Control Theory (RMS-1-612-s ; RIMA-1-612-s)

Exercises

Part 1 – Modelling of Dynamic Systems and Linearization of Nonlinear Systems

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Phong Ba Dao, dr hab. inż., prof. AGH
Department of Robotics and Mechatronics
AGH University of Science and Technology

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Remarks:

(1) Students are required to complete the following exercises **without using MATLAB/Simulink**.

(2) Short instructions are provided to support you in solving exercises. You should read the instructions carefully. Although some parts of the exercises were explained and guided in the instructions but they are final results or not yet complete. **Therefore, you are required to elaborate in the report your solutions and results obtained in detail for all highlighted paragraphs**.

(3) For exercises 1.2 and 1.3, you are recommended to read the following book (chapters 3 & 6), which is available in the "Textbooks" directory of the "Course materials" link.

David Morin, *Introduction to Classical Mechanics with Problems and Solutions*, first edition, Cambridge University Press, 2008.

(4) For exercises 1.4 and 1.5, you are referred to documents given in the "Linearization of nonlinear systems" directory of the "Course materials" link.

(5) The exercise report must be typed on a word-processor and submitted as a single PDF file via **UPeL platform**.

1.1. DC motor

Consider a DC motor shown in Figure 1. The rotor and the shaft are assumed to be rigid. The input is the armature voltage V in [volts] (i.e. the motor is driven by a voltage source). The measured output variables are the shaft angle θ in [radians] and the angular velocity of the shaft ω in [radians per second].

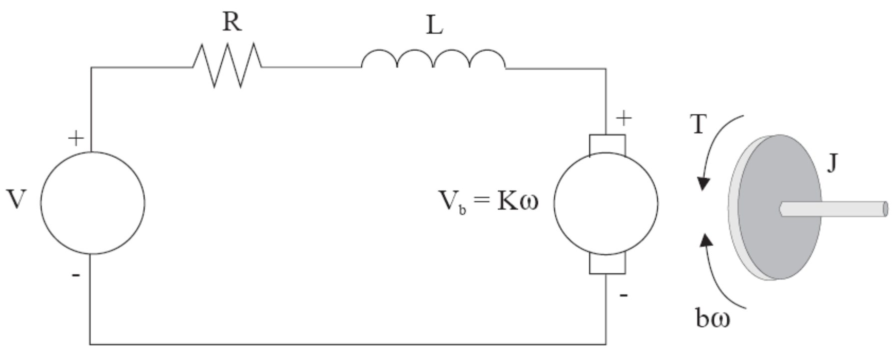

Figure 1. Schematic representation of the considered DC motor.

The physical parameters of the motor are given below:

moment of inertia of the rotor:	$J = 0.02\ kgm^2$
damping (friction) of the mechanical system:	$b = 0.05\ Nms$
(back-)electromotive force constant:	$K = 0.02\ Nm/A$
electric resistance:	$R = 1\ \Omega$
electric inductance:	$L = 0.03\ H$

Questions: (1) develop a linear mathematical model for the DC motor in terms of transfer functions from the input voltage $V(s)$ to the output angle $\theta(s)$ and from the input voltage $V(s)$ to the output angular velocity $\omega(s)$; (2) draw/sketch a block diagram of the DC motor.

Instructions:

The motor torque T is related to the armature current i by the constant factor K by

$$T = Ki \tag{1}$$

The back electromotive force (emf) V_b is related to the angular velocity ω by

$$V_b = K\omega = K\frac{d\theta}{dt} \tag{2}$$

Then, the following equations can be achieved based on the Newton's law and the Kirchhoff's law:

$$J\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} = Ki \tag{3}$$

$$L\frac{di}{dt} + Ri = V - K\frac{d\theta}{dt} \tag{4}$$

Next, apply the Laplace transform for Equations (3) and (4) and after some computation and substitution steps you should get

$$Js^2\theta(s) + bs\theta(s) = K\frac{V(s) - Ks\theta(s)}{R + Ls} \tag{5}$$

From Equation (5) you will be able to obtain:

- The transfer function from the input voltage $V(s)$ to the output angle $\theta(s)$

$$G_\theta(s) = \frac{\theta(s)}{V(s)} = \ldots? \tag{6}$$

- The transfer function from the input voltage $V(s)$ to the output angular velocity $\omega(s)$

$$G_\omega(s) = \frac{\omega(s)}{V(s)} = \ldots? \tag{7}$$

Based on Equation (5), a block diagram – describing the DC motor – can be achieved, as shown in Figure 2, where $\omega(s) = s\theta(s)$, $V_b(s) = K\omega(s) = Ks\theta(s)$, $T(s) = KI(s) = K\frac{V(s) - V_b(s)}{R + Ls}$.

Now, you are required to fill in the transfer functions (described by question marks in Figure 2) for the blocks "Armature" and "Load" to complete this block diagram.

