Katedra Robotyki i Mechatroniki

Akademia Górniczo-Hutnicza w Krakowie

Industrial Robots

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2.
Mechanics of manipulators 3
Geometrical model
Inverse Kinematics (Inverse Kinematic Problem)

Problems:

- formulation of the inverse kinematics problem
- Analytical solution methods to the inverse kinematics

Problem formulation

Input data - Given:

Position and orientation: $\boldsymbol{x, y, z, \varphi, \theta, \psi}$

Results - Find:

Joint variables: $\boldsymbol{q_1, q_2, \ldots, q_n}$

Classification of the inverse kinematics solution methods

- **Analytical - algebraic**
  - geometric
- Numerical

A set of equations:

Assumed position and orientation

$$ {}^{o}\underline{\tau}_e = \begin{bmatrix} N_x & O_x & A_x & P_x \\ N_y & O_y & A_y & P_y \\ N_z & O_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Geometrical model

$$ {}^{0}\underline{T}_e = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

e.g. for a SCARA manipulator

$$ {}^{0}\tau_4 = \begin{bmatrix} -0.259 & -0.966 & 0 & 0.933 \\ -0.966 & 0.259 & 0 & 0.250 \\ 0 & 0 & -1 & -0.200 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{0}\underline{T}_4 = \begin{bmatrix} C_{12}C_4 + S_{12}S_4 & -C_{12}S_4 + S_{12}C_4 & 0 & a_1C_1 + a_2C_{12} \\ S_{12}C_4 - C_{12}S_4 & -C_{12}C_4 - S_{12}S_4 & 0 & a_1S_1 + a_2S_{12} \\ 0 & 0 & -1 & -d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

A set of 12 nonlinear algebraic equations
only 6 of equations are mutually independent

$$N_x = f_{11}(q_1, q_2, \ldots, q_e) \qquad O_x = f_{12}(q_1, q_2, \ldots, q_e)$$
$$N_y = f_{21}(q_1, q_2, \ldots, q_e) \qquad O_y = f_{22}(q_1, q_2, \ldots, q_e)$$
$$N_z = f_{31}(q_1, q_2, \ldots, q_e) \qquad O_z = f_{32}(q_1, q_2, \ldots, q_e)$$

$$A_x = f_{13}(q_1, q_2, \ldots, q_e) \qquad P_x = f_{14}(q_1, q_2, \ldots, q_e)$$
$$A_y = f_{23}(q_1, q_2, \ldots, q_e) \qquad P_y = f_{24}(q_1, q_2, \ldots, q_e)$$
$$A_z = f_{33}(q_1, q_2, \ldots, q_e) \qquad P_z = f_{34}(q_1, q_2, \ldots, q_e)$$

There is no general solution method of such the set
One should determine:

- does a solution exist?
- how many solutions are there?

A general solution method:

The assumed position and orientation:

$$^{o}\underline{\tau}_{e} = \begin{bmatrix} N_x & O_x & A_x & P_x \\ N_y & O_y & A_y & P_y \\ N_z & O_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}\underline{\tau}_{e} = {}^{0}\underline{T}_{e} = \underline{A}_1 \underline{A}_2 \ldots \underline{A}_e$$

$$\underline{A}_1^{-1}\,{}^{0}\underline{\tau}_6 = \underline{A}_2 \underline{A}_3 \underline{A}_4 \underline{A}_5 \underline{A}_6$$

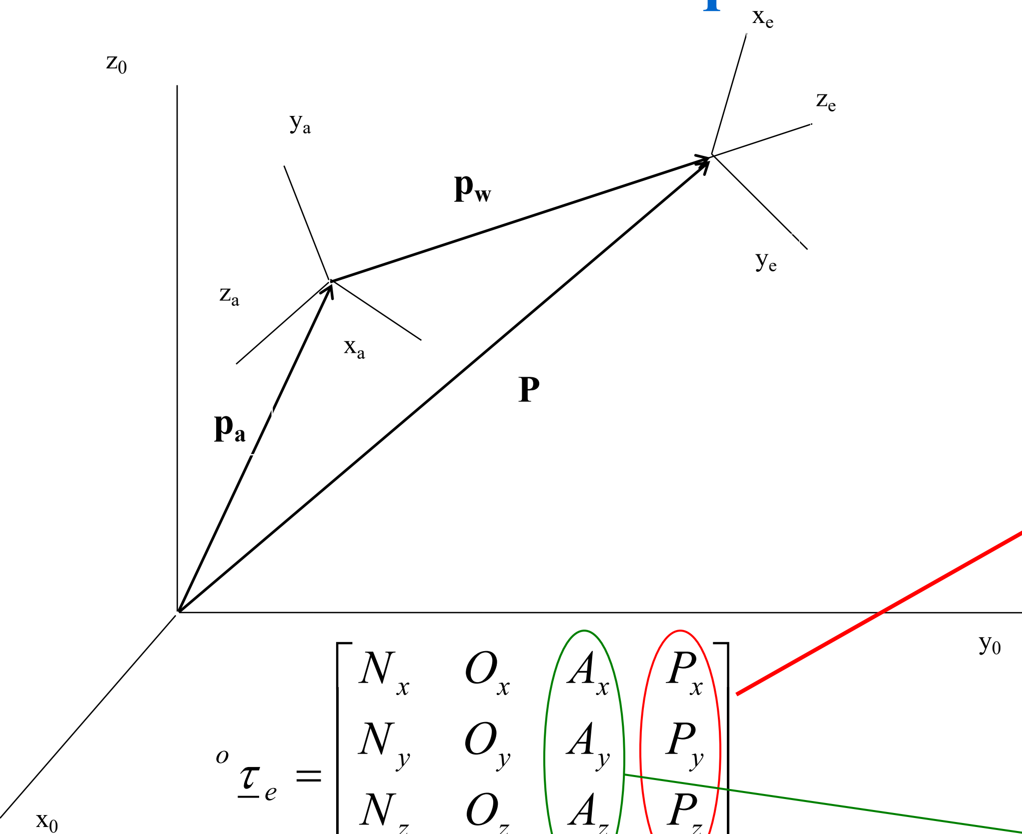
$$\underline{A}_2^{-1}\left(\underline{\alpha}_1^{-1}\,{}^{0}\underline{\tau}_6\right) = \underline{A}_3 \underline{A}_4 \underline{A}_5 \underline{A}_6$$

A 2 step method applicable to manipulators with intersecting 3 wrist motion axes

$$ {}^{0}\underline{T}_{e} = {}^{0}\underline{T}_{a}\,{}^{a}\underline{T}_{e} = \begin{bmatrix} \underline{R}_{a} & \overline{P}_{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{a}\underline{R}_{w} & {}^{a}\overline{P}_{w} \\ 0 & 1 \end{bmatrix} = $$

$$ = \begin{bmatrix} \underline{R}_{a}\,{}^{a}\underline{R}_{w} & \overline{P}_{a} + \underline{R}_{a}\,{}^{a}\overline{P}_{w} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{R}_{w} & \overline{P}_{a} + \overline{P}_{w} \\ 0 & 1 \end{bmatrix} $$

Formulation of a set of equations – IDEA and the INITIAL STEP



$$\overline{P} = \overline{p}_a + \overline{p}_w$$

$$\overline{p}_a = \overline{P} - \overline{p}_w$$

$$\overline{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

$$^{o}\underline{\tau}_e = \begin{bmatrix} N_x & O_x & A_x & P_x \\ N_y & O_y & A_y & P_y \\ N_z & O_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A length of $\boldsymbol{p_w}$ vector is known
Its direction corresponds to the axis z_e

$$\overline{p}_w = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \left| \overline{p}_w \right|$$

$$\begin{bmatrix} p_{ax} \\ p_{ay} \\ p_{az} \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} - \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \left| \overline{p}_w \right|$$

The first step

The following set of 3 equations should be solved:

$$p_{ax} = g_1(q_1, q_2, q_3)$$

$$p_{ay} = g_2(q_1, q_2, q_3)$$

$$p_{az} = g_3(q_1, q_2, q_3)$$

SCARA

$$\left|\overline{p}_w\right| = \overline{0}$$

$$0.933 = a_1 C_1 + a_2 C_{12}$$

$$0.25 = a_1 S_1 + a_2 S_{12}$$

$$-0.2 = -d_3$$

In the result values of: q_1, q_2, q_3 variables are calculated.

They should be substituted into the 3rd link orientation matrix $\underline{R}_3$
In order to evaluate a $\underline{r}_3$ matrix (it is composed of numbers only!)

The second step

$$q_1, q_2, q_3 \longrightarrow \underline{R}_3$$

$$^{o}\underline{\tau}_e = \begin{bmatrix} N_x & O_x & A_x & P_x \\ N_y & O_y & A_y & P_y \\ N_z & O_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}\underline{r}_e = {}^{0}\underline{r}_3 \, {}^{3}\underline{R}_e$$

$$^{3}\underline{R}_e = {}^{0}\underline{r}_3{}^{T} \, {}^{0}\underline{r}_e = \begin{bmatrix} n_{wx} & o_{wx} & a_{wx} \\ n_{wy} & o_{wy} & a_{wy} \\ n_{wz} & o_{wz} & a_{wz} \end{bmatrix}$$

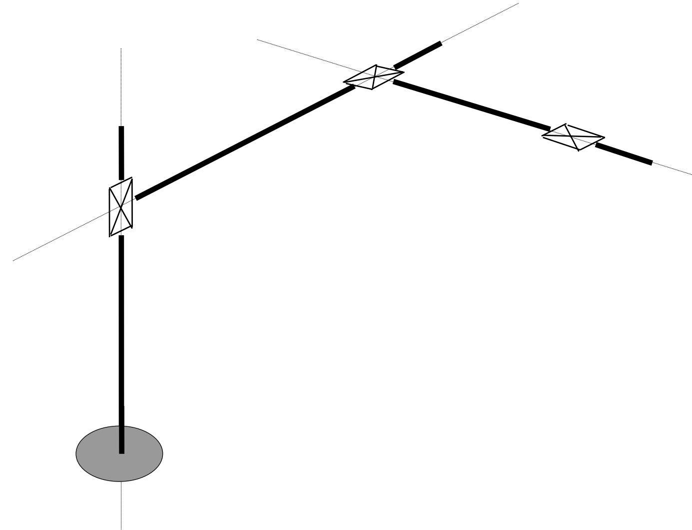
The following set of equations should be solved (only 3 equations are independent)

$$\begin{bmatrix} n_{wx} & o_{wx} & a_{wx} \\ n_{wy} & o_{wy} & a_{wy} \\ n_{wz} & o_{wz} & a_{wz} \end{bmatrix} = \underline{R}_4 \underline{R}_5 \underline{R}_6 = \begin{bmatrix} h_{11}(q_4, q_5, q_6) & h_{12}(q_4, q_5, q_6) & h_{13}(q_4, q_5, q_6) \\ h_{21}(q_4, q_5, q_6) & h_{22}(q_4, q_5, q_6) & h_{23}(q_4, q_5, q_6) \\ h_{31}(q_4, q_5, q_6) & h_{32}(q_4, q_5, q_6) & h_{33}(q_4, q_5, q_6) \end{bmatrix}$$

