Akademia Górniczo-Hutnicza
im. Stanisława Staszica w Krakowie

WHEEL MOBILE ROBOTS
LECTURE– 03 – DYNAMICS
Lagrange

dr hab. inż. Tomasz Buratowski

Faculty of Mechanics and Robotics
Department of Robotics and Mechatronics

Dynamics modeling of the two-wheeled mobile robot

In dynamics modeling of the mobile robots we often use Langrange equation of type II with multipliers or Maggie's equation with are based on Lagrange's ones.

Dynamic equation of motion of the mobile robots can be used in order to solve simple and inverse dynamics problem.

In simple dynamics problem we obtain parameters of motion.

In inverse dynamics problem we obtain forces and torques acting on a robot.

Dynamics – Lagrange II

One of the mathematical formalism used in dynamics modeling is Lagrange of type II.

Its general form:

$$\frac{d}{dt}\left(\frac{\partial E}{\partial \dot{q}}\right)^{T} - \left(\frac{\partial E}{\partial q}\right)^{T} = Q + J^{T}(q)\lambda \qquad (3.1)$$

where:

- q – vector of generalized coordinates
- $E = E(q, \dot{q})$ - energy of the system
- Q – vector of generalized forces
- J(q) – jacobian resulting from constrains equations
- λ - vector of Lagrange's multipliers

Dynamics – Lagrange II

Resulting dynamics equation can be difficult to solve, that is why we use a transformation allowing to decouple multiplier from driving torques. As a result we write equation 3.1 in matrix form:

$$M(q)\ddot{q} + C(q,\dot{q})q = B(q)\tau + J^T(q)\lambda \qquad (3.2)$$

where:

- M – inertia matrix
- C – centrifugal and Coriolis force matrix
- B – forces and torques coefficient matrix
- τ – vector of forces and torques

Dynamics – Lagrange II

Next we decompose vector Q of generalized coordinates to a form of:

$$q = [q_1, q_2]^T \qquad q \in R^n, \quad q_1 \in R^m, \quad q_2 \in R^{n-m} \tag{3.3}$$

Now we can write constrain equation in a form of:

$$[J_1(q), J_2(q)] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = 0 \quad \det \qquad J_1(q) \neq 0 \tag{3.4}$$

Vector q_2 must be selected in such a manner to correspond with number of DOFs:

$$\dot{q} = \begin{bmatrix} J_{12}(q) \\ I_{n-m} \end{bmatrix} \dot{q}_2 = T(q)\dot{q}_2$$

$$\ddot{q} = \dot{T}(q)\dot{q}_2 + T(q)\ddot{q}_2$$

where:

$$J_{12} = -J_1^{-1}(q)J_2(q) \tag{3.5}$$

I_{n-m} – unit matix

Dynamics – Lagrange II

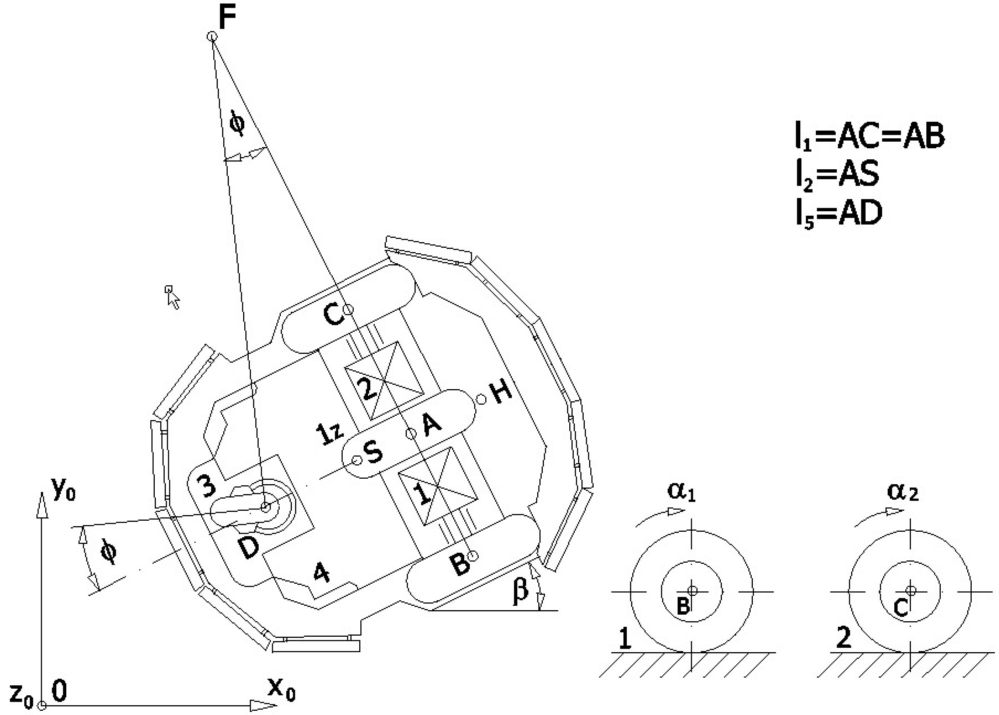
After decoupling procedure described before we obtain dynamic equation of motion:

