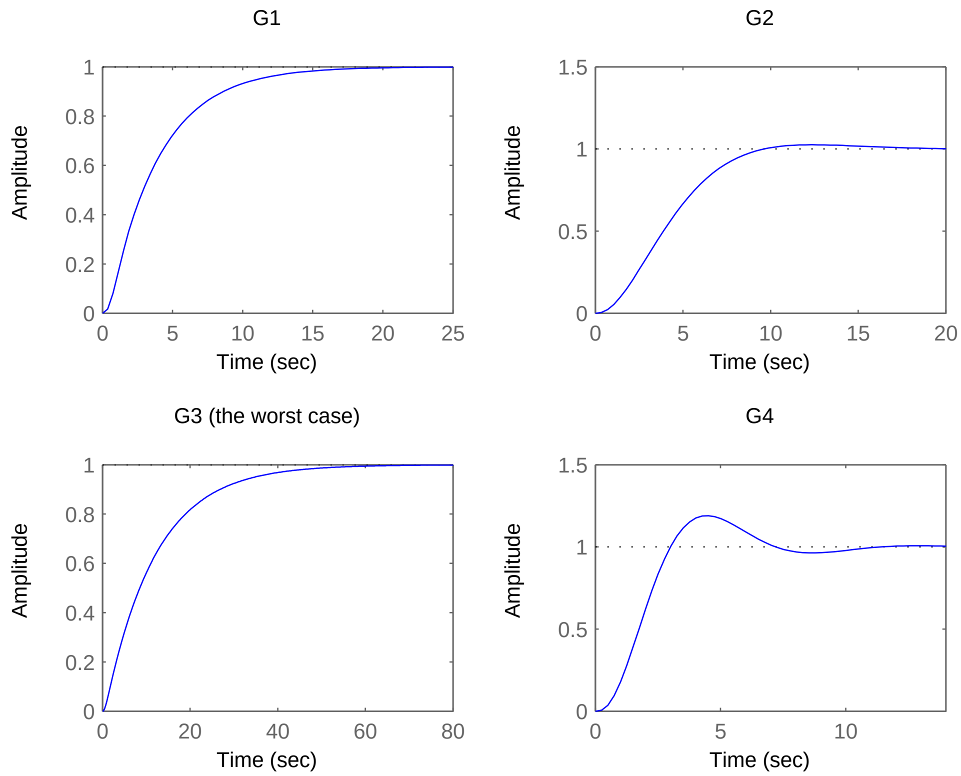


Figure 27. Step responses of the four closed-loop systems created with transfer functions $G_{1...4}(s)$.