THE FIRST STEP - Examples

A. The Cartesian arm PPP

$$\underline{T}_3 = \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



To be calculated: d_1, a_2, d_3

$$p_{ax} = a_2$$
$$p_{ay} = -d_3$$
$$p_{az} = d_1$$

$$\Longrightarrow$$

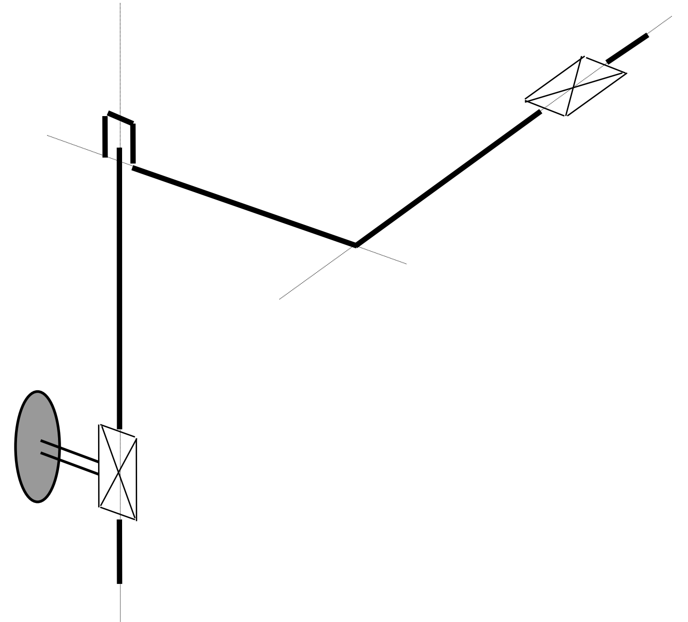
$$d_1 = p_{az}$$
$$a_2 = p_{ax}$$
$$d_3 = -p_{ay}$$

There is only 1 solution

The solution is always achievable

B. The cylindrical arm PRP

$$\underline{T}_3 = \begin{bmatrix} C_2 & 0 & -S_2 & a_2C_2 - d_3S_2 \\ S_2 & 0 & C_2 & a_2S_2 + d_3C_2 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



To be calculated: d_1, θ_2, d_3.

$$p_{ax} = a_2C_2 - d_3S_2$$
$$p_{ay} = a_2S_2 + d_3C_2$$
$$p_{az} = d_1$$

$$\Rightarrow \boxed{d_1 = p_{az}}$$

$$d_3^2S_2^2 + a_2^2C_2^2 - 2a_2d_3S_2C_2 = p_{ax}^2$$
$$d_3^2C_2^2 + a_2^2S_2^2 + 2a_2d_3S_2C_2 = p_{ay}^2$$

$$\Rightarrow d_3^2 + a_2^2 = p_{ax}^2 + p_{ay}^2$$

$$\boxed{d_3 = \pm\sqrt{p_{ax}^2 + p_{ay}^2 - a_2^2}}$$

The 1st equation: $p_{ax} = a_2 C_2 - d_3 S_2$

Is in a form of: $-A\sin\theta + B\cos\theta = D$

where: $A = d_3$ $D = p_{ax}$

$B = a_2$ $\theta = \theta_2$

Substitution:

$$A = r\cos\phi \qquad r = \sqrt{A^2 + B^2}$$

$$B = r\sin\phi \qquad \phi = \arctan\frac{B}{A}$$

The general form of the 1st equation:

$$-\cos\phi\sin\theta + \sin\phi\cos\theta = \frac{D}{r}$$

$$\sin(\phi-\theta)=\frac{D}{r} \quad \Rightarrow \quad \cos(\phi-\theta)=\pm\sqrt{1-\frac{D^2}{r^2}}$$

$$\tan(\phi-\theta)=\frac{\frac{D}{r}}{\pm\sqrt{1-\left(\frac{D}{r}\right)^2}}=\frac{D}{\pm\sqrt{r^2-D^2}}=\frac{D}{\pm\sqrt{A^2+B^2-D^2}}$$

$$\theta=\arctan\left(\frac{B}{A}\right)-\arctan\left(\frac{D}{\pm\sqrt{A^2+B^2-D^2}}\right)$$

$$\boxed{\theta_2=\arctan\left(\frac{a_2}{d_3}\right)-\arctan\left(\frac{p_{ax}}{\pm\sqrt{d_3^{\,2}+a_2^{\,2}-p_{ax}^{\,2}}}\right)}$$

$$d_1 = p_{az}$$

$$\theta_2 = \arctan\left(\frac{a_2}{d_3}\right) - \arctan\left(\frac{p_{ax}}{\pm\sqrt{d_3^{\ 2} + a_2^{\ 2} - p_{ax}^{\ 2}}}\right)$$

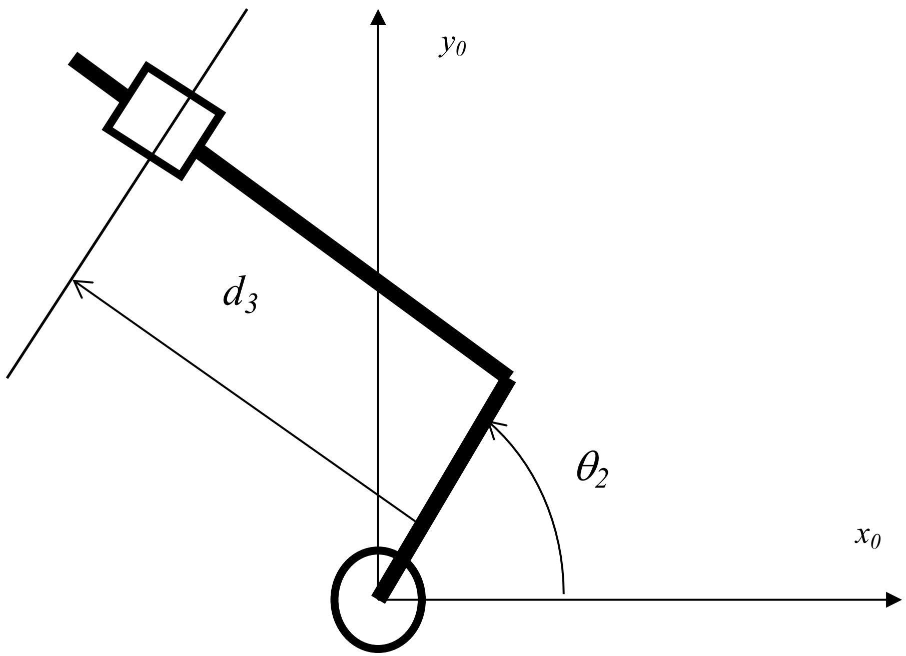
$$d_3 = \pm\sqrt{p_{ax}^{\ 2} + p_{ay}^{\ 2} - a_2^{\ 2}}$$

4 solutions:

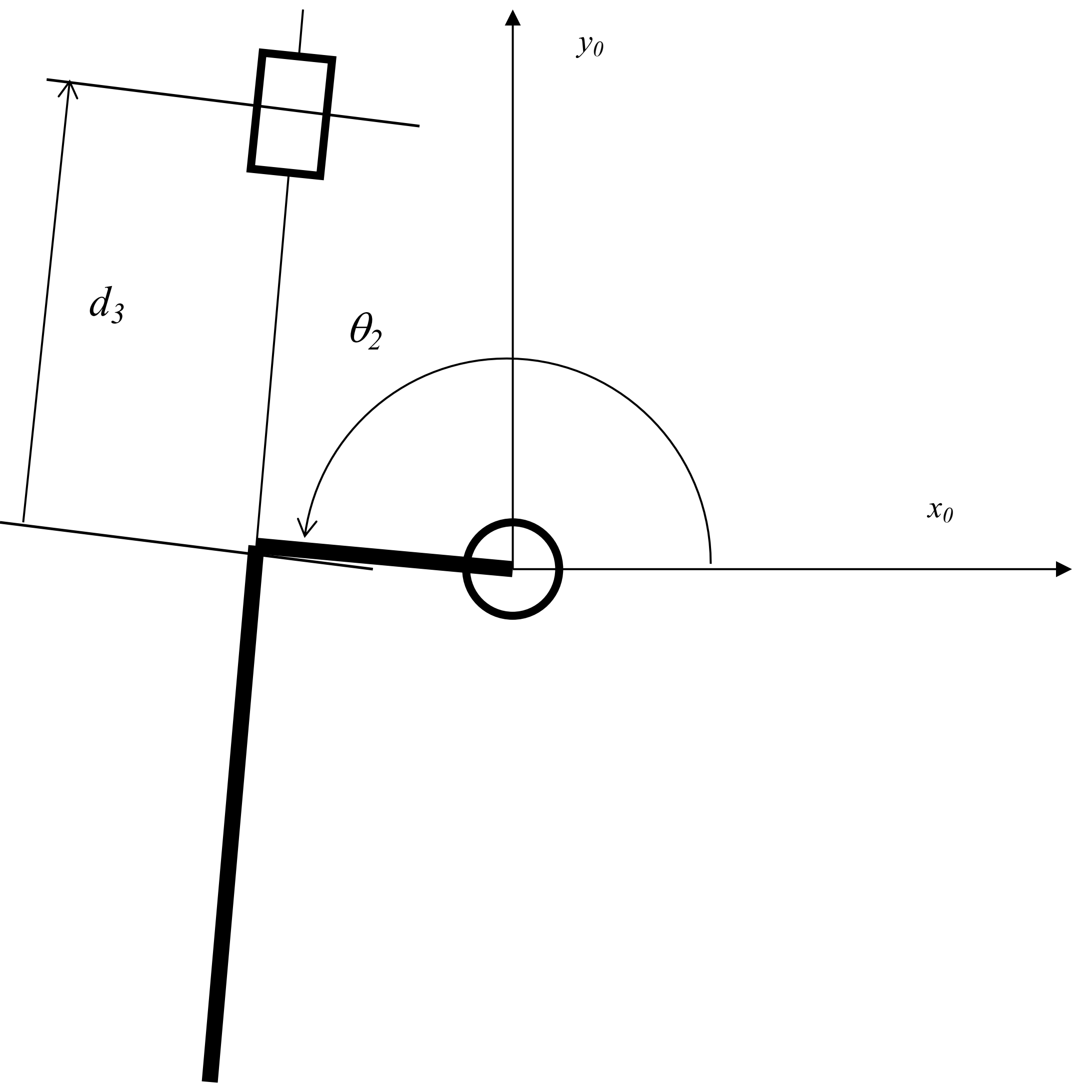
d_1	d_1	d_1	d_1
θ_{21}	θ_{22}	θ_{23}	θ_{24}
d_{31}	d_{31}	d_{32}	d_{32}