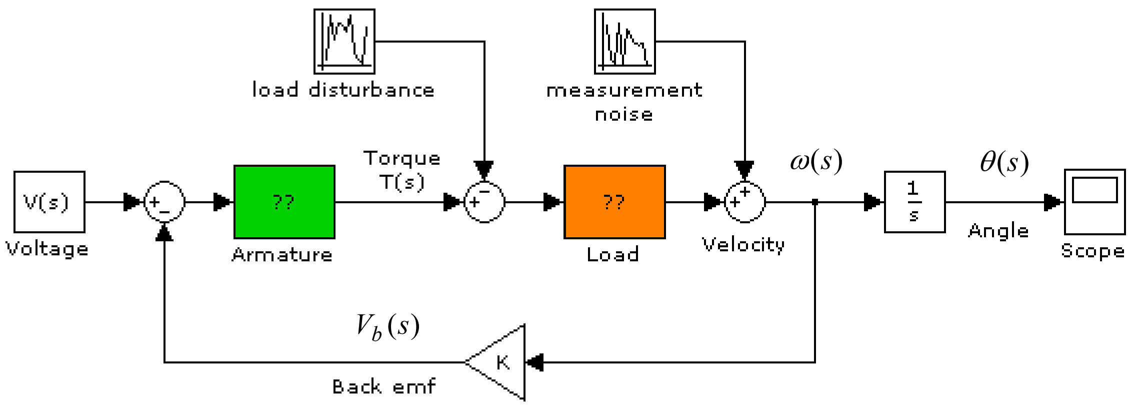

Figure 2. Block diagram of the DC motor.

Finally, substitute the given physical parameters of the motor to the transfer functions found in Equations (6) & (7) then you should obtain the results:

$$G_\theta(s) = \frac{\theta(s)}{V(s)} = \frac{0.02}{0.0006s^3 + 0.0215s^2 + 0.0504s}$$

$$G_\omega(s) = \frac{\omega(s)}{V(s)} = \frac{0.02}{0.0006s^2 + 0.0215s + 0.0504}$$

1.2. Inverted pendulum on a cart

In this exercise, you will build a mathematical model in terms of *nonlinear state differential equations* for the inverted pendulum on a cart in Figure 3 by using two methods: (1) the Newton's laws of motion (or the Newtonian approach) and (2) the Euler-Lagrange method (or the Lagrangian approach). All friction effects (b_1, b_2) are ignored in this exercise. What are the advantages and disadvantages of each approach?

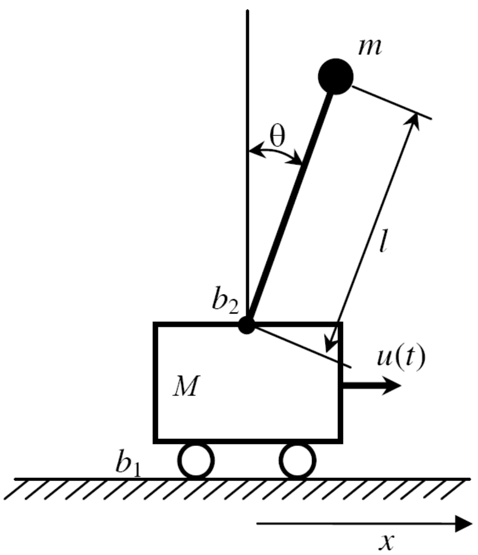

Figure 3. Inverted pendulum on a cart.

M : mass of the cart [kg];

m : mass at the tip of the pendulum [kg];

l : pendulum length [m];

The model parameters: $M = 1.0kg$, $m = 0.3kg$, $l = 0.3m$ and $g = 9.81m/s^2$.

$u(t)$: external force acting on the cart [N];

$x(t)$: cart position [m];

$\theta(t)$: angle between the pendulum and the vertical axis [rad].

Instructions:

(1) Using the Newton's laws of motion

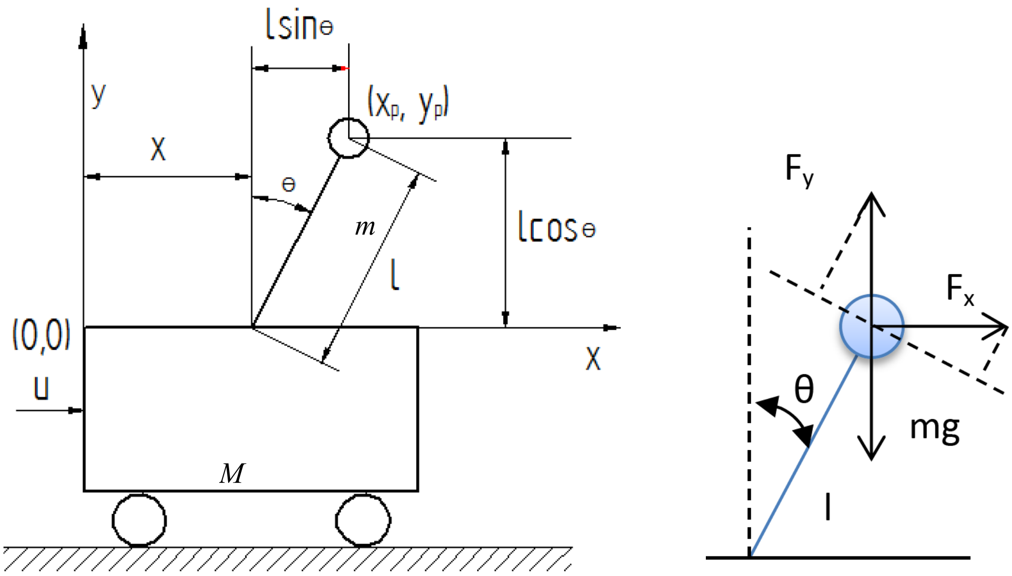

Figure 4. Free-body diagram of the inverted pendulum on a cart.