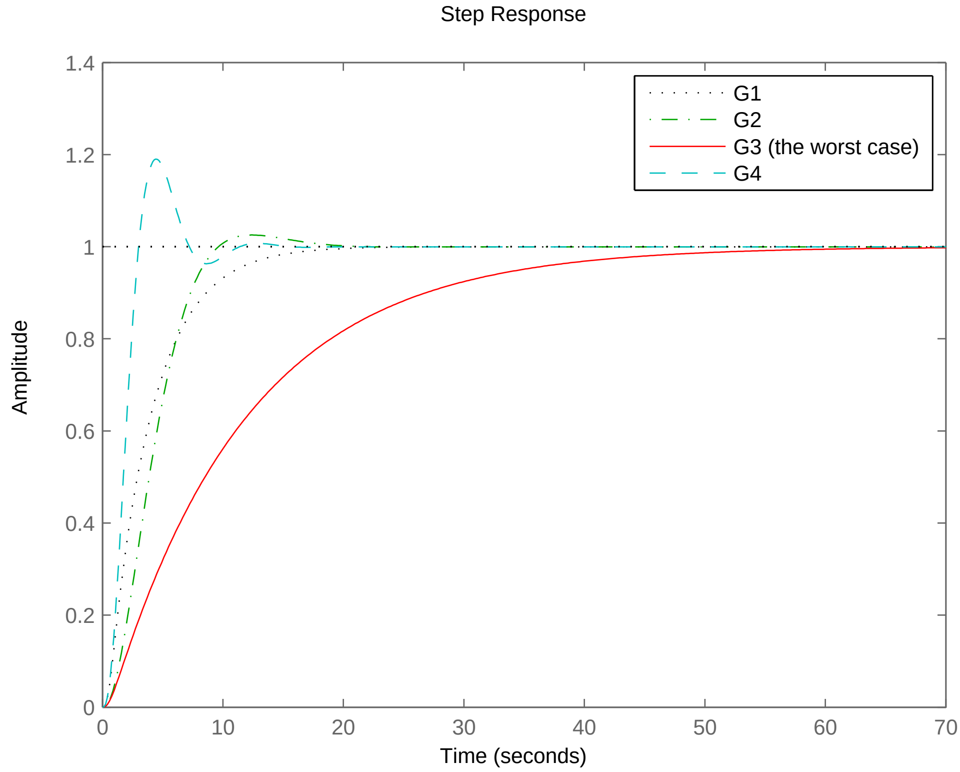


Figure 28. Comparison of the step responses shown in Figure 27.

Transfer function	Peak time [s]	Overshoot [%]	Rise time [s]	Settling time [s]
$G_1(s)$	37.32	0	7.79	14.36
$G_2(s)$	12.34	2.54	5.88	14.41
$G_3(s)$	**90.60**	0	**25.04**	**45.18**
$G_4(s)$	4.51	**18.99**	1.91	10.08

Table 1. Summary of the step response characteristics for four closed-loop systems using $G_{1...4}(s)$.

Step 3: Follow Section 12.7 in [2] to design a robust PID controller for the worst-case plant – presented by the transfer function $G_w(s) = G_3(s)$ – using the ITAE criterion. You are referred to Section 5.9 in [2]. The optimum coefficients given in Table 5.6 on page 259 for a step input should be employed for the design. Three parameters of the PID controller will be selected to minimize the ITAE performance index, which produces an excellent response to a step input (as shown in Figure 5.30c on page 261 in [2]).

The robust PID controller design procedure for the worst-case plant $G_3(s)$ consists of three sub-steps:

1. Step 3(a): Select the natural frequency ω_n of the closed-loop system based on the required performance such as the settling time (T_s), peak time (T_p), damping ratio (ξ), and percent overshoot.
2. Step 3(b): Determine the three PID controller parameters using the appropriate optimum equation (Table 5.6, page 259) and the ω_n from the previous step to obtain the controller $G_c(s)$.

3. Step 3(c): Determine a prefilter $G_p(s)$ so that the closed-loop transfer function $T(s)$ does not have any zeros, as required by Equation (19).

$$T(s) = \frac{Y(s)}{R(s)} = \frac{b_0}{s^2 + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0} \tag{19}$$

It should be noted that this transfer function $T(s)$ has a steady-state error equal to zero for a step input.

Here, one critical question is how to select/find the natural frequency ω_n? There are two solutions:

- In case if ω_n is selected in order to meet the settling time requirement then one can use the formula $T_s = 4/\xi\omega_n$ with the optimum value of the damping ratio ξ chosen on the basis of ITAE is 0.7 (see Figure 29). For example, if we want to obtain a settling time of less than 0.5 second (i.e. $T_s \le 0.5$) then with $\xi = 0.7$ we have $T_s = \dfrac{4}{\xi\omega_n} \le 0.5 \Rightarrow \xi\omega_n \ge 8 \Rightarrow \omega_n \ge \dfrac{8}{0.7} = 11.42$.
- The second way is to use the step responses from Figure 5.30c in [2] (which is also shown in Figure 30) for ITAE systems. The responses are provided for normalized time ($\omega_n t$). Based on the given order of the ITAE system ($n = 2,3\ldots,6$) the corresponding plot should be used. If ω_n is selected in order to meet a desired settling time (T_s) then one can use the approximation $\omega_n T_s \approx N$, where N is the value obtained from the relevant plot. By analogy, the approximation $\omega_n T_p \approx N$ is used for the peak time (T_p) requirement. For example, for a third-order system (i.e. $n = 3$) we can estimate from Figure 30 that $\omega_n T_s \approx 8$ and $\omega_n T_p \approx 4$. Then ω_n can be found based on the requirement for T_s or T_p.