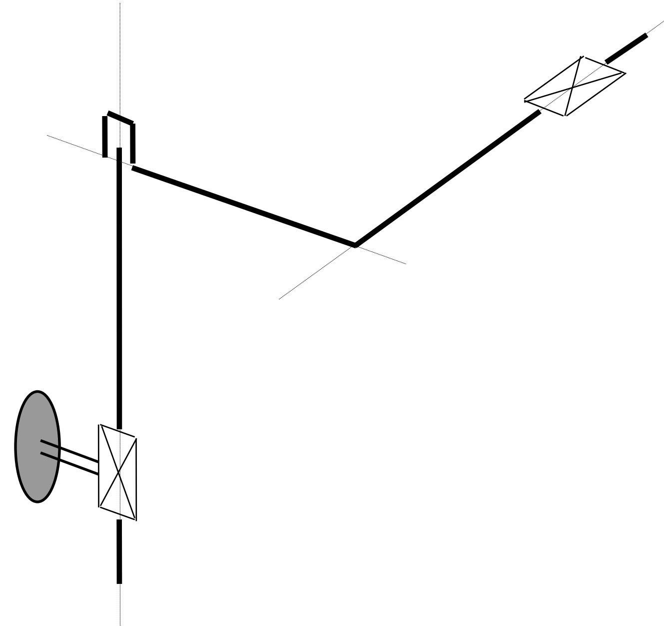
2 appropriate solutions

$d_3 \geq 0$

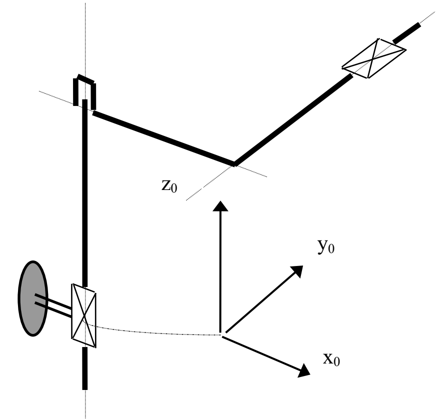


$d_3 < 0$





Analysis of existence of a solution:



$$\theta_2 = \arctan\left(\frac{a_2}{d_3}\right) - \arctan\left(\frac{p_{ax}}{\pm\sqrt{d_3^{\ 2} + a_2^{\ 2} - p_{ax}^{\ 2}}}\right)$$

$$\arctan\left(\frac{a_2}{d_3}\right) = \arctan\left(\frac{0}{0}\right) \Rightarrow \boxed{a_2 = 0 \qquad d_3 = 0}$$

$$p_{ax} = 0$$

$$\arctan\left(\frac{p_{ax}}{\pm\sqrt{d_3^{\ 2} + a_2^{\ 2} - p_{ax}^{\ 2}}}\right) = \arctan\left(\frac{0}{0}\right)$$

The solution of the inverse problem is indefinite

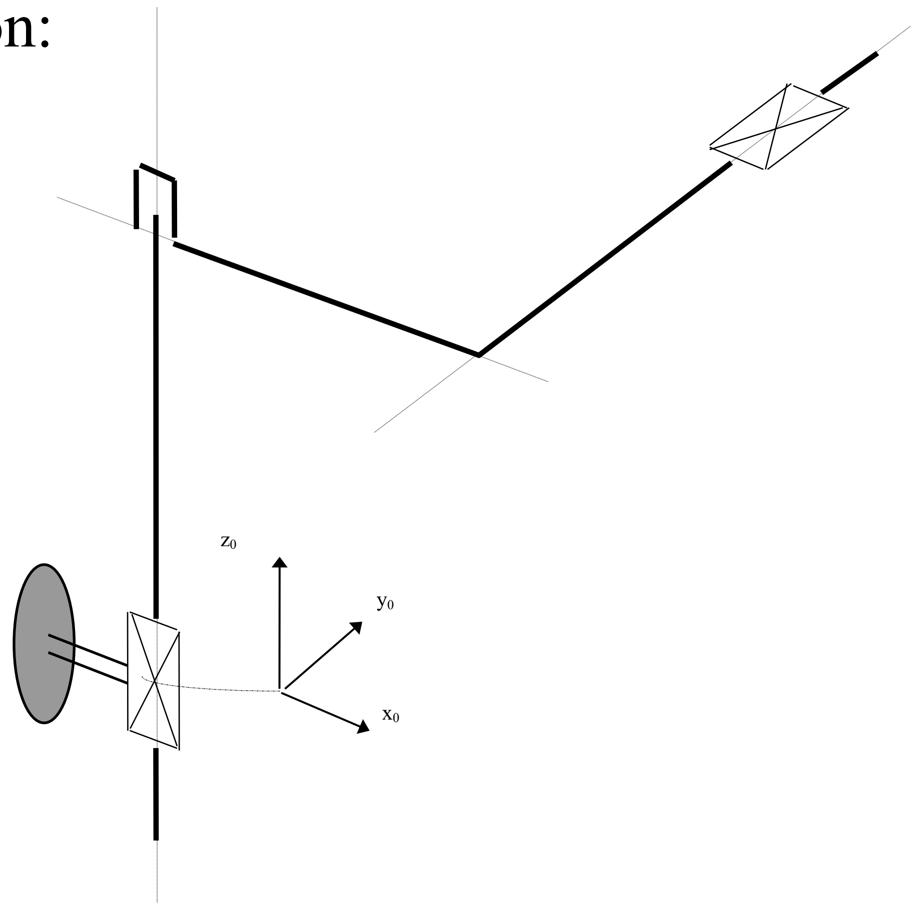
There is infinite number of values of θ_2

Analysis of existence of a solution:

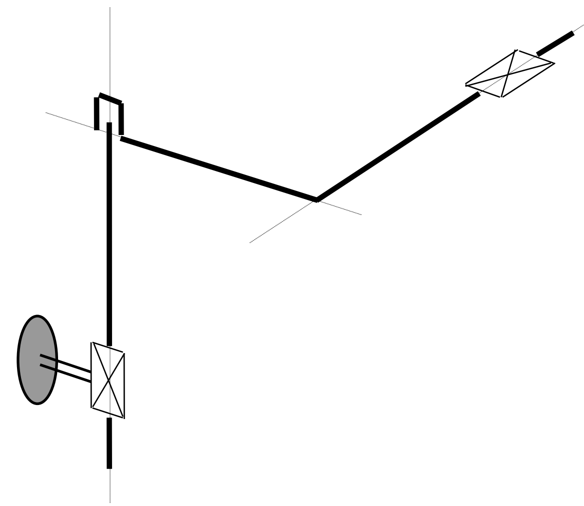
$$d_3 = \pm\sqrt{p_{ax}^{\,2} + p_{ay}^{\,2} - a_2^{\,2}}$$

The solution always exists as:

$$p_{ax}^{\,2} + p_{ay}^{\,2} \geq a_2^{\,2}$$



A numerical example, a PRP arm:



Known values of parameters:

$a_2 = 0.1$ m $\qquad d_1 \in [0,1]$ m

$\theta_2 \in [-90°, 135°] \qquad d_3 \in [0.3,1]$ m

$$
{}^{0}\underline{T}_{3} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -0.2\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0.3\sqrt{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

To be determined:

d_1, θ_2, d_3.

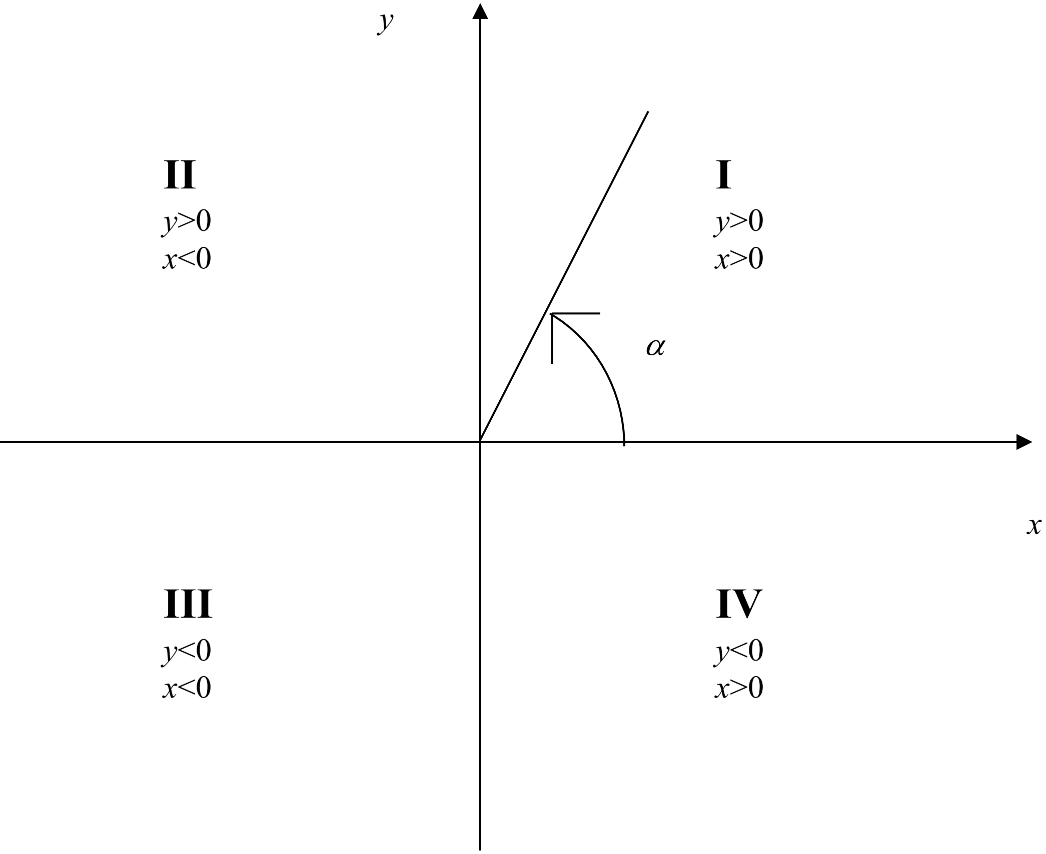
The solution:

$$
{}^{0}\underline{T}_{3} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -0.2\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0.3\sqrt{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$d_1 = p_{az} = 0$$

$$d_3 = \pm\sqrt{p_{ax}^{\;2} + p_{ay}^{\;2} - a_2^{\;2}} = \pm\sqrt{0.08 + 0.18 - 0.01} = \pm\sqrt{0.25} = \pm 0.5$$