Firstly, it is assumed that the inverted pendulum on a cart is a system of two point masses connected by a massless rod. We can see in Figure 4 that the cart can only move in one direction (the displacement along the x-axis), whereas the pendulum can rotate (the displacement is possible in both x and y axes).

The drawing in Figure 4 represents the position of the mass at the tip of the pendulum (x_p, y_p) with respect to the origin of the coordinate system $(0,0)$. The coordinates (x_p, y_p) can be found using trigonometry rules, i.e.

$$\begin{cases} x_p = x + l\sin\theta \\ y_p = l\cos\theta \end{cases} \tag{8}$$

Apply the Newton's second law for the translational motion along the x-axis that yields

$$M\frac{d^2x}{dt^2} + m\frac{d^2x_p}{dt^2} = u \tag{9}$$

Substitute x_p from Equation (8) to Equation (9) we have

$$M\frac{d^2x}{dt^2} + m\frac{d^2}{dt^2}(x + l\sin\theta) = u \tag{10}$$

Expand derivatives in Equation (10) by yourself and make substitution then you should obtain

$$(M+m)\ddot{x} - ml(\sin\theta)\dot{\theta}^2 + ml(\cos\theta)\ddot{\theta} = u \tag{11}$$

Next, based on the free-body diagram in Figure 4 that shows all of the forces acting on the mass at the tip of the pendulum we can apply the Newton's second law for the rotational motion of the inverted pendulum and obtain the following torque-balancing equation, i.e.

$$F_x l\cos\theta - F_y l\sin\theta = Pl\sin\theta \tag{12}$$

where $F_x = m\ddot{x}_p$, $F_y = m\ddot{y}_p$ and $P = mg$

$$\Rightarrow m\frac{d^2x_p}{dt^2}l\cos\theta - m\frac{d^2y_p}{dt^2}l\sin\theta = mgl\sin\theta \tag{13}$$

Substitute x_p, y_p from Equation (8) to Equation (13) we have

$$\left[m\frac{d^2}{dt^2}(x + l\sin\theta)\right]l\cos\theta - \left[m\frac{d^2}{dt^2}(l\cos\theta)\right]l\sin\theta = mgl\sin\theta \tag{14}$$

Expand derivatives in Equation (14) by yourself and make reduction then you should get

$$\ddot{x}\cos\theta + l\ddot{\theta} = g\sin\theta \tag{15}$$

Finally, by solving Equations (11) and (15) for $\ddot{x}$ and $\ddot{\theta}$ by yourself then you should obtain

$$\begin{cases} \ddot{x} = \dfrac{u + ml(\sin\theta)\dot{\theta}^2 - mg\cos\theta\sin\theta}{M + m - m(\cos\theta)^2} \\ \ddot{\theta} = \dfrac{u\cos\theta - (M+m)g(\sin\theta) + ml(\cos\theta\sin\theta)\dot{\theta}^2}{ml(\cos\theta)^2 - (M+m)l} \end{cases} \tag{16}$$

which describe dynamic characteristics of the inverted pendulum on a cart.

(2) Using the Euler-Lagrange method

Kinetic energy of the mass at the tip of the pendulum:

$$K_p = \frac{1}{2}m\dot{x}_p^2 + \frac{1}{2}m\dot{y}_p^2 \tag{17}$$

Substitute x_p, y_p from Equation (8) to Equation (17) we have

$$K_p = \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 \tag{18}$$

Kinetic energy of the cart:

$$K_C = \frac{1}{2}M\dot{x}^2 \tag{19}$$

Kinetic energy of the whole system:

$$K = K_P + K_C = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 \tag{20}$$

Potential energy of the cart:

$$P_C = 0 \tag{21}$$

Hence, potential energy of the system equals to that of the mass at the tip of the pendulum, i.e.

$$P = P_p = mgl\cos\theta \tag{22}$$

Lagrangian:

$$L = K - P = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \tag{23}$$

It should be noted that the Lagrangian is a quantity that describes the balance between no dissipative energies. The generalized coordinates are selected as $q = [q_1 \quad q_2]^T = [x \quad \theta]^T$ so that the Euler-Lagrange equations have the form

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = u \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \end{aligned} \tag{24}$$

Now, apply Equation (24) for Equation (23) through performing a series of calculations:

$$\frac{\partial L}{\partial x}, \frac{\partial L}{\partial \dot{x}}, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial \dot{\theta}}, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)$$

then you will obtain the following nonlinear state differential equations, which are similar to that obtained by using the Newton's laws of motion, i.e.