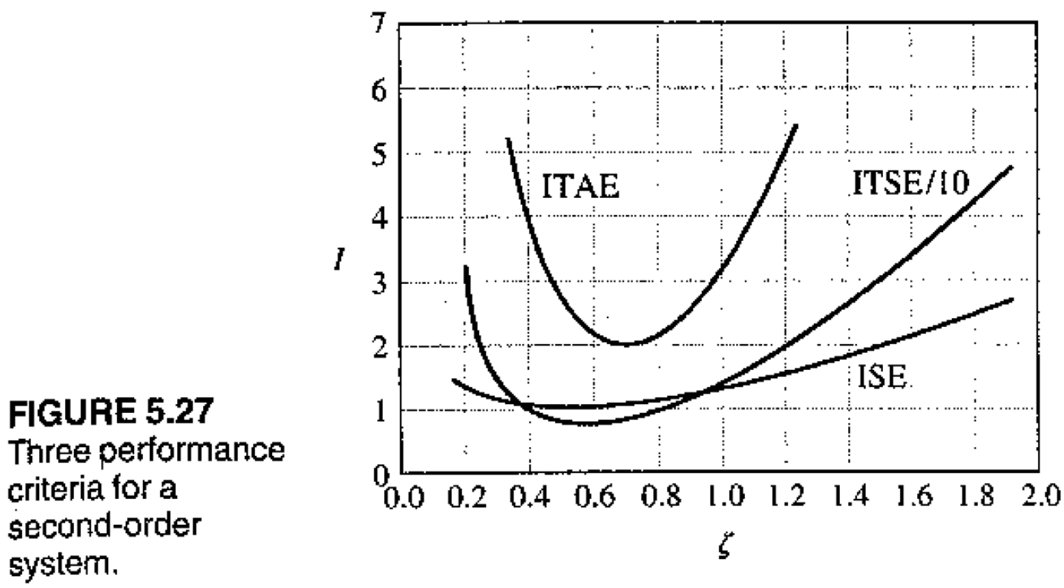


Figure 29. Three performance indices (ISE, ITSE, and ITAE) calculated for various values of the damping ratio ξ and for a step input [2].