Atan2



quadrants I and IV:

$$\operatorname{atan} 2\left(\frac{y}{x}\right) = \arctan\left(\frac{y}{x}\right)$$

quadrants II and III:

$$\operatorname{atan} 2\left(\frac{y}{x}\right) = \arctan\left(\frac{y}{x}\right) \pm \pi$$

$$\theta_2 = \arctan\left(\frac{a_2}{d_{31}}\right) - \arctan\left(\frac{p_{ax}}{\pm\sqrt{d_{31}{}^2 + a_2{}^2 - p_{ax}{}^2}}\right) =$$

$$= \arctan\left(\frac{0.1}{0.5}\right) - \arctan\left(\frac{-0.2\sqrt{2}}{\pm\sqrt{0.25+0.01-0.08}}\right) = 11.31° - \arctan\left(\frac{-2}{\pm 3}\right)$$

$$\boxed{d_{31}=0.5}$$

$$\operatorname{atan2}\left(\frac{-2}{3}\right) = \arctan\left(-\frac{2}{3}\right) = -33.69°$$

IV quadrant

$$\operatorname{atan2}\left(\frac{-2}{-3}\right) = \arctan\left(\frac{2}{3}\right) + \pi = 33.69° + 180° = 213.69°$$

III quadrant

$$\boxed{\begin{aligned} \theta_{21} &= 11.31° - (-33.69°) = 45° \\ \theta_{22} &= 11.31° - 213.69° = -202.38° \end{aligned}}$$

Verification of the results by substitution of θ_{21}, θ_{22}, d_{31} to:

$$p_{ax} = a_2 C_2 - d_3 S_2$$
$$p_{ay} = a_2 S_2 + d_3 C_2$$

Results:

$$p_{ax}(\theta_{21}) \approx -0.283 \approx -0.2\sqrt{2}$$
$$p_{ax}(\theta_{22}) \approx -0.283 \approx -0.2\sqrt{2}$$
$$p_{ay}(\theta_{21}) = 0.424 \approx 0.3\sqrt{2}$$
$$p_{ay}(\theta_{22}) = -0.424 \approx -0.3\sqrt{2}$$

$${}^{0}\underline{T}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -0.2\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0.3\sqrt{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\theta_2 = \theta_{21} = 45^\circ}$$

$$d_{32}=-0.5$$

$$\theta_2 = \arctan\left(\frac{a_2}{d_{32}}\right) - \arctan\left(\frac{p_{ax}}{\pm\sqrt{d_{32}{}^2 + a_2{}^2 - p_{ax}{}^2}}\right) =$$

$$= \arctan\left(\frac{0.1}{-0.5}\right) - \arctan\left(\frac{-0.2\sqrt{2}}{\pm\sqrt{0.25+0.01-0.08}}\right) = \arctan\left(\frac{1}{-5}\right) - \arctan\left(\frac{-2}{\pm 3}\right)$$

$$\operatorname{atan2}\left(\frac{1}{-5}\right) = \arctan\left(-\frac{1}{5}\right) + \pi = -11.31^\circ + 180^\circ = 168.69^\circ$$

$$\theta_{23} = 168.69^\circ - \left(-33.69^\circ\right) = 202.38^\circ$$

$$\theta_{24} = 168.69^\circ - 213.69^\circ = -45^\circ$$

Verification of the results by substitution of θ_{23}, θ_{24}, d_{32} to:

$$p_{ax} = a_2 C_2 - d_3 S_2$$
$$p_{ay} = a_2 S_2 + d_3 C_2$$

results:

$$p_{ax}(\theta_{23}) \approx -0.283 \approx -0.2\sqrt{2}$$
$$p_{ax}(\theta_{24}) \approx -0.283 \approx -0.2\sqrt{2}$$
$$p_{ay}(\theta_{23}) = 0.424 \approx 0.3\sqrt{2}$$
$$p_{ay}(\theta_{24}) = -0.424 \approx -0.3\sqrt{2}$$

$$^{0}\underline{T}_{3} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -0.2\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0.3\sqrt{2} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\theta_2 = \theta_{23} = 202.38^\circ}$$

2 solutions:

$$\boxed{\begin{aligned} d_1 &= 0 \\ \theta_2 &= 45^\circ \\ d_3 &= 0.5 \end{aligned} \qquad \begin{aligned} d_1 &= 0 \\ \theta_2 &= 202{,}38^\circ \\ d_3 &= -0.5 \end{aligned}}$$

$a_2 = 0.1$ m $\qquad d_1 \in [0,1]$ m

$\theta_2 \in [-90^\circ, 135^\circ] \qquad d_3 \in [0.3,1]$ m

The alternative method of solution:

$$p_{ax} = a_2 C_2 - d_3 S_2$$
$$p_{ay} = a_2 S_2 + d_3 C_2$$

$$\Rightarrow \begin{bmatrix} -d_3 & a_2 \\ a_2 & d_3 \end{bmatrix} \begin{bmatrix} S_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} p_{ax} \\ p_{ay} \end{bmatrix}$$

$$S_2 = \frac{a_2 p_{ay} - d_3 p_{ax}}{a_2^{\,2} + d_3^{\,2}} \qquad C_2 = \frac{a_2 p_{ax} + d_3 p_{ay}}{a_2^{\,2} + d_3^{\,2}}$$

$$\boxed{\tan(\theta_2) = \frac{a_2 p_{ay} - d_3 p_{ax}}{a_2 p_{ax} + d_3 p_{ay}}}$$

Another alternative method of solution:

$$\theta_2 = \arcsin\left(\frac{a_2 p_{ay} - d_3 p_{ax}}{a_2{}^2 + d_3{}^2}\right) \qquad \theta_2 = \arccos\left(\frac{a_2 p_{ax} + d_3 p_{ay}}{a_2{}^2 + d_3{}^2}\right)$$

$\boxed{d_3=0.5}$

$$\theta_2 = \arcsin(0.707) = 45^\circ \quad \text{or} \quad \theta_2 = 180^\circ - 45^\circ = 135^\circ$$

$$\theta_2 = \arccos(0.707) = 45^\circ \quad \text{or} \quad \theta_2 = -45^\circ$$

$$\Rightarrow \theta_2 = 45^\circ$$

$\boxed{d_3=-0.5}$

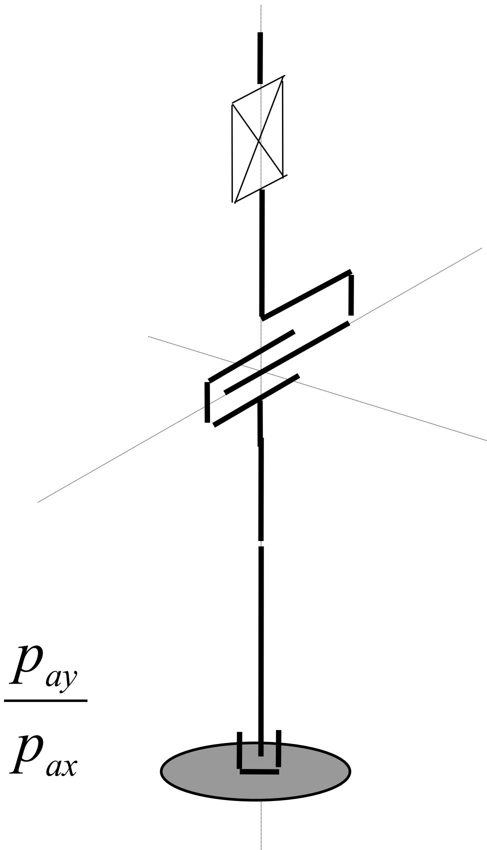
$$\theta_2 = \arcsin(-0.381) = -22.38^\circ \quad \text{or} \quad \theta_2 = 180^\circ - (-22.38)^\circ = 202.38^\circ$$

$$\theta_2 = \arccos(-0.925) = 157.61^\circ \quad \text{or} \quad \theta_2 = -157.61^\circ = 202.38^\circ$$

$$\Rightarrow \theta_2 = 202.38^\circ$$

C. The spherical RRP arm

$$\underline{T}_3 = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & d_3C_1S_2 \\ S_1C_2 & C_1 & S_1S_2 & d_3S_1S_2 \\ -S_2 & 0 & C_2 & d_1 + d_3C_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



To be determined: θ_1, θ_2, d_3.

$$p_{ax} = d_3C_1S_2$$
$$p_{ay} = d_3S_1S_2$$
$$p_{az} = d_1 + d_3C_2$$

$$\frac{d_3S_1S_2}{d_3C_1S_2} = \frac{p_{ay}}{p_{ax}} \Rightarrow \tan(\theta_1) = \frac{p_{ay}}{p_{ax}}$$

$$\boxed{\theta_1 = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) + n\pi} \qquad n = -1,0,1$$

$$p_{ax}^{2} = d_3^{2} C_1^{2} S_2^{2}$$
$$p_{ay}^{2} = d_3^{2} S_1^{2} S_2^{2}$$

$$\Rightarrow$$

$$p_{ax}^{2} + p_{ay}^{2} = d_3^{2} S_2^{2}$$
$$\left(p_{az} - d_1\right)^2 = d_3^{2} C_2^{2}$$

$$\frac{d_3^{2} S_2^{2}}{d_3^{2} C_2^{2}} = \frac{p_{ax}^{2} + p_{ay}^{2}}{\left(p_{az} - d_1\right)^2} \Rightarrow \mathrm{tg}(\theta_2) = \frac{\pm\sqrt{p_{ax}^{2} + p_{ay}^{2}}}{p_{az} - d_1}$$