$$\begin{aligned} &(M + m)\ddot{x} - ml(\sin\theta)\dot{\theta}^2 + ml(\cos\theta)\ddot{\theta} = u \\ &\ddot{x}\cos\theta + l\ddot{\theta} = g\sin\theta \end{aligned} \tag{25}$$

Solve Equation (25) for $\ddot{x}$ and $\ddot{\theta}$ $\Rightarrow \begin{cases} \ddot{x} = \dfrac{u + ml(\sin\theta)\dot{\theta}^2 - mg\cos\theta\sin\theta}{M + m - m(\cos\theta)^2} \\ \ddot{\theta} = \dfrac{u\cos\theta - (M+m)g(\sin\theta) + ml(\cos\theta\sin\theta)\dot{\theta}^2}{ml(\cos\theta)^2 - (M+m)l} \end{cases}$ (26)

1.3. One-link planar manipulator

Build a mathematical model in terms of *nonlinear state differential equations* for the one-link planar manipulator with load shown in Figure 5 by using the Euler-Lagrange method (or the Lagrangian approach).

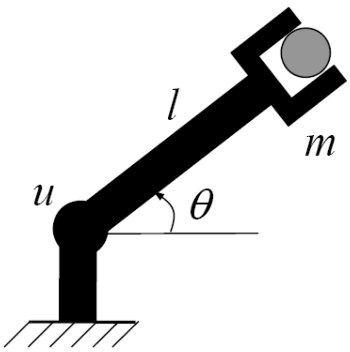

Figure 5. One-link planar manipulator.

M : mass of link [kg];

m : mass of load [kg];

l : link length [m];

l_c : distance from the centre of gravity of the manipulator to the rotation axis [m];

B : viscous friction coefficient at the rotation axis [Nms];

Input $u(t)$ is the external torque acting on the rotation axis of the manipulator [Nm];

Output $\theta(t)$ is the rotation angle between the link and the horizontal axis [rad];

Manipulator parameters: $M = 3.5kg$, $m = 0.6kg$, $l = 1.4m$, $l_c = 0.5m$, $B = 0.01Nms$ and $g = 9.81m/s^2$.

Instructions:

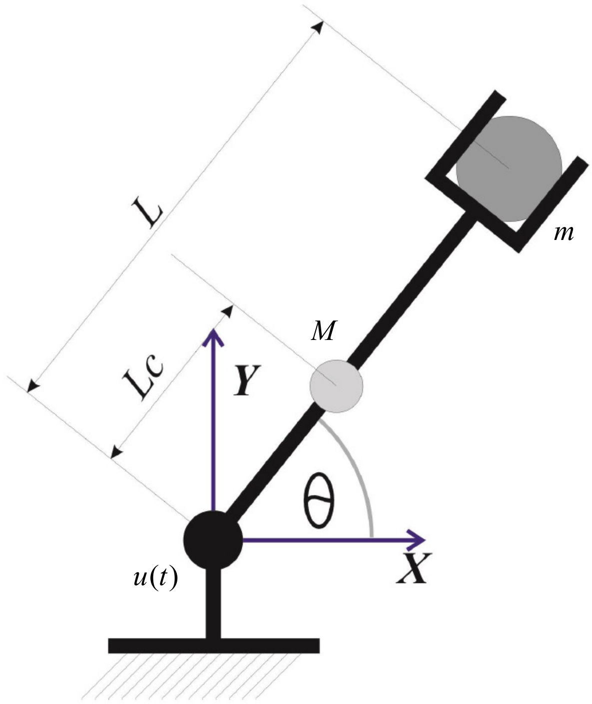

Figure 6. Distribution of mass for the one-link planar manipulator with load.

The dynamic characteristics of the manipulator are defined by determining its inertia, kinetic and potential energy, and dissipative energy.

Inertia:

$$I = Ml_c^2 + ml^2 \tag{27}$$

Kinetic energy:

$$K = \frac{1}{2} I\omega^2 = \frac{1}{2}(Ml_c^2 + ml^2)\dot{\theta}^2 \tag{28}$$

Potential energy:

$$P = (Ml_c + ml)g\sin\theta \tag{29}$$

Dissipative energy in terms of the viscous friction at the rotation axis:

$$D = \frac{1}{2} B\dot{\theta}^2 \tag{30}$$

Lagrangian:

$$L = K - P = \frac{1}{2}(Ml_c^2 + ml^2)\dot{\theta}^2 - (Ml_c + ml)g\sin\theta \tag{31}$$

Notice that the Lagrangian is a quantity that describes the balance between no dissipative energies.

The Euler-Lagrange equation for the generalized coordinate θ has the form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} + \frac{\partial D}{\partial \dot{\theta}} = u \tag{32}$$

Now, apply Equation (32) for Equation (31) by performing a series of calculations:

$$\frac{\partial L}{\partial \theta}, \frac{\partial L}{\partial \dot{\theta}}, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right), \frac{\partial D}{\partial \dot{\theta}}$$

then you will obtain Equation (33) that describes dynamic characteristics of the manipulator, i.e.

$$(Ml_c^2 + ml^2)\ddot{\theta} + (Ml_c + ml)g\cos\theta + B\dot{\theta} = u \tag{33}$$

1.4. Linearization of the inverted pendulum on a cart

This exercise uses the results obtained from **"Exercise 1.2. Inverted pendulum on a cart"**.