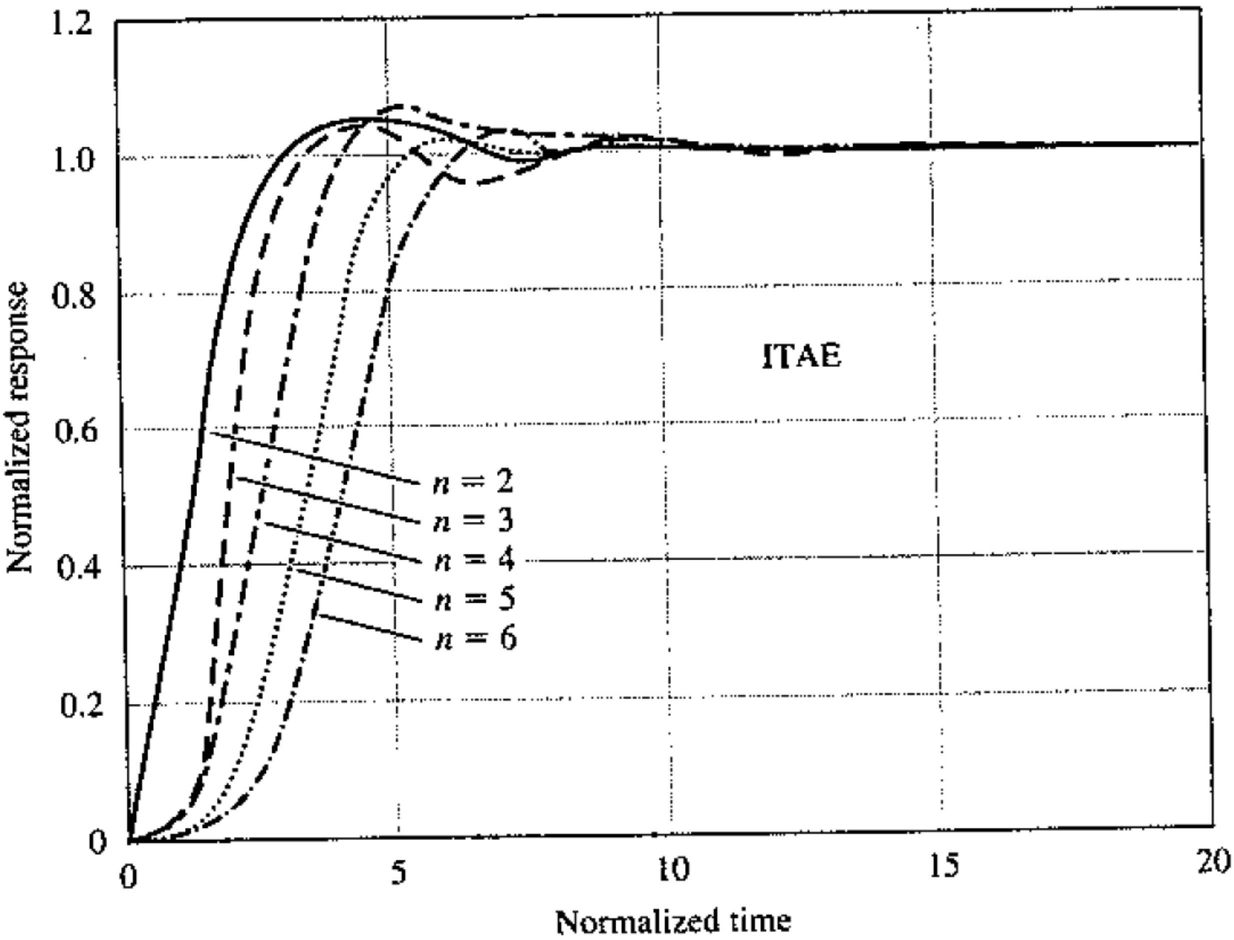


Figure 30. Step responses of a normalized transfer function using optimum coefficients for ITAE. The responses are for normalized time ($\omega_n t$) [2].

The PID controller is given by

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s} \tag{20}$$

Therefore the closed-loop transfer function without a prefilter (i.e. $G_p(s) = 1$) is

$$T_1(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_w(s)}{1 + G_c(s)G_w(s)} = \frac{\dfrac{K_D s^2 + K_P s + K_I}{s^4 + 7s^3 + 12s^2}}{1 + \dfrac{K_D s^2 + K_P s + K_I}{s^4 + 7s^3 + 12s^2}} = \frac{K_D s^2 + K_P s + K_I}{s^4 + 7s^3 + (12 + K_D)s^2 + K_P s + K_I} \tag{21}$$

where $G_w(s) = G_3(s) = \dfrac{1}{s^3 + 7s^2 + 12s}$.

Equation (21) shows that we have the 4th-order characteristic polynomial. We need to select ω_n in order to meet the requirement that the peak time response to a step input is less than 2.5 seconds for the worst-case performance. Using the step response in Figure 30 for the fourth-order ITAE system (i.e. $n = 4$) we obtain the approximation $\omega_n T_p \approx 5.5$. With the required performance $T_p \le 2.5 \Rightarrow \omega_n \ge 5.5/2.5 = 2.2$. As a result, we may select $\omega_n = 2.5$ [rad/s] and move to the next step.

It is important to note here that the choice of ω_n strongly influences the system performance. A (too) large or (too) small value of ω_n could make the system less robust and may even become unstable.

Using the ITAE method (with the optimum coefficients given in Table 5.6 on page 259 in [2]), we desire the characteristic polynomial to be

$$q(s) = s^4 + 2.1\omega_n s^3 + 3.4\omega_n^2 s^2 + 2.7\omega_n^3 s + \omega_n^4 \tag{22}$$

Equating the denominator of Equation (21) to Equation (22) and solving for the controller gains using $\omega_n = 2.5$ [rad/s] we obtain:

$$\begin{aligned} &12 + K_D = 3.4\omega_n^2 \Rightarrow K_D = 3.4 * 2.5^2 - 12 = 9.25 \\ &K_P = 2.7\omega_n^3 \Rightarrow K_P = 2.7 * 2.5^3 = 42.18 \\ &K_I = \omega_n^4 \Rightarrow K_I = 2.5^4 = 39.06 \end{aligned} \tag{23}$$

As a result, we obtain the PID controller

$$G_c(s) = 42.18 + \frac{39.06}{s} + 9.25s = \frac{9.25s^2 + 42.18s + 39.06}{s} \tag{24}$$

Then Equation (21) becomes

$$T_1(s) = \frac{9.25s^2 + 42.18s + 39.06}{s^4 + 7s^3 + 21.25s^2 + 42.18s + 39.06} \tag{25}$$

Step response of the closed-loop system without a prefilter for the worst-case plant is shown in Figure 31. This response has a big overshoot of 54.6%. Hence, we need to select a prefilter $G_p(s)$ in order to achieve the desired optimum ITAE response, where the closed-loop transfer function $T(s)$ will not have any zeros, as required by Equation (19), i.e.