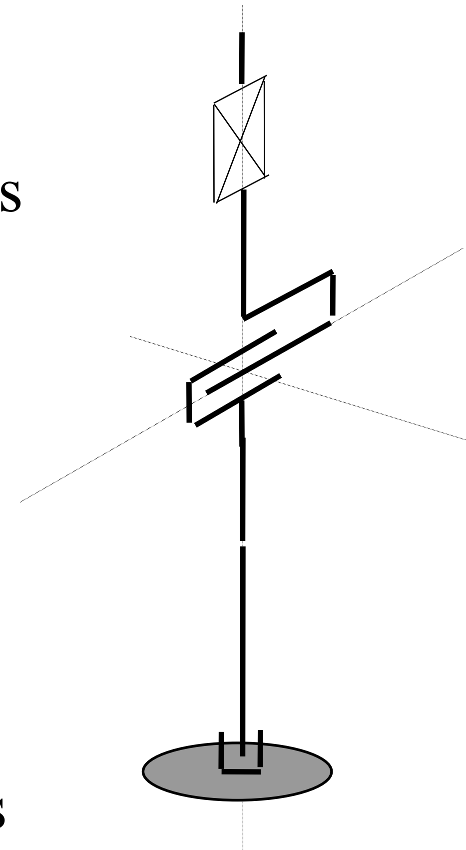
$$\boxed{\theta_2 = \arctan\left(\frac{\pm\sqrt{p_{ax}^{2} + p_{ay}^{2}}}{p_{az} - d_1}\right) + n\pi} \qquad n = -1,0,1$$

$$\boxed{d_3 = \pm\sqrt{p_{ax}^{2} + p_{ay}^{2} + \left(p_{az} - d_1\right)^2}}$$

The spherical RRP arm - solutions

$$\theta_1 = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) + n\pi \qquad n = -1,0,1$$

2 solutions



$$\theta_2 = \arctan\left(\frac{\pm\sqrt{p_{ax}{}^2 + p_{ay}{}^2}}{p_{az} - d_1}\right) + n\pi \qquad n = -1,0,1$$

4 solutions

$$d_3 = \sqrt{p_{ax}^2 + p_{ay}^2 + (p_{az} - d_1)^2}$$

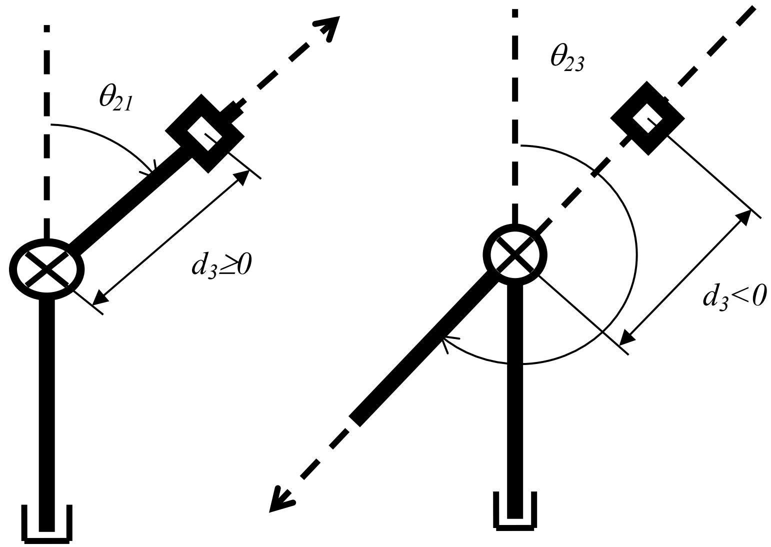
2 solutions

Altogether: 2x4x2=16 solutions

Analysis of number of solutions – 4 appropriate results:

$$\theta_{11} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right)$$

$$\theta_{21} = \arctan\left(\frac{\sqrt{p_{ax}^{\ 2} + p_{ay}^{\ 2}}}{p_{az} - d_1}\right)$$

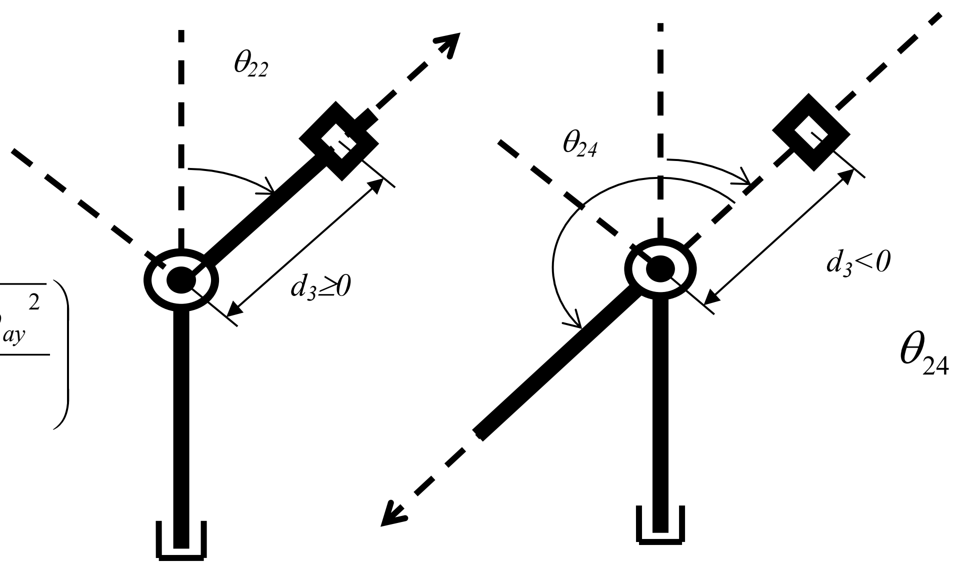


$$\theta_{11} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right)$$

$$\theta_{23} = \arctan\left(\frac{\sqrt{p_{ax}^{\ 2} + p_{ay}^{\ 2}}}{p_{az} - d_1}\right) \pm \pi$$

$$\theta_{12} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) \pm \pi$$

$$\theta_{22} = \arctan\left(\frac{-\sqrt{p_{ax}^{\ 2} + p_{ay}^{\ 2}}}{p_{az} - d_1}\right)$$



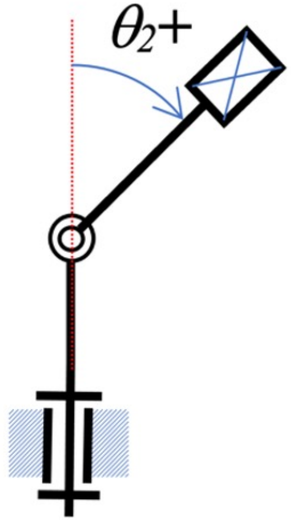
$$\theta_{12} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) \pm \pi$$

$$\theta_{24} = \arctan\left(\frac{-\sqrt{p_{ax}^{\ 2} + p_{ay}^{\ 2}}}{p_{az} - d_1}\right) \pm \pi$$

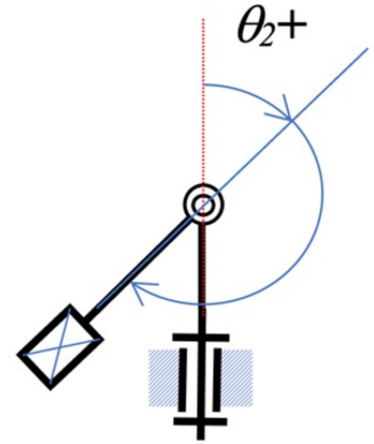
For $\boldsymbol{\theta}_{11}$ and $d_3>0$ there are 4 solutions

$$\theta_{21} = \frac{\sqrt{{p_{ax}}^2 + {p_{ay}}^2}}{p_{az} - d_1}$$

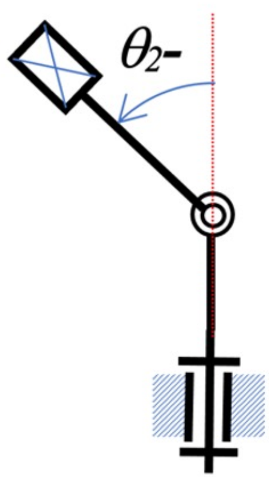
The only correct solution



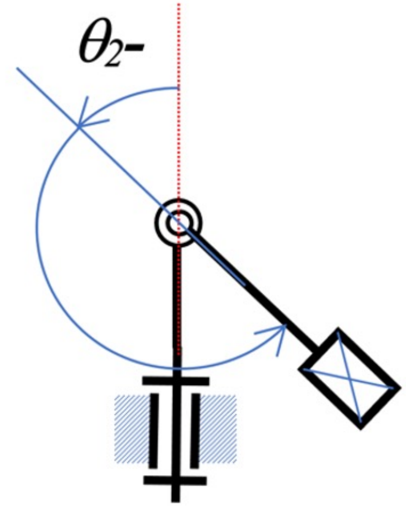
$$\theta_{23} = \frac{\sqrt{{p_{ax}}^2 + {p_{ay}}^2}}{p_{az} - d_1} \pm \pi$$



$$\theta_{22} = \frac{-\sqrt{{p_{ax}}^2 + {p_{ay}}^2}}{p_{az} - d_1}$$



$$\theta_{24} = \frac{-\sqrt{{p_{ax}}^2 + {p_{ay}}^2}}{p_{az} - d_1} \pm \pi$$

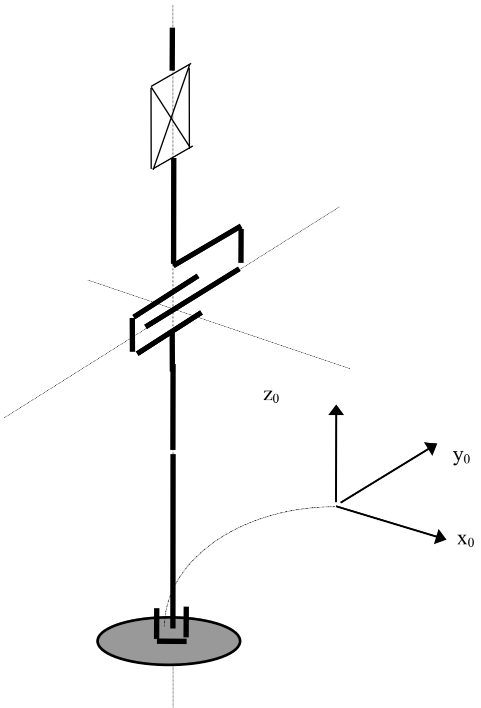


Analysis of existence of a solution:

$$\arctan\left(\frac{p_{ay}}{p_{ax}}\right) = \arctan\left(\frac{0}{0}\right) \Rightarrow \begin{array}{l} p_{ax} = 0 \\ p_{ay} = 0 \end{array}$$