Suppose that the state variables are selected as follows:

$$\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \\ x_3(t) = x(t) \\ x_4(t) = \dot{x}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = \dot{\theta}(t) = x_2(t) \\ \dot{x}_2(t) = \ddot{\theta}(t) \\ \dot{x}_3(t) = \dot{x}(t) = x_4(t) \\ \dot{x}_4(t) = \ddot{x}(t) \end{cases} \tag{34}$$

and the system has one input $u(t)$ and two outputs $y(t) = \begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}$ (35)

It should be noted that in Equations (34) and (35) $x_1(t), x_2(t), x_3(t), x_4(t)$ are state variables; whereas $x(t), \dot{x}(t)$ and $\ddot{x}(t)$ are cart position, cart velocity and cart acceleration, respectively.

You are now required to

1. obtain a linearized model in terms of *linear state-space equations* from the nonlinear model of the inverted pendulum on a cart at the "upright" position (i.e. an equilibrium point), where $(\bar{x}, \bar{u}) = (0,0)$. Note that, in this context, $\bar{x}$ is the vector of state variables, i.e. $\bar{x} = [\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4]^T = [0,0,0,0]^T$;
2. build a state-space model for the linearized model and then transform (or convert) it into the representation of transfer function for each output: (i) the transfer function $G_1(s)$ from the input external force $u(t)$ to the output angle $\theta(t)$; and (ii) the transfer function $G_2(s)$ from the input external force $u(t)$ to the output cart position $x(t)$.

Instructions:

One can perform linearization of a system – described by nonlinear differential equations around an equilibrium point – by using the *Taylor's series expansion* with a condition is that the signals involved are small. For more detailed description of the linearization method, you are referred to the documents in the "Linearization of nonlinear systems" directory of the course materials provided during this course.

Given the results obtained in Equation (16) or Equation (26) and based on the state variables selected and input-output described in Equations (34) and (35) we have

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2(t) \\ \dfrac{u\cos x_1 - (M+m)g(\sin x_1) + ml(\sin x_1 \cos x_1)x_2^2}{ml(\cos x_1)^2 - (M+m)l} \\ x_4(t) \\ \dfrac{u + ml(\sin x_1)x_2^2 - mg\cos x_1 \sin x_1}{M + m - m(\cos x_1)^2} \end{bmatrix}}_{f(\cdots)} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ f_4(t) \end{bmatrix} \tag{36}$$

$$y(t) = \begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \underbrace{\begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}}_{h(\cdots)} = \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix} \tag{37}$$

It is well known that linearization of a nonlinear system around an equilibrium point using the Taylor's series expansion results in a linearized system described by linear state-space equations (or in other words, a linear state-space model)

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t) \end{cases} \tag{38}$$

In this exercise, the equilibrium point is defined in the context that the inverted pendulum is at the "upright" position and the external force acting on the cart is equal to zero, i.e.

$$\begin{cases} \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \bar{u} = 0 \end{cases} \tag{39}$$

Since we have investigated a system of four state variables, the matrices (A, B, C, D) in Equation (38) are calculated as

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} \end{bmatrix}_{(0,0)} \tag{40}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix}_{(0,0)} \tag{41}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial x_4} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_4} \end{bmatrix}_{(0,0)} \tag{42}$$

$$D = \left.\frac{\partial h}{\partial u}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial u} \end{bmatrix}_{(0,0)} \tag{43}$$

where

$$\begin{cases} f_1(t) = x_2(t) \\ f_2(t) = \dfrac{u\cos x_1 - (M+m)g(\sin x_1) + ml(\sin x_1 \cos x_1)x_2^2}{ml(\cos x_1)^2 - (M+m)l} \\ f_3(t) = x_4(t) \\ f_4(t) = \dfrac{u + ml(\sin x_1)x_2^2 - mg\cos x_1 \sin x_1}{M + m - m(\cos x_1)^2} \end{cases} \tag{44}$$

and

$$\begin{cases} h_1(t) = x_1(t) \\ h_2(t) = x_3(t) \end{cases} \tag{45}$$

The matrices (A, B, C, D) can be derived by means of calculating a series of partial derivatives. The following basic differentiation formulas can be used to calculate the partial derivatives:

1. Derivative of constant multiple: $\frac{d}{dx}(cu) = c\frac{du}{dx}$

2. Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

3. Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

where $u = f(x)$ and $v = g(x)$ represent differentiable functions of x.

For examples, consider calculating the elements of the state matrix A. Recall that the nonlinear system is linearized around the equilibrium point that satisfies Equation (39).

Two special values: $\sin 0 = 0$ and $\cos 0 = 1$.