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_w(s)G_p(s)}{1 + G_c(s)G_w(s)} = \frac{39.06}{s^4 + 7s^3 + 21.25s^2 + 42.18s + 39.06} \tag{26}$$

From Equations (21) and (26) we have

$$\begin{aligned} &T(s) = T_1(s)G_p(s) \\ &\Rightarrow \frac{39.06}{s^4 + 7s^3 + 21.25s^2 + 42.18s + 39.06} = \frac{K_D s^2 + K_P s + K_I}{s^4 + 7s^3 + 21.25s^2 + 42.18s + 39.06} G_p(s) \end{aligned} \tag{27}$$

Hence we expect to have

$$G_p(s) = \frac{K_I}{K_D s^2 + K_P s + K_I} = \frac{39.06}{9.25s^2 + 42.18s + 39.06} \tag{28}$$

in order to eliminate the zeros in Equation (25) and bring the overall numerator to 39.06.

Step response of the closed-loop system with the prefilter $G_p(s)$ for the worst-case plant is shown in Figure 32. This response has a smaller overshoot of 10.3%, but it is still much larger than the desired value of 2%. The peak time requirement is fairly met, because $T_p = 2.24 < 2.5$.

<u>Remark</u>: On equating the denominator of Equation (21) to Equation (22) we notice that ω_n could be calculated directly from $\omega_n = 7/2.1 = 3.33$ [rad/s]. This value satisfies the condition $\omega_n \geq 2.2$, which was found using the peak time requirement. However, this straightforward calculation does not exist in every cases so that it is advised to use the required performance such as the settling time (T_s) or peak time (T_p) to calculate ω_n.

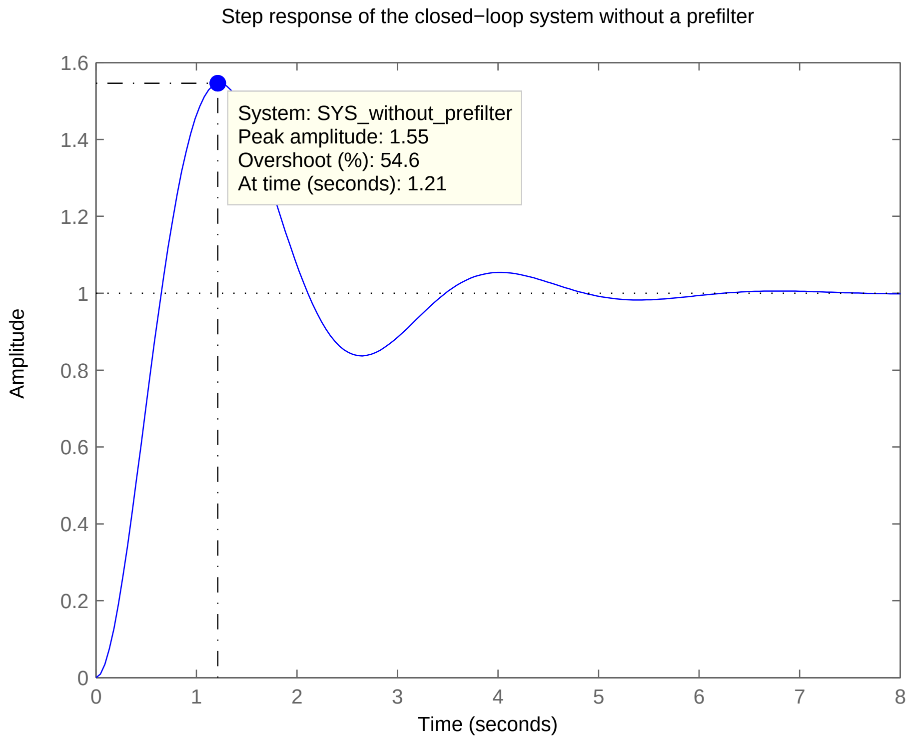


Figure 31. Step response of the closed-loop system without a prefilter for the worst-case plant $G_w(s) = G_3(s)$. This response is for $\omega_n = 2.5$ [rad/s].