Infinite number of solutions.
The θ_1 angle can take any value

$$\arctan\left(\frac{\sqrt{p_{ax}^2 + p_{ay}^2}}{p_{az} - d_1}\right) = \arctan\left(\frac{0}{0}\right) \Rightarrow \begin{array}{l} p_{ax} = 0 \\ p_{ay} = 0 \\ p_{az} = d_1 \end{array}$$



Infinite number of solutions.
The θ_2 angle can take any value

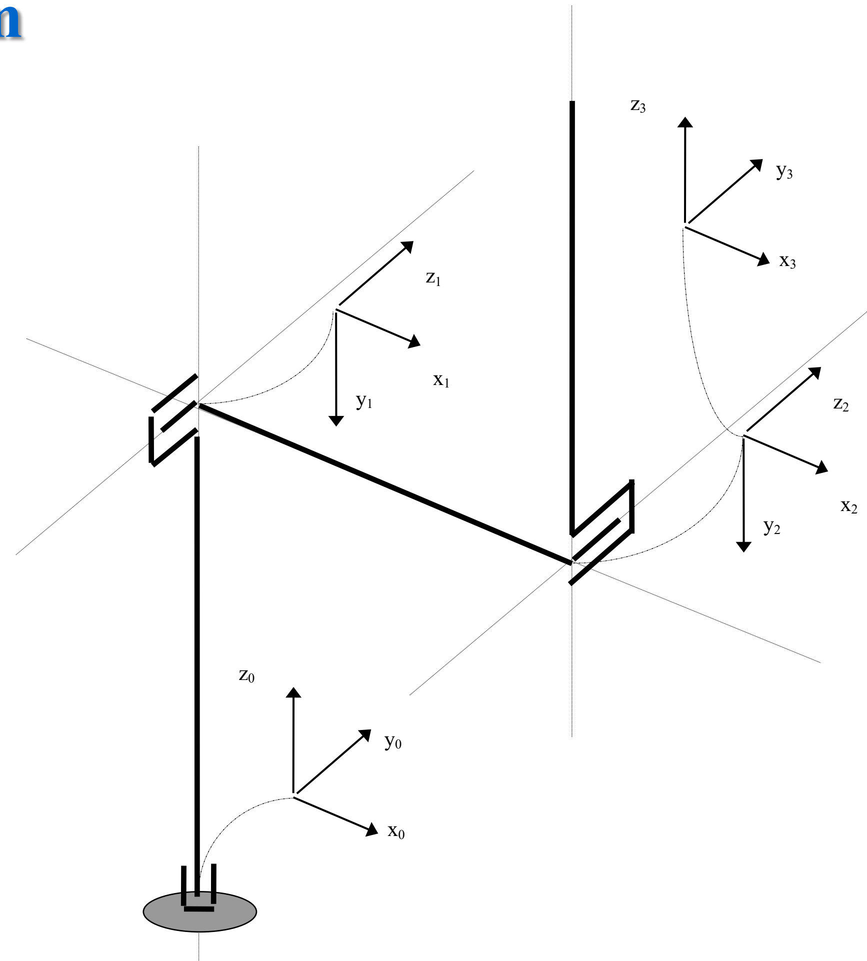
$$p_{ax}^2 + p_{ay}^2 + (p_{az} - d_1)^2 \geq 0$$

A solution for d_3 always exists

D. The anthropomorphic RRR arm

$$\underline{T}_3 = \begin{bmatrix} C_1C_{23} & -S_1 & C_1S_{23} & a_2C_1C_2 \\ S_1C_{23} & C_1 & S_1S_{23} & a_2S_1C_2 \\ -S_{23} & 0 & C_{23} & d_1 - a_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To be determined: θ_1, θ_2, θ_3.



Remark !

Coordinate frame No. 3 is located at the beginning of the 3rd link

Coordinates of the 3rd link tip with respect to the 3rd coordinate system:

$$\overline{p}_4^{\,*} = \begin{bmatrix} 0 \\ 0 \\ d_4 \\ 1 \end{bmatrix}$$

Coordinates of the arm's tip with respect to the reference frame

$$\overline{p}_a = {}^0\underline{T}_3 \overline{p}_4^{\,*}$$

$$\begin{bmatrix} p_{ax} \\ p_{ay} \\ p_{az} \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 C_{23} & -S_1 & C_1 S_{23} & a_2 C_1 C_2 \\ S_1 C_{23} & C_1 & S_1 S_{23} & a_2 S_1 C_2 \\ -S_{23} & 0 & C_{23} & d_1 - a_2 S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ d_4 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 S_{23} d_4 + a_2 C_1 C_2 \\ S_1 S_{23} d_4 + a_2 S_1 C_2 \\ C_{23} d_4 + d_1 - a_2 S_2 \\ 1 \end{bmatrix}$$

A set of equations to be solved:

$$p_{ax} = C_1 S_{23} d_4 + a_2 C_1 C_2$$
$$p_{ay} = S_1 S_{23} d_4 + a_2 S_1 C_2$$
$$p_{az} = C_{23} d_4 + d_1 - a_2 S_2$$

$$\Rightarrow$$

$$p_{ax} = C_1 \left(S_{23} d_4 + a_2 C_2 \right)$$
$$p_{ay} = S_1 \left(S_{23} d_4 + a_2 C_2 \right)$$

$$\boxed{\theta_1 = \arctan\left(\frac{p_{ay}}{p_{ax}} \right) + n\pi \qquad n = -1,0,1}$$

$$p_{ax} = C_1 \left(S_{23} d_4 + a_2 C_2 \right)$$
$$p_{ay} = S_1 \left(S_{23} d_4 + a_2 C_2 \right)$$

$$\left(S_{23} d_4 + a_2 C_2 \right)^2 = p_{ax}{}^2 + p_{ay}{}^2$$

$$p_{az} = C_{23} d_4 + d_1 - a_2 S_2$$

$$S_{23} d_4 = \alpha - a_2 C_2$$
$$C_{23} d_4 = p_{az} - d_1 + a_2 S_2$$
$$\boxed{\alpha = \pm \sqrt{p_{ax}{}^2 + p_{ay}{}^2}}$$

$$d_4{}^2 = \alpha^2 + a_2{}^2 C_2{}^2 - 2 a_2 \alpha C_2 + p_{az}{}^2 + d_1{}^2 + a_2{}^2 S_2{}^2 - 2 p_{az} d_1 + 2 a_2 p_{az} S_2 - 2 a_2 d_1 S_2$$

$$2 a_2 \left(-\left(p_{az} - d_1 \right) S_2 + \alpha C_2 \right) = \alpha^2 + p_{az}{}^2 + d_1{}^2 + a_2{}^2 - 2 p_{az} d_1 - d_4{}^2$$

$$-\left(p_{az} - d_1 \right) S_2 + \alpha C_2 = \frac{\alpha^2 + p_{az}{}^2 + d_1{}^2 + a_2{}^2 - 2 p_{az} d_1 - d_4{}^2}{2 a_2} = \beta$$

$$-A \sin\theta + B \cos\theta = D$$

$$\theta = \arctan\left(\frac{B}{A}\right) - \arctan\left(\frac{D}{\pm\sqrt{A^2 + B^2 - D^2}}\right)$$

$$A = p_{az} - d_1$$

$$B = \alpha$$

$$D = \beta$$

$$\boxed{\theta_2 = \arctan\left(\frac{\alpha}{p_{az} - d_1}\right) - \arctan\left(\frac{\beta}{\pm\sqrt{(p_{az} - d_1)^2 + (\alpha)^2 - \beta^2}}\right)}$$

$$S_{23} d_4 = \alpha - a_2 C_2$$

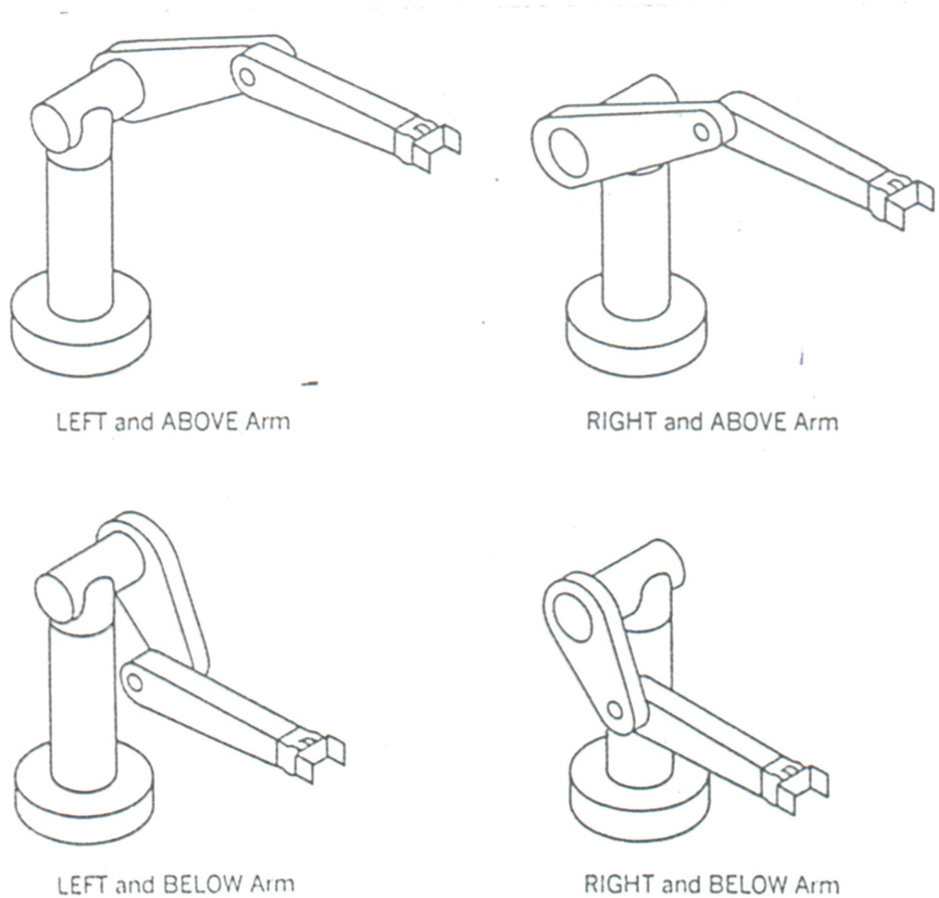
$$C_{23} d_4 = p_{az} - d_1 + a_2 S_2$$

$$\alpha = \pm\sqrt{p_{ax} \quad + p_{ay}{}^{2}}$$

$$\tan(\theta_2 + \theta_3) = \frac{\alpha - a_2 C_2}{p_{az} - d_1 + a_2 S_2}$$

$$\boxed{\theta_3 = \arctan\left(\frac{\alpha - a_2 C_2}{p_{az} - d_1 + a_2 S_2}\right) - \theta_2}$$