- The first row (i.e. partial derivatives of function f_1) only relates to x_2 therefore we have:

  $\frac{\partial f_1}{\partial x_2} = 1$ and $\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial x_4} = 0$

- The second row (i.e. partial derivatives of function f_2) only relates to x_1 and x_2 so that we obtain:

  $$\frac{\partial f_2}{\partial x_1} = \frac{-u\sin x_1 - (M+m)g(\cos x_1) - mlx_2^2(\sin x_1)^2 + mlx_2^2(\cos x_1)^2}{ml(\cos x_1)^2 - (M+m)l}$$

$$-\frac{[u\cos x_1-(M+m)g(\sin x_1)+ml(\cos x_1\sin x_1)x_2^2][2ml\cos x_1(-\sin x_1)]}{[ml(\cos x_1)^2-(M+m)l]^2}$$

$$\Rightarrow \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)}=\frac{-(M+m)g}{ml-(M+m)l}-0=\frac{(M+m)g}{Ml}$$

$$\frac{\partial f_2}{\partial x_2}=\frac{2mlx_2\sin x_1\cos x_1}{ml(\cos x_1)^2-(M+m)l}\Rightarrow \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)}=0$$

$$\frac{\partial f_2}{\partial x_3}=\frac{\partial f_2}{\partial x_4}=0$$

- The third row (i.e. partial derivatives of function f_3) only relates to x_4 therefore we get:

$$\frac{\partial f_3}{\partial x_4}=1 \text{ and } \frac{\partial f_3}{\partial x_1}=\frac{\partial f_3}{\partial x_2}=\frac{\partial f_3}{\partial x_3}=0$$

- The fourth row (i.e. partial derivatives of function f_4) only relates to x_1 and x_2 so that we obtain:

$$\frac{\partial f_4}{\partial x_1}=\frac{mlx_2^2\cos x_1+mg(\sin x_1)^2-mg(\cos x_1)^2}{M+m-m(\cos x_1)^2}$$

$$-\frac{[u+ml(\sin x_1)x_2^2-mg\cos x_1\sin x_1][2m\cos x_1\sin x_1]}{[M+m-m(\cos x_1)^2]^2}$$

$$\Rightarrow \left.\frac{\partial f_4}{\partial x_1}\right|_{(0,0)}=\frac{-mg}{M+m-m}-0=-\frac{mg}{M}$$

$$\frac{\partial f_4}{\partial x_2}=\frac{2mlx_2\sin x_1}{M+m-m(\cos x_1)^2}\Rightarrow \left.\frac{\partial f_4}{\partial x_2}\right|_{(0,0)}=0$$

$$\frac{\partial f_4}{\partial x_3}=\frac{\partial f_4}{\partial x_4}=0$$

As a result, we obtain the state matrix A:

$$A=\left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})}=\left.\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4}\\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4}\\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4}\end{bmatrix}\right|_{(0,0)}=\begin{bmatrix}0 & 1 & 0 & 0\\ \frac{(M+m)}{Ml}g & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -\frac{m}{M}g & 0 & 0 & 0\end{bmatrix} \tag{46}$$

Now, the matrices (B, C, D) are needed to calculate by yourself and you should obtain

$$B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial u} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 \\ -\frac{1}{Ml} \\ 0 \\ \frac{1}{M} \end{bmatrix} \tag{47}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} & \frac{\partial h_1}{\partial x_4} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} & \frac{\partial h_2}{\partial x_4} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{48}$$

$$D = \left.\frac{\partial h}{\partial u}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \frac{\partial h_1}{\partial u} \\ \frac{\partial h_2}{\partial u} \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{49}$$

Substitute the physical parameters of the system given you will obtain the results:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 42.51 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -2.943 & 0 & 0 & 0 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ -3.333 \\ 0 \\ 1 \end{bmatrix}; \; C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \; D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{50}$$

To obtain the transfer function from the state-space model you can use the following formula

$$G(s) = C(sI - A)^{-1}B + D \tag{51}$$

Recall that the system has one input $u(t)$ and two outputs $y(t) = \begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix}$,

therefore you need to calculate two transfer functions:

$$G_1(s) = C_1(sI - A)^{-1}B + D_1 \tag{52}$$

$$G_2(s) = C_2(sI - A)^{-1}B + D_2 \tag{53}$$

where

$G_1(s)$ is the transfer function from the input external force $u(t)$ to the output angle $\theta(t)$;

$G_2(s)$ is the transfer function from the input external force $u(t)$ to the output cart position $x(t)$;

$C_1 = [1 \quad 0 \quad 0 \quad 0]$; $C_2 = [0 \quad 0 \quad 1 \quad 0]$; $D_1 = D_2 = 0$.

It is important to note that in order to reduce the workload on calculating the part $(sI - A)^{-1}$ in Equations (52) and (53) by hand, you are allowed to use computing software tools such as MATLAB or MAPLE. For example, the Matlab code below can be used.

```
syms s
A = [s -1 0 0; -42.51 s 0 0; 0 0 s -1; 2.943 0 0 s];
inv(A)
```

However, you should bear in mind that MATLAB or MAPLE could automatically perform the so-called pole-zero cancellation; therefore you might obtain the wrong answer.