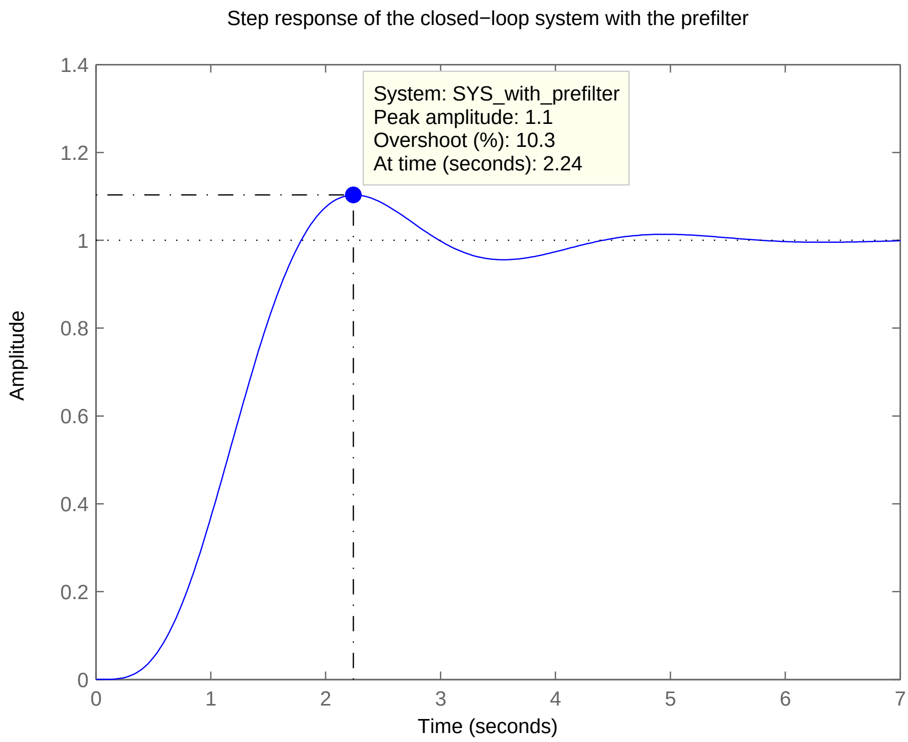


Figure 32. Step response of the closed-loop system with the prefilter $G_p(s)$ for the worst-case plant $G_w(s) = G_3(s)$. This response is for $\omega_n = 2.5$ [rad/s].

To meet both two design requirements we need to come back to Step 3(a) to select another value of the natural frequency ω_n (where $\omega_n \geq 2.2$) and then repeat the whole design procedure described in Step 3. After a number of iterations an optimum value of ω_n has been achieved that gives the results shown in Figure 33. With the peak time $T_p = 1.57 < 2.5$ seconds and the maximum peak overshoot is equal to 1.38%, we can conclude that both two design requirements are well met for the worst-case plant.

Figure 34 shows the step responses of the closed-loop system with the prefilter $G_p(s)$ for the worst-case plant $G_w(s) = G_3(s)$ and for the nominal plant (when $K = 2$; $p = 2$) using the optimum value of ω_n.