There are 4 correct solutions



(Fu)

$$\theta_1 = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) + n\pi$$

$$n = -1,0,1$$

$$\theta_2 = \arctan\left(\frac{\alpha}{p_{az} - d_1}\right) - \arctan\left(\frac{\beta}{\pm\sqrt{(p_{az} - d_1)^2 + (\alpha)^2 - \beta^2}}\right) \qquad \theta_3 = \arctan\left(\frac{\alpha - a_2 C_2}{p_{az} - d_1 + a_2 S_2}\right) - \theta_2$$

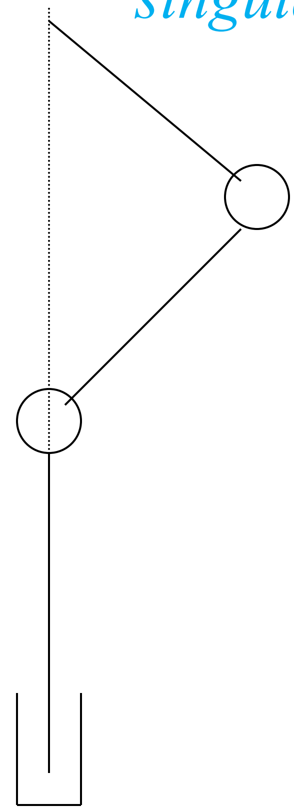
Analysis of existence and a number of solutions

$$\theta_1 = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) + n\pi$$

$$n = -1,0,1$$

$$\arctan\left(\frac{p_{ay}}{p_{ax}}\right) = \arctan\left(\frac{0}{0}\right) \Rightarrow \boxed{\begin{aligned} p_{ax} &= 0 \\ p_{ay} &= 0 \end{aligned}}$$

Shoulder singularity



Infinite number of solutions.
The angle θ_1 can take any value

THE SECOND STEP

$$q_1, q_2, q_3 \longrightarrow \underline{R}_3$$

$${}^{o}\underline{T}_e = \begin{bmatrix} N_x & O_x & A_x & P_x \\ N_y & O_y & A_y & P_y \\ N_z & O_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}\underline{r}_3^{T}\,{}^{0}\underline{r}_e = \begin{bmatrix} n_{wx} & o_{wx} & a_{wx} \\ n_{wy} & o_{wy} & a_{wy} \\ n_{wz} & o_{wz} & a_{wz} \end{bmatrix}$$

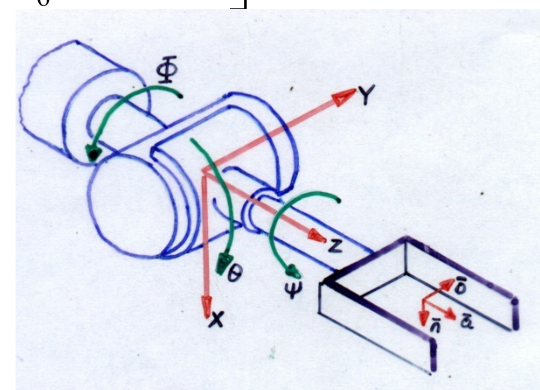
A set of equations (only 3 equations are independent)

$$\begin{bmatrix} n_{wx} & o_{wx} & a_{wx} \\ n_{wy} & o_{wy} & a_{wy} \\ n_{wz} & o_{wz} & a_{wz} \end{bmatrix} = \underline{R}_4 \underline{R}_5 \underline{R}_6 \quad \Rightarrow \quad \underline{R}_4^{-1} \begin{bmatrix} n_{wx} & o_{wx} & a_{wx} \\ n_{wy} & o_{wy} & a_{wy} \\ n_{wz} & o_{wz} & a_{wz} \end{bmatrix} = R_5 R_6$$

THE SECOND STEP - Examples

A. The Spherical (Euler) wrist:

$$\begin{bmatrix} C_4 n_{wx} + S_4 n_{wy} & C_4 o_{wx} + S_4 o_{wy} & C_4 a_{wx} + S_4 a_{wy} \\ -n_{wz} & -o_{wz} & -a_{wz} \\ -S_4 n_{wx} + C_4 n_{wy} & -S_4 o_{wx} + C_4 o_{wy} & -S_4 a_{wx} + C_4 a_{wy} \end{bmatrix} = \begin{bmatrix} C_5 C_6 & -C_5 S_6 & S_5 \\ S_5 C_6 & -S_5 S_6 & -C_5 \\ S_6 & C_6 & 0 \end{bmatrix}$$



[3,3] $\Rightarrow$ $$\theta_4 = \arctan\left(\frac{a_{wy}}{a_{wx}}\right)$$

[1,3], [2,3] $\Rightarrow$ $$\theta_5 = \arctan\left(\frac{C_4 a_{wx} + S_4 a_{wy}}{a_{wz}}\right)$$

[3,1], [3,2] $\Rightarrow$ $$\theta_6 = \arctan\left(\frac{-S_4 n_{wx} + C_4 n_{wy}}{-S_4 o_{wx} + C_4 o_{wy}}\right)$$

$$\theta_4 = \arctan\left(\frac{a_{wy}}{a_{wx}}\right) \Rightarrow \quad \sin(\theta_4) = \frac{a_{wy}}{\pm\sqrt{a_{wx}^2 + a_{wy}^2}}$$

$$\cos(\theta_4) = \frac{a_{wx}}{\pm\sqrt{a_{wx}^2 + a_{wy}^2}}$$

If a_{wz}=1, then:

$$a_{wx} = 0 \quad \text{and} \quad a_{wy} = 0$$

$$\theta_5 = \arctan\left(\frac{C_4 a_{wx} + S_4 a_{wy}}{a_{wz}}\right) = \arctan\left(\frac{\pm\sqrt{a_{wx}^2 + a_{wy}^2}}{a_{wz}}\right) = \arctan\left(\frac{\pm 0}{1}\right) = 0 \quad \text{or} \quad \pm\pi$$

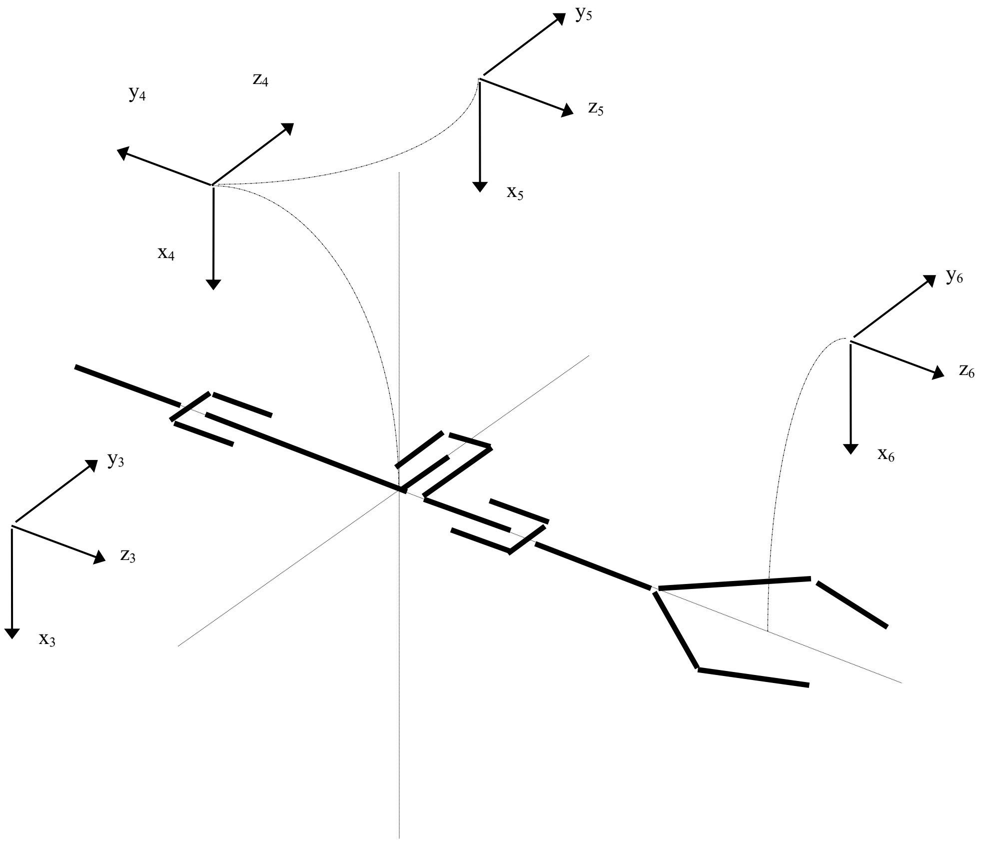
a_{wz}=-1

$$\theta_4 = \arctan\left(\frac{0}{0}\right) \Rightarrow \quad \theta_4 = ?$$

$$\theta_6 = ?$$

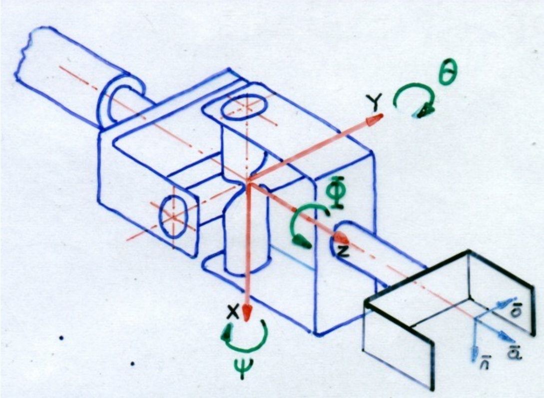
for θ_5=0° the 4th and 6th axes of motion are collinear

There is an infinite number of the: θ_4 and θ_6 values of pairs corresponding to the command end-effector orientation.