Perform the remaining calculation by yourself and finally you should obtain the results:

$$G_1(s) = \frac{\theta(s)}{U(s)} = \frac{-3.333s^2}{s^4 - 42.51s^2} \quad \text{and} \quad G_2(s) = \frac{X(s)}{U(s)} = \frac{s^2 - 32.7}{s^4 - 42.51s^2} \tag{54}$$

1.5. Linearization of the one-link planar manipulator

This exercise uses the results obtained from **"Exercise 1.3. One-link planar manipulator"**.

Suppose that the state variables are selected as follows:

$$\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = \dot{\theta}(t) = x_2(t) \\ \dot{x}_2(t) = \ddot{\theta}(t) \end{cases} \tag{55}$$

and the system has one input $u(t)$ and one output $y(t) = \theta(t)$.

Questions:

1. Obtain a linearized model in terms of *linear state-space equations* from the nonlinear model of the manipulator around the operation point $\bar{\theta} = \pi/4$ (i.e. an equilibrium point or a stationary point).
2. Build a state-space model for the linearized model and then transform (or convert) it into the representation of transfer function $G_P(s)$ from the input external torque $u(t)$ to the output rotation angle $\theta(t)$.
3. Is the linearized system stable?

Instructions:

First, the concept of equilibrium (or stationary) points is introduced:

Consider a nonlinear differential equation

$$\dot{x}(t) = f(x(t), u(t)) \tag{56}$$

where f is a function mapping $\Re^n \times \Re^m \to \Re^n$. A point $\bar{x} \in \Re^n$ is called an equilibrium point if there is a specific $\bar{u} \in \Re^m$ (called the equilibrium input) such that

$$f(\bar{x}, \bar{u}) = 0_n$$

Now, suppose that $\bar{x}$ is an equilibrium point (with equilibrium input $\bar{u}$). Consider the system given by Equation (56) from the initial condition $x(t_0) = \bar{x}$ and apply the input $u(t) \equiv \bar{u}$ for all $t \geq t_0$. The resulting solution $x(t)$ satisfies $x(t) = \bar{x}$ for all $t \geq t_0$. That is why $\bar{x}$ is called an equilibrium point. In other words, a state point $\bar{x}$ is called an equilibrium/stationary point of a nonlinear system if and only if the system is staying at $\bar{x}$ and is activated by an input control signal $\bar{u}$ – which is unchanged – then the system will keep staying at the state point $\bar{x}$.

In summary, if $(\bar{x}, \bar{u})$ is an equilibrium/stationary point of a nonlinear system then

$$f(x(t), u(t))\big|_{x=\bar{x}, u=\bar{u}} = f(\bar{x}, \bar{u}) = 0 \tag{57}$$

Concerning the investigated problem you need to obtain a set of linear differential equations

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t)) \end{cases} \tag{58}$$

From Equation (33) that describes dynamic characteristics of the manipulator and based on the state variables selected and the input-output described above we have

$$\begin{cases} \dot{x}_1(t) = x_2(t) = f_1(t) \\ \dot{x}_2(t) = \ddot{\theta}(t) = -\dfrac{(Ml_c + ml)g}{Ml_c^2 + ml^2}\cos x_1(t) - \dfrac{B}{Ml_c^2 + ml^2}x_2(t) + \dfrac{1}{Ml_c^2 + ml^2}u(t) = f_2(t) \end{cases} \tag{59}$$

$$y(t) = x_1(t) = h(t) \tag{60}$$

Next, you need to determine the static operation mode/point at the equilibrium point $(\bar{x}, \bar{u})$.

Perform the calculations using Equations (57) and (59) by yourself and subsequently you should obtain the static operation mode at the equilibrium point $(\bar{x}, \bar{u})$ given by

$$\begin{cases} \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \pi/4 \\ 0 \end{bmatrix} \\ \bar{u} = (Ml_c + ml)g\cos\bar{x}_1 \end{cases} \tag{61}$$

Substitute the physical parameters of the manipulator you will obtain $\bar{u} = 17.9661$.

Then, the nonlinear system can be linearized around this equilibrium point $(\bar{x}, \bar{u})$ by using the Taylor's series expansion and you will obtain a linearized system described by linear state-space equations, or in other words, a linear state-space model that has the form

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + B\tilde{u}(t) \\ \tilde{y}(t) = C\tilde{x}(t) + D\tilde{u}(t) \end{cases} \tag{62}$$

where $\tilde{x}(t) = x(t) - \bar{x}$, $\tilde{u}(t) = u(t) - \bar{u}$, $\tilde{y}(t) = y(t) - \bar{y}$ (where $\bar{y} = h(\bar{x}, \bar{u})$.