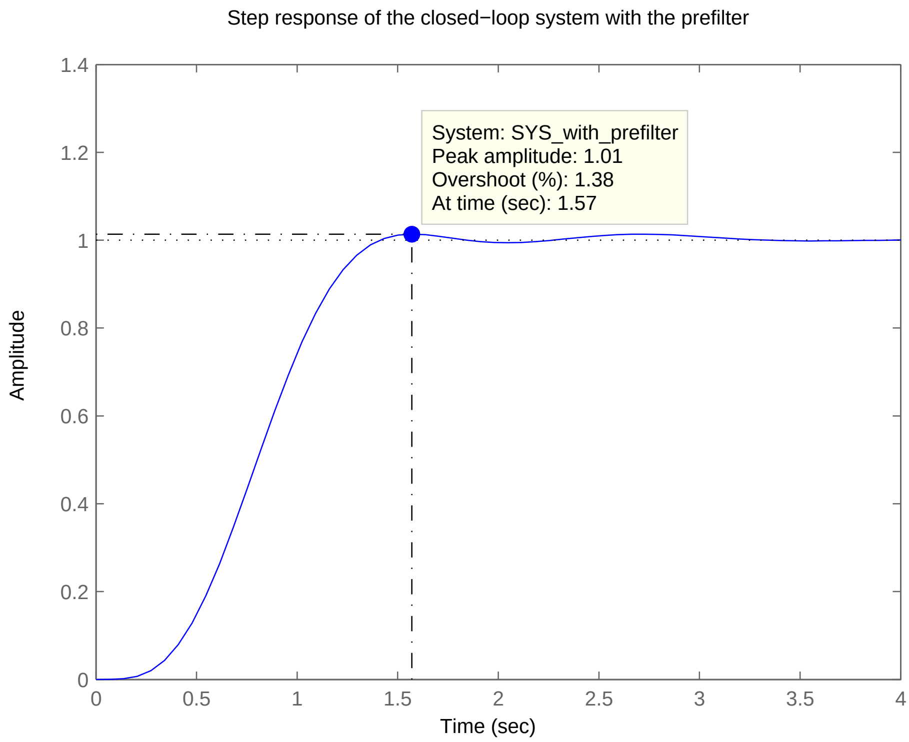


Figure 33. Step response of the closed-loop system with the prefilter $G_p(s)$ for the worst-case plant $G_w(s) = G_3(s)$ using the optimum value of ω_n.

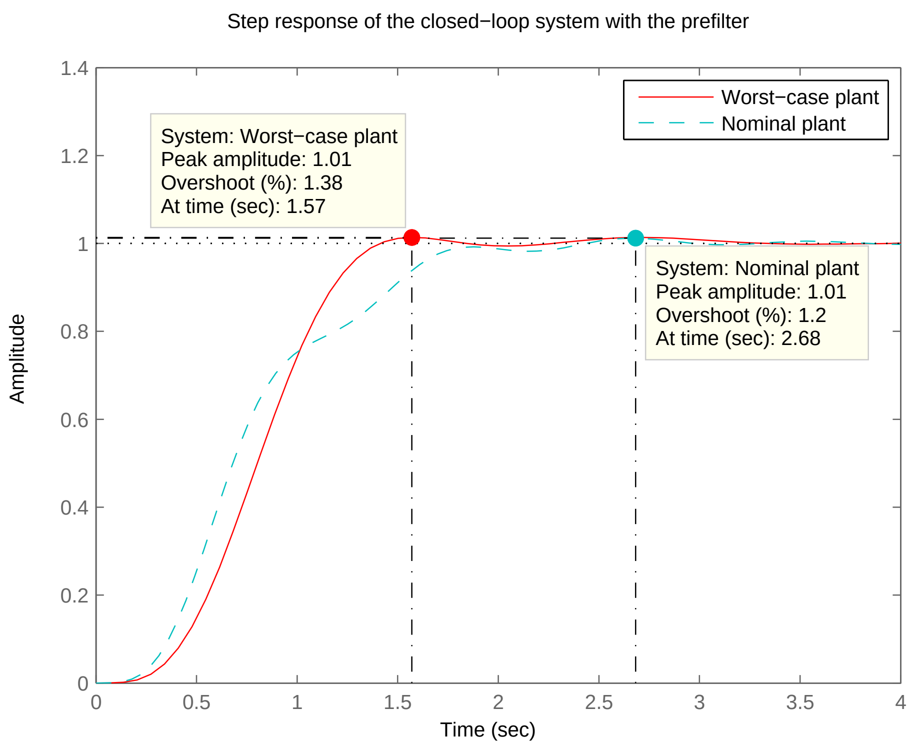


Figure 34. Step responses of the closed-loop system with the prefilter $G_p(s)$ for the worst-case plant $G_w(s) = G_3(s)$ and the nominal plant (when $K = 2$; $p = 2$) using the optimum value of ω_n .