B. The Spherical (RPY) wrist:

$$\begin{bmatrix} C_4 n_{wx} + S_4 n_{wy} & C_4 o_{wx} + S_4 o_{wy} & C_4 a_{wx} + S_4 a_{wy} \\ -n_{wz} & -o_{wz} & -a_{wz} \\ -S_4 n_{wx} + C_4 n_{wy} & -S_4 o_{wx} + C_4 o_{wy} & -S_4 a_{wx} + C_4 a_{wy} \end{bmatrix} = \begin{bmatrix} C_5 & S_5 S_6 & S_5 C_6 \\ S_5 & -C_5 S_6 & -C_5 C_6 \\ 0 & C_6 & -S_6 \end{bmatrix}$$



[3,1] $\Longrightarrow$ $$\theta_4 = \arctan\left(\frac{n_{wy}}{n_{wx}}\right)$$

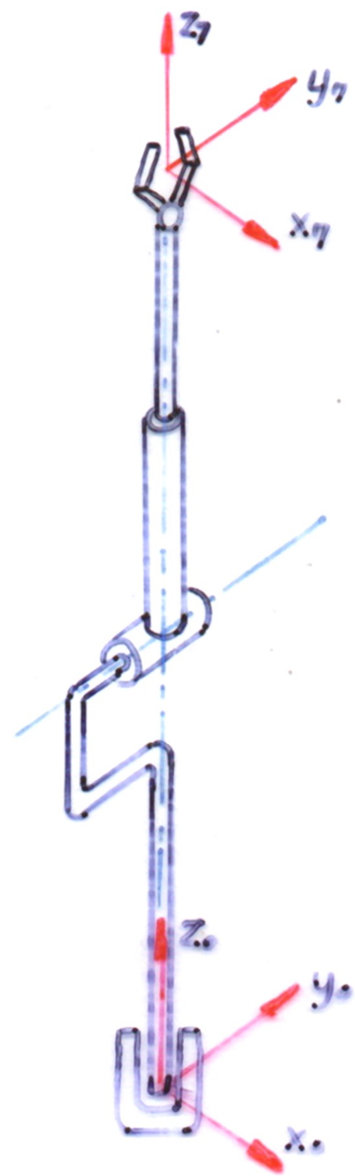
[1,1], [2,1] $\Longrightarrow$ $$\theta_5 = \arctan\left(\frac{-n_{wz}}{C_4 n_{wx} + S_4 n_{wy}}\right)$$

[3,2], [3,3] $\Longrightarrow$ $$\alpha_6 = \arctan\left(\frac{S_4 a_{wx} - C_4 a_{wy}}{-S_4 o_{wx} + C_4 o_{wy}}\right)$$

Example of a RRP+RPY manipulator

Geometrical model:

Link No.	θ	d	a	α	joint motion range
1	θ_1 v	d_1	0	-90°	-150°÷150°
2	θ_2 v	0	0	90°	30°÷120°
3	0	d_3 v	0	0	d_p÷d_k (d_p>0 i d_k>0)
4	θ_4 v	d_4	0	-90°	-120°÷120°
5	θ_5 v	0	0	90°	-60°÷60°
6	0	0	0	α_6 v	-90°÷90°
6	0	d_7	0	0	-



Problem formulation:

The following pose of an end-effector is specified:

$$\underline{\boldsymbol{\tau}}_t = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} d_7 + L \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} d_7 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{1}{2} d_7 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d_1, d_7 and L –
- positive constant values

To be determined:

Values of θ_1, θ_2, d_3, θ_4, θ_5, α_6 joint coordinates

Determination of the arm tip position:

$$A_x = \frac{\sqrt{3}}{2\sqrt{2}}$$

$$A_y = \frac{\sqrt{3}}{2\sqrt{2}}$$

$$A_z = -\frac{1}{2}$$

$$\left|\overline{p_w}\right| = d_7$$

$$\Rightarrow$$

$$p_{wx} = \frac{\sqrt{3}}{2\sqrt{2}} d_7$$

$$p_{wy} = \frac{\sqrt{3}}{2\sqrt{2}} d_7$$

$$p_{wz} = -\frac{1}{2} d_7$$

$$p_{ax} = P_x - p_{wx} = \frac{\sqrt{3}}{2\sqrt{2}} d_7 + L - \frac{\sqrt{3}}{2\sqrt{2}} d_7 = L$$

$$p_{ay} = P_y - p_{wy} = \frac{\sqrt{3}}{2\sqrt{2}} d_7 - \frac{\sqrt{3}}{2\sqrt{2}} d_7 = 0$$

$$p_{az} = P_z - p_{wz} = -\frac{1}{2} d_7 + d_1 + \frac{1}{2} d_7 = d_1$$

Solution No 1

$$p_{ax} = L \qquad p_{ay} = 0 \qquad p_{az} = d_1$$

$$\theta_{11} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) = \arctan\left(\frac{0}{L}\right) = 0^\circ$$

$$\theta_{21} = \arctan\left(\frac{\sqrt{p_{ax}{}^2 + p_{ay}{}^2}}{p_{az} - d_1}\right) = \arctan\left(\frac{\sqrt{L^2 + 0^2}}{d_1 - d_1}\right) = a\tan 2\left(\frac{L}{0}\right) = 90^\circ$$

$$d_{31} = \sqrt{p_{ax}{}^2 + p_{ay}{}^2 + (p_{az} - d_1)^2} = \sqrt{L^2 + 0^2 + (d_1 - d_1)^2} = L$$

Solution No 2

$$\theta_{12} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) \pm 180^\circ = \mathrm{atan2}\left(\frac{0}{L}\right) \pm 180^\circ = \pm 180^\circ$$

$$\theta_{22} = \arctan\left(\frac{-\sqrt{p_{ax}{}^2 + p_{ay}{}^2}}{p_{az} - d_1}\right) = \arctan\left(\frac{-\sqrt{L^2 + 0^2}}{d_1 - d_1}\right) = \mathrm{atan2}\left(\frac{-L}{0}\right) = -90^\circ$$

$$d_{31} = \sqrt{p_{ax}{}^2 + p_{ay}{}^2 + (p_{az} - d_1)^2} = \sqrt{L^2 + 0^2 + (d_1 - d_1)^2} = L$$

Solution No 3

$$p_{ax} = L$$
$$p_{ay} = 0$$
$$p_{az} = d_1$$

$$\theta_{11} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) = \mathrm{arctg}\left(\frac{0}{L}\right) = 0^\circ$$

$$\theta_{23} = \arctan\left(\frac{\sqrt{p_{ax}{}^2 + p_{ay}{}^2}}{p_{az} - d_1}\right) \pm 180^\circ = \arctan\left(\frac{\sqrt{L^2 + 0^2}}{d_1 - d_1}\right) \pm 180^\circ = \mathrm{atan2}\left(\frac{L}{0}\right) \pm 180^\circ = -90^\circ$$

$$d_{32} = -\sqrt{p_{ax}{}^2 + p_{ay}{}^2 + (p_{az} - d_1)^2} = -\sqrt{L^2 + 0^2 + (d_1 - d_1)^2} = -L$$

Solution No 4

$$\theta_{12} = \arctan\left(\frac{p_{ay}}{p_{ax}}\right) \pm 180^\circ = \mathrm{atan2}\left(\frac{0}{L}\right) \pm 180^\circ = \pm 180^\circ$$

$$\theta_{24} = \arctan\left(\frac{-\sqrt{p_{ax}{}^2 + p_{ay}{}^2}}{p_{az} - d_1}\right) \pm 180^\circ = \arctan\left(\frac{-\sqrt{L^2 + 0^2}}{d_1 - d_1}\right) \pm 180^\circ = \mathrm{atan2}\left(\frac{-L}{0}\right) \pm 180^\circ = 90^\circ$$

$$d_{32} = -\sqrt{p_{ax}{}^2 + p_{ay}{}^2 + (p_{az} - d_1)^2} = -\sqrt{L^2 + 0^2 + (d_1 - d_1)^2} = -L$$

θ_1	**0°**	180°	**0°**	180	-150°-150°
θ_2	**90°**	-90°	-90°	**90°**	30°-120°
d_3	**L**	**L**	-L	-L	>0

The only result consistent with the assumed motion ranges:

$$\boxed{\theta_1 = 0^\circ \qquad \theta_2 = 90^\circ \qquad d_3 = L}$$

$$\theta_1 = 0^\circ \qquad \theta_2 = 90^\circ \qquad d_3 = L$$

A substitution to $\underline{R}_3$

$$\underline{R}_3 = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 \\ S_1C_2 & C_1 & S_1S_2 \\ -S_2 & 0 & C_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \underline{r}_3^{-1} = r_3^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$${}^3\underline{r}_6 = \underline{r}_3^{-1}\underline{r}_6 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

$$\begin{bmatrix} n_{wx} & o_{wx} & a_{wx} \\ n_{wy} & o_{wy} & a_{wy} \\ n_{wz} & o_{wz} & a_{wz} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} \end{bmatrix}$$

$$\theta_4 = \arctan\left(\frac{n_{wy}}{n_{wx}}\right) = \operatorname{atan2}\left(\frac{\frac{1}{\sqrt{2}}}{0}\right) = 90^\circ$$

$$\theta_5 = \arctan\left(\frac{-n_{wz}}{C_4 n_{wx} + S_4 n_{wy}}\right) = \operatorname{atan2}\left(\frac{\frac{1}{\sqrt{2}}}{0 + \frac{1}{\sqrt{2}}}\right) = 45^\circ$$

$$\alpha_6 = \arctan\left(\frac{S_4 a_{wx} - C_4 a_{wy}}{-S_4 o_{wx} + C_4 o_{wy}}\right) = \arctan\left(\frac{1 * \frac{1}{2} - 0 * \frac{\sqrt{3}}{2\sqrt{2}}}{-1 * \left(-\frac{\sqrt{3}}{2}\right) + 0 * \frac{1}{2\sqrt{2}}}\right) = \operatorname{atan2}\left(\frac{1}{\sqrt{3}}\right) = 30^\circ$$

2.
Mechanics of manipulators 3
Geometrical model
Inverse Kinematics (Inverse Kinematic Problem)