Since we have a system of two state variables so that the matrices (A, B, C, D) in Equation (62) are calculated as

$$A = \left.\frac{\partial f}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}_{(\bar{x},\bar{u})} \tag{63}$$

$$B = \left.\frac{\partial f}{\partial u}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \end{bmatrix}_{(\bar{x},\bar{u})} \tag{64}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(\bar{x},\bar{u})} = \begin{bmatrix} \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} \end{bmatrix}_{(\bar{x},\bar{u})} \tag{65}$$

$$D = \left.\frac{\partial h}{\partial u}\right|_{(\bar{x},\bar{u})} \tag{66}$$

where $f(t)$ and $h(t)$ are given in Equations (59) and (60).

Perform calculations of these partial derivatives by yourself and then substitute the physical parameters given for the manipulator you will obtain the state matrices of the linearized system:

$$A = \begin{bmatrix} 0 & 1 \\ a_{21} & a_{22} \end{bmatrix} =; \; B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}; \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}; \; D = 0 \tag{67}$$

where $a_{21} = 8.7597$, $a_{22} = \text{-}0.0049$, $b_2 = 0.4876$.

However, it is essential to remind that we need to take the condition $\tilde{u}(t) = u(t) - \bar{u}$ into account. This means that some state matrices given in Equation (67) are needed to be modified. Consider the continuous-time 2nd-order system model in Figure 7 and Equation (59) it can be noticed that the element b_2 is directly related to $\dot{x}_2$ and u. Hence, the state matrix B is modified as

$$B = \begin{bmatrix} 0 \\ b_2 - \bar{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4876 - 17.9661 \end{bmatrix} = \begin{bmatrix} 0 \\ \text{-}17.4785 \end{bmatrix} \tag{68}$$

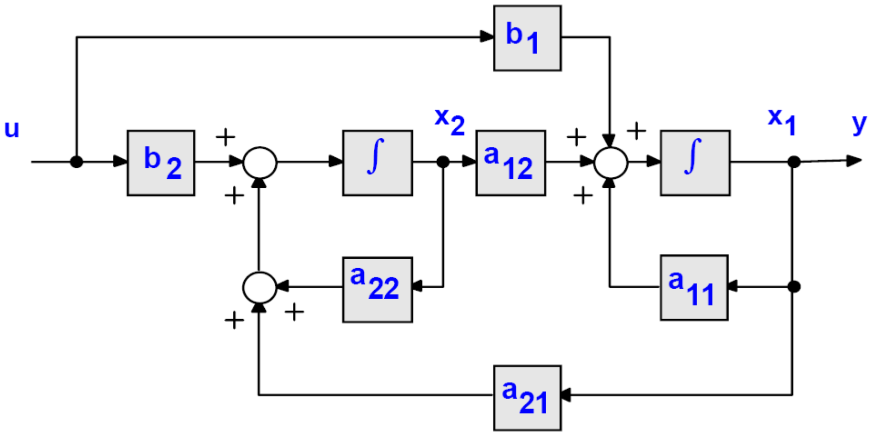

Figure 7. Continuous-time 2nd-order system model.

The transfer function $G_P(s)$ from the input external torque $u(t)$ to the output rotation angle $\theta(t)$ can be calculated by using the formula

$$G_P(s) = C(sI - A)^{-1}B + D \tag{69}$$

Perform the calculation by yourself and finally you will obtain the result:

$$G_P(s) = \frac{\theta(s)}{U(s)} = \frac{-17.4785}{s^2 + 0.004876s - 8.76} \tag{70}$$

Finally, you should prove that the linearized robot model is unstable.

Appendix

Instead of using the linearization method – based on the Taylor's series expansion as presented above – for simplicity in the case of **"Exercise 1.4. Linearization of the inverted pendulum on a cart"** one can use approximations which are valid near the origin, i.e. $\sin\theta = \sin x_1 \approx x_1$ and $\cos\theta = \cos x_1 \approx 1$. In addition, all squared state components in Equation (36) are very small and thus are set equal to zero. This yields the linear state equation

$$\underbrace{\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} x_2(t) \\ \dfrac{(M+m)gx_1 - u}{Ml} \\ x_4(t) \\ \dfrac{u - mgx_1}{M} \end{bmatrix}}_{f(\cdots)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(M+m)g}{Ml} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{Ml} \\ 0 \\ \dfrac{1}{M} \end{bmatrix} u = Ax + Bu \tag{71}$$

The output equation depends on the measurement taken, which depends on the sensors available. The cart position can be measured by placing one or two optical encoders on the wheels and the rod angle can be measured by placing an encoder at the rod pivot point. With regard to the cart velocity and the rod angle velocity, they may be obtained by placing tachometers on a wheel and at the rod pivot point.

With an assumption that measurements of the cart position and the rod angle are available then the output equation is

$$y(t) = \begin{bmatrix} \theta(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_3(t) \end{bmatrix} \Rightarrow y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = Cx + Du \tag{72}$$

In case if the cart position and velocity as well as the rod angle and velocity can be measured then the output equation becomes

$$y(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ x(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \Rightarrow y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = Cx + Du \tag{73}$$

Given the linear state-space equations, a controller can be designed to keep the rod "upright". Although the controller is designed for the linearized system, it should be tested and simulated in a closed-loop system using the full nonlinear model of the inverted pendulum on a cart.