$$\begin{aligned} M_{12}(q)\ddot{q} + C_{12}(q,\dot{q})\dot{q}_2 &= B\ (q)\tau + J_1^T(q)\lambda \\ M_{22}(q_2)\ddot{q} + C_{22}(q_2,\dot{q}_2)\dot{q}_2 &= B_2(q_2)\tau \end{aligned} \tag{3.6}$$

Those equations allow us to solve simple and inverse dynamics problem. Dynamics modeling is crucial for proper steering of the system discussed in this course

Lagrange II – dynamics modeling

In order to simplify calculation wheels 1 and 2 were substituted with one common wheel, for which angle of rotation is α



Lagrange II – dynamics modeling

Generalized vector of coordinates has a form of:

$$q = [x_A, y_A, \beta, \alpha]^T \tag{3.7}$$

Kinetic energy of the system (in a function of generalized coordinates) without taking into account idler wheel has a form of:

$$\begin{aligned} E_k = {} & \frac{1}{2}(m_1 + m_2 + m_4)\dot{x}_A^2 + ((m_1 - m_2)l_1 \cos(\beta) + m_4 l_2 \sin(\beta))\dot{\beta}\dot{x}_A + \\ & + \frac{1}{2}(m_1 + m_2 + m_4)\dot{y}_A^2 + ((m_1 - m_2)l_1 \sin(\beta) - m_4 l_2 \cos(\beta))\dot{\beta}\dot{x}_A + \\ & + \frac{1}{2}((m_1 + m_2)l_1^2 + Ix_1 + Iz_1 h^2 + Ix_2 + Iz_2 h^2 + Iz_4 + m_4 l_2^2)\dot{\beta}^2 + (Iz_1 h - Iz_2 h)\dot{\beta}\dot{\alpha} + \frac{1}{2}(Iz_1 + Iz_2)\dot{\alpha}^2 \end{aligned} \tag{3.8}$$

Lagrange II – dynamics modeling

Taking into account idler wheel, kinetic energy has a form of:

$$
\begin{aligned}
E_k = {} & \frac{1}{2}(m_1+m_2+m_3+m_4)\dot{x}_A^2 + ((m_1-m_2)l_1\cos(\beta)+(m_3l_5+m_4l_2)\sin(\beta))\dot{\beta}\dot{x}_A + \\
& + \frac{1}{2}(m_1+m_2+m_3+m_4)\dot{y}_A^2 + ((m_1-m_2)l_1\sin(\beta)-(m_3l_5+m_4l_2)\cos(\beta))\dot{\beta}\dot{y}_A + \\
& + \frac{1}{2}((m_1+m_2)l_1^2 + Ix_1 + Iz_1h^2 + Ix_2 + Iz_2h^2 + Ix_3 + m_3l_5^2 + Iz_4 + m_4l_2^2)\dot{\beta}^2 + \\
& + (Iz_1h - Iz_2h)\dot{\beta}\dot{\alpha} + \frac{1}{2}(Iz_1+Iz_2)\dot{\alpha}^2 + \frac{1}{2}\frac{Iz_3(v_A^2+l_1^2\dot{\beta}^2)}{r_3^2} + \frac{Ix_3\dot{\beta}l_1\ddot{\beta}v_A}{v_A^2+l_1^2\dot{\beta}^2} + \frac{1}{2}\frac{Ix_3l_1^2\ddot{\beta}^2v_A^2}{(v_A^2+l_1^2\dot{\beta}^2)^2}
\end{aligned}
\tag{3.9}
$$

where: m_1, m_2, m_3, m_4 – mases of elements of the system

- Ix_1, Ix_2, Ix_3, Iz_1, Iz_2, Iz_3, Iz_4 – inertia moments w.r.t. particular axis
- h –l_1/r_1 ratio

Lagrange II – dynamics modeling

Assuming $m_1=m_2$, $Ix_1=Ix_2$, $Iz_1=Iz_2$, $r=r_1=r_2$ inertia matrix for a system without idler wheel has a form of:

$$M=\begin{bmatrix} 2m_1+m_4 & 0 & m_4 l_2\sin(\beta) & 0 \\ 0 & 2m_1+m_4 & -m_4 l_2\cos(\beta) & 0 \\ m_4 l_2\sin(\beta) & -m_4 l_2\cos(\beta) & 2m_1 l_1^2+2Ix_1+2Iz_1 h^2+Iz_4+m_4 l_2^2 & 0 \\ 0 & 0 & 0 & 2Iz_1 \end{bmatrix} \quad (3.10)$$

For a system with idler wheel has a form of:

$$M=\begin{bmatrix} 2m_1+m_3+m_4 & 0 & (m_3 l_5+m_4 l_2)\sin(\beta) & 0 \\ 0 & 2m_1+m_3+m_4 & -(m_3 l_5+m_4 l_2)\cos(\beta) & 0 \\ (m_3 l_5+m_4 l_2)\sin(\beta) & -(m_3 l_5+m_4 l_2)\cos(\beta) & 2m_1 l_1^2+2Ix_1+2Iz_1 h^2+Iz_4+m_4 l_2^2+Ix_3+m_3 l_5^2+\%1 & 0 \\ 0 & 0 & 0 & 2Iz_1 \end{bmatrix}$$

$$\%1=\frac{Ix_3 l_1 v_A}{v_A^2+l_1^2\dot{\beta}^2}-\frac{8Ix_3 l_1^3\dot{\beta}^2 v_A}{(v_A^2+l_1^2\dot{\beta}^2)^2}+\frac{8Ix_3 l_1^5\dot{\beta}^4 v_A-6Ix_3 l_1^4\dot{\beta}^2 v_A^2}{(v_A^2+l_1^2\dot{\beta}^2)^3}+\frac{12Ix_3 l_1^6\dot{\beta}^4 v_A^2}{(v_A^2+l_1^2\dot{\beta}^2)^4}+\frac{Iz_3 l_1^2}{r_3^2} \quad (3.11)$$

Lagrange II – dynamics modeling

Coriolis and centrifugal force matrix for a system without idler wheel has a form of:

$$C = \begin{bmatrix} 0 & 0 & m_4 l_2 \dot{\beta} \cos(\beta) & 0 \\ 0 & 0 & m_4 l_2 \dot{\beta} \sin(\beta) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.12}$$

For a system with idler wheel has a form of: *(3.13)*

$$C = \begin{bmatrix} 0 & 0 & (m_3 l_5 + m_4 l_2)\dot{\beta}\cos(\beta) & 0 \\ 0 & 0 & (m_3 l_5 + m_4 l_2)\dot{\beta}\sin(\beta) & 0 \\ 0 & 0 & \dfrac{-(l_1^6 \dot{\beta}^6 + (12 l_1^5 v_A - 13 v_A^2 l_1^4)\dot{\beta}^4 - (30 l_1^3 v_A^3 + 5 l_1^2 v_A^4)\dot{\beta}^2 + 6 l_1 v_A^5 + 9 v_A^6)\ddot{\beta}\dot{\beta} I x_3 l_1^3 v_A^3}{(v_A^2 + l_1^2 \dot{\beta}^2)^5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lagrange II – dynamics modeling

Vector of forces and torques for a system without idler wheel:

$$\tau = \begin{bmatrix} 0 \\ 0 \\ (M_1 - N_1 f_1 \operatorname{sgn}(\dot{\alpha}))h - (M_2 - N_2 f_2 \operatorname{sgn}(\dot{\alpha}))h \\ M_1 + M_2 - N_1 f_1 \operatorname{sgn}(\dot{\alpha}) - N_2 f_2 \operatorname{sgn}(\dot{\alpha}) \end{bmatrix} \qquad (3.14)$$

For a system with idler wheel has a form of:

$$\tau = \begin{bmatrix} -\dfrac{N_3 f_3 \operatorname{sgn}(\dot{\alpha}_3)}{r_3} \\ -\dfrac{N_3 f_3 \operatorname{sgn}(\dot{\alpha}_3)}{r_3} \\ (M_1 - N_1 f_1 \operatorname{sgn}(\dot{\alpha}))h - (M_2 - N_2 f_2 \operatorname{sgn}(\dot{\alpha}))h \\ M_1 + M_2 - N_1 f_1 \operatorname{sgn}(\dot{\alpha}) - N_2 f_2 \operatorname{sgn}(\dot{\alpha}) \end{bmatrix} \qquad (3.15)$$

gdzie: M_1, M_2 – driving torques

- N_1, N_2, N_3 – wheel loads
- f_1, f_2, f_3 – friction coefficient of particular wheel

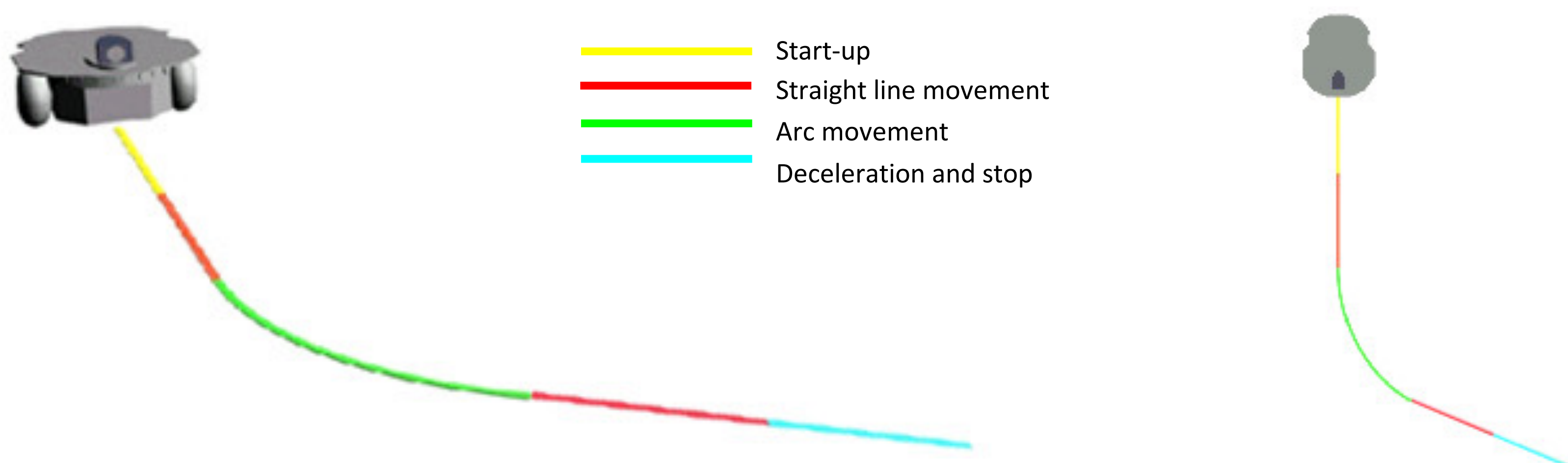
Lagrange II – dynamics modeling

Using decoupling transformation we write Langrange's equation in a form of (for a system without idler wheel):

$$
\begin{aligned}
&m_4 l_2 \sin(\beta)\ddot{\beta} + (m_4 + 2m_1) r \cos(\beta)\ddot{\alpha} + m_4 l_2 \cos(\beta)\dot{\beta}^2 - (m_4 + 2m_1) r \sin(\beta)\dot{\beta}\dot{\alpha} = \lambda_1 \\
&-m_4 l_2 \cos(\beta)\ddot{\beta} + (m_4 + 2m_1) r \sin(\beta)\ddot{\alpha} + m_4 l_2 \sin(\beta)\dot{\beta}^2 + (m_4 + 2m_1) r \cos(\beta)\dot{\beta}\dot{\alpha} = \lambda_2 \\
&(2Iz_1 h^2 + 2m_1 l_1^2 + 2Ix_1 + Iz_4 + m_4 l_2^2)\ddot{\beta} - m_4 l_2 r \dot{\beta}\dot{\alpha} = (M_1 - N_1 f_1 \operatorname{sgn}(\dot{\alpha}) - M_2 + N_2 f_2 \operatorname{sgn}(\dot{\alpha}))h \\
&(r^2 m_4 + 2r^2 m_1 + 2Iz_1)\ddot{\alpha} + m_4 l_2 r \dot{\beta}^2 = M_1 + M_2 - N_1 f_1 \operatorname{sgn}(\dot{\alpha}) - N_2 f_2 \operatorname{sgn}(\dot{\alpha})
\end{aligned}
\tag{3.16}
$$

Lagrange II – dynamics simulation

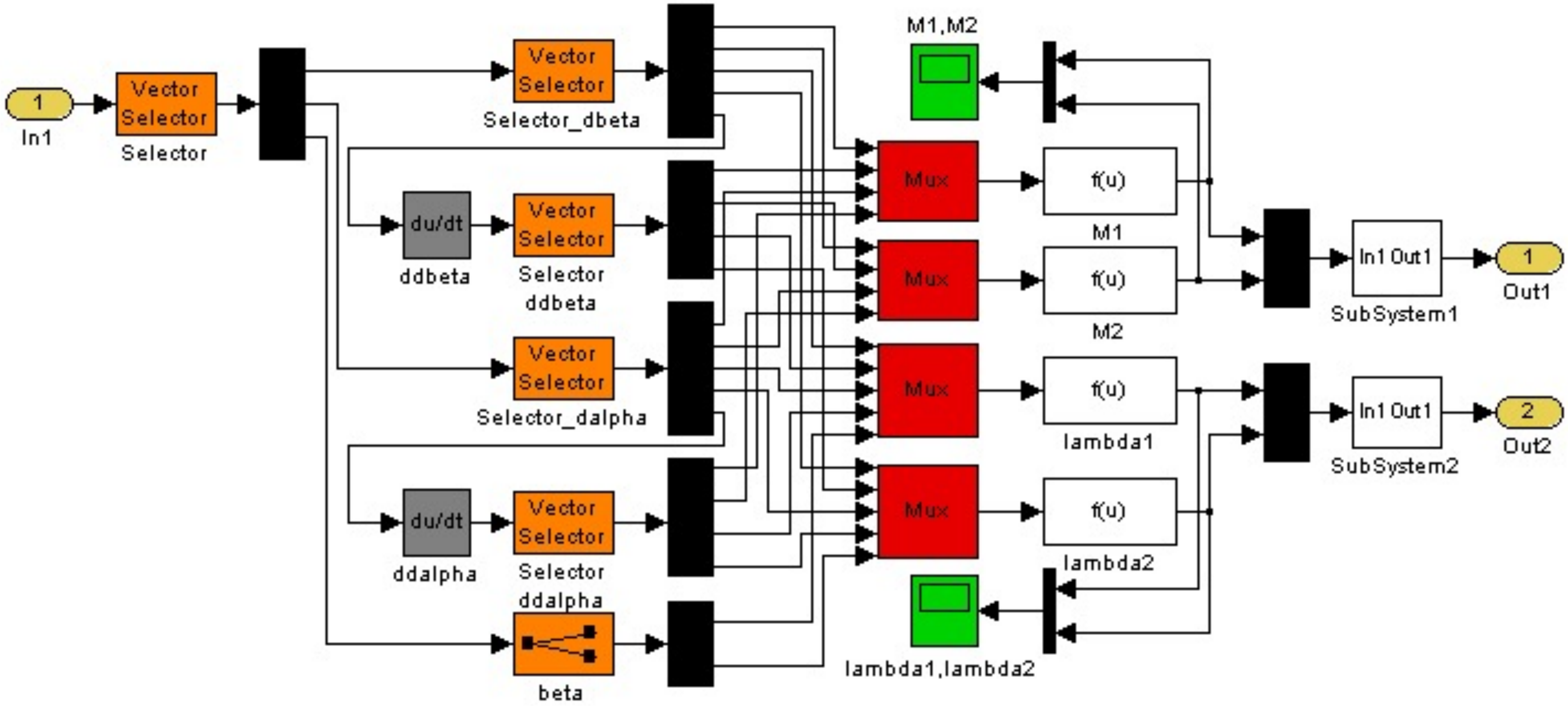
Based on a path proposed below and physical properties of a robot in a table below simulation model was prepared:



l_2 [m]	m_1 [kg]	m_2 [kg]	m_3 [kg]	m_4 [kg]	Ix_1[kgm^2]	Ix_2[kgm^2]	Ix_3[kgm^2]	Iz_1[kgm^2]
0.07	1.5	1.5	0.5	5.67	0.02	0.02	0.005	0.051
Iz_2[kgm^2]	Iz_3[kgm^2]	Iz_4[kgm^2]	N_1 [N]	N_2 [N]	N_3 [N]	f_1	f_2	f_3
0.051	0.002	0.154	31.25	31.25	29.20	0.01	0.01	0.001

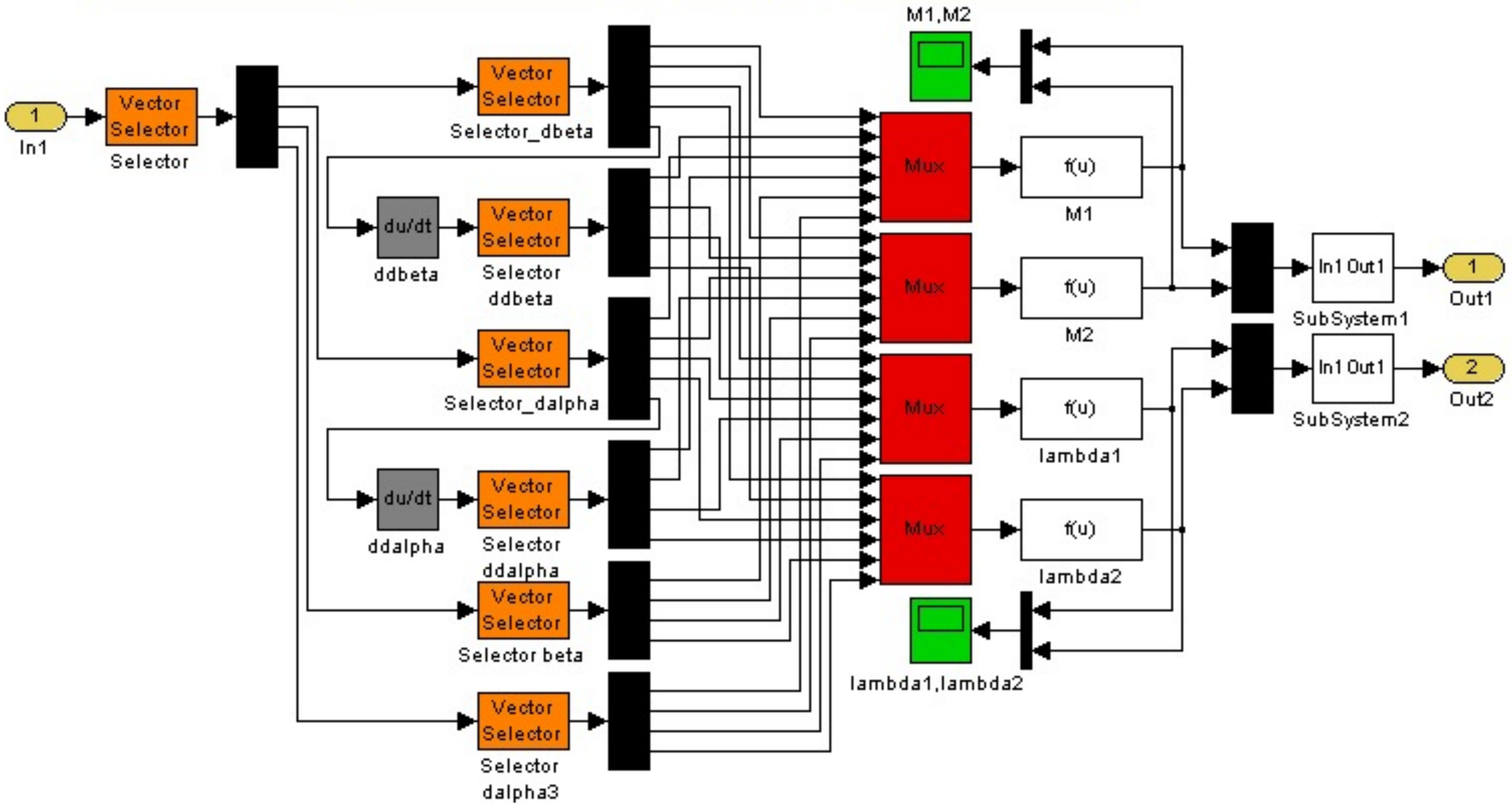
Lagrange II – dynamics simulation

As an input to the model we use output from inverse kinematic solver (case without idler wheel):



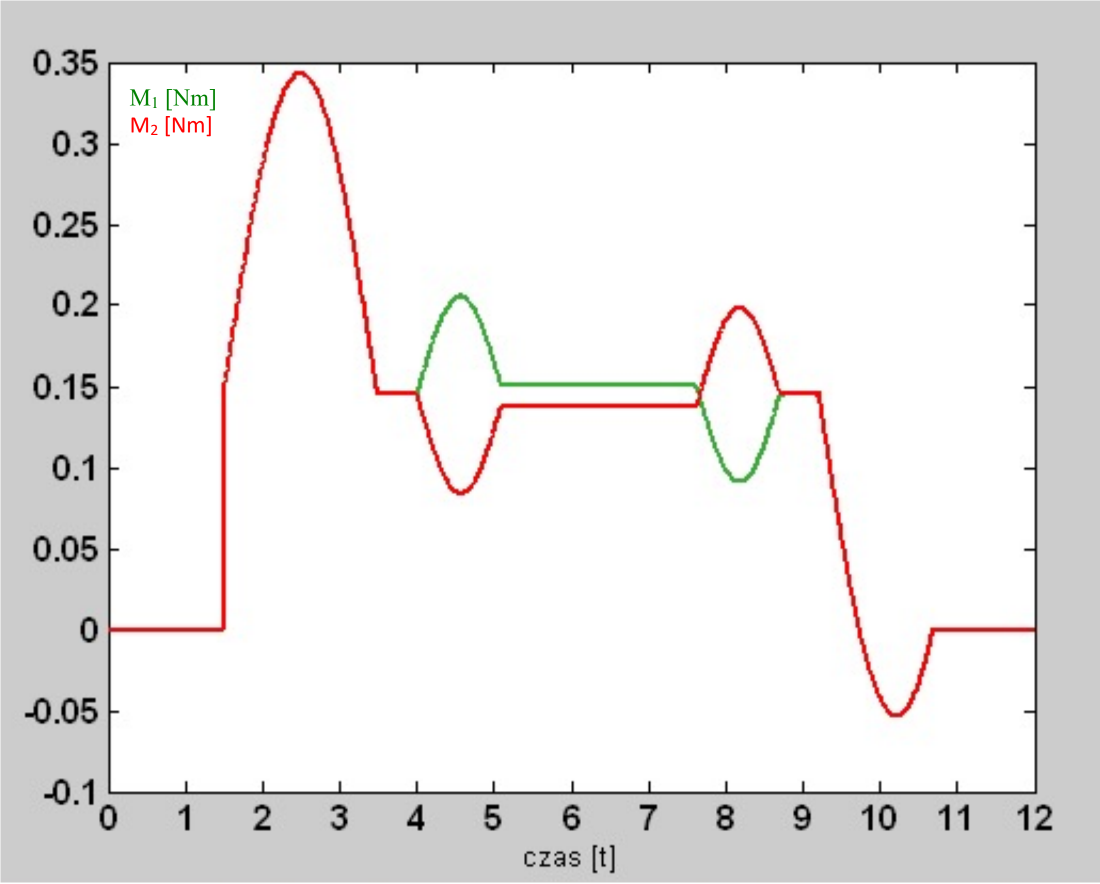
Lagrange II – dynamics simulation

(case with idler wheel):