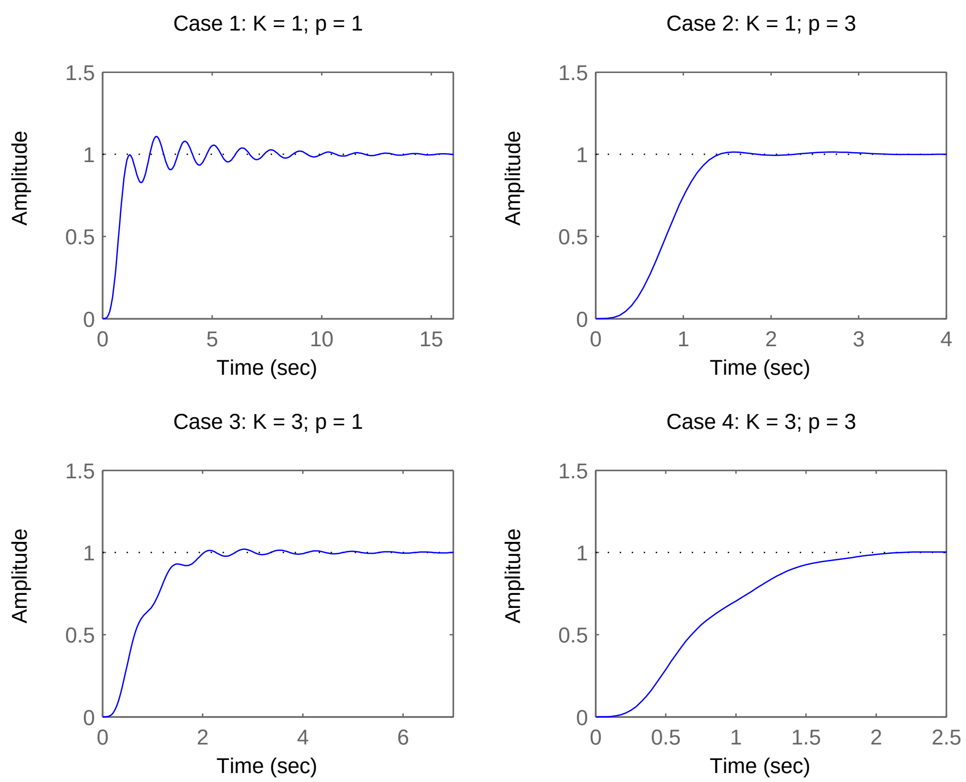


Figure 35. Step responses of the closed-loop system with the prefilter $G_p(s)$ for the four special cases.

Step 4: Test the designed robust PID controller with the prefilter $G_p(s)$ for the four special conditions ($K=1$, $p=1$; $K=1$, $p=3$; $K=3$, $p=1$; $K=3$, $p=3$) to see if the designed control system achieves robust behaviour (i.e. still attaining the peak time response to a step input is less than 2.5 seconds and the maximum peak overshoot is less than 2%) while the plant $G(s)$ varies in the range indicated.

Figure 35 shows the step responses of the closed-loop system with the prefilter for the four special cases. A summary of the step response characteristics for these four investigated cases is given in Table 2. We notice that the overshoot requirement is not satisfied for Case 1, whereas the peak time for Cases 3 and 4 is a bit larger than the desired value of 2.5 seconds. Besides these drawbacks the designed control system has achieved system stability and robustness characteristics for all four cases, as shown in Figure 35, together with the desired performances (peak time and percent overshoot) are well met.

Transfer function	Peak time [s]	Overshoot [%]	Rise time [s]	Settling time [s]
Case 1: $K=1$, $p=1$	2.48	**10.85**	0.61	8.44
Case 2: $K=1$, $p=3$	1.56	1.37	0.73	1.33
Case 3: $K=3$, $p=1$	**2.82**	1.99	1.02	2.51
Case 4: $K=3$, $p=3$	**2.87**	0.55	1.06	1.91

Table 2. Summary of the step response characteristics for the four special cases.

Based on the instructions provided you are now required to write a Matlab program that performs the robust PID control system design procedure with the prefilter based on the ITAE criterion, which are described in Step 3 and Step 4. You should elaborate and discuss in the report your solutions with Matlab code and results obtained in detail.

Your Matlab program must be included in the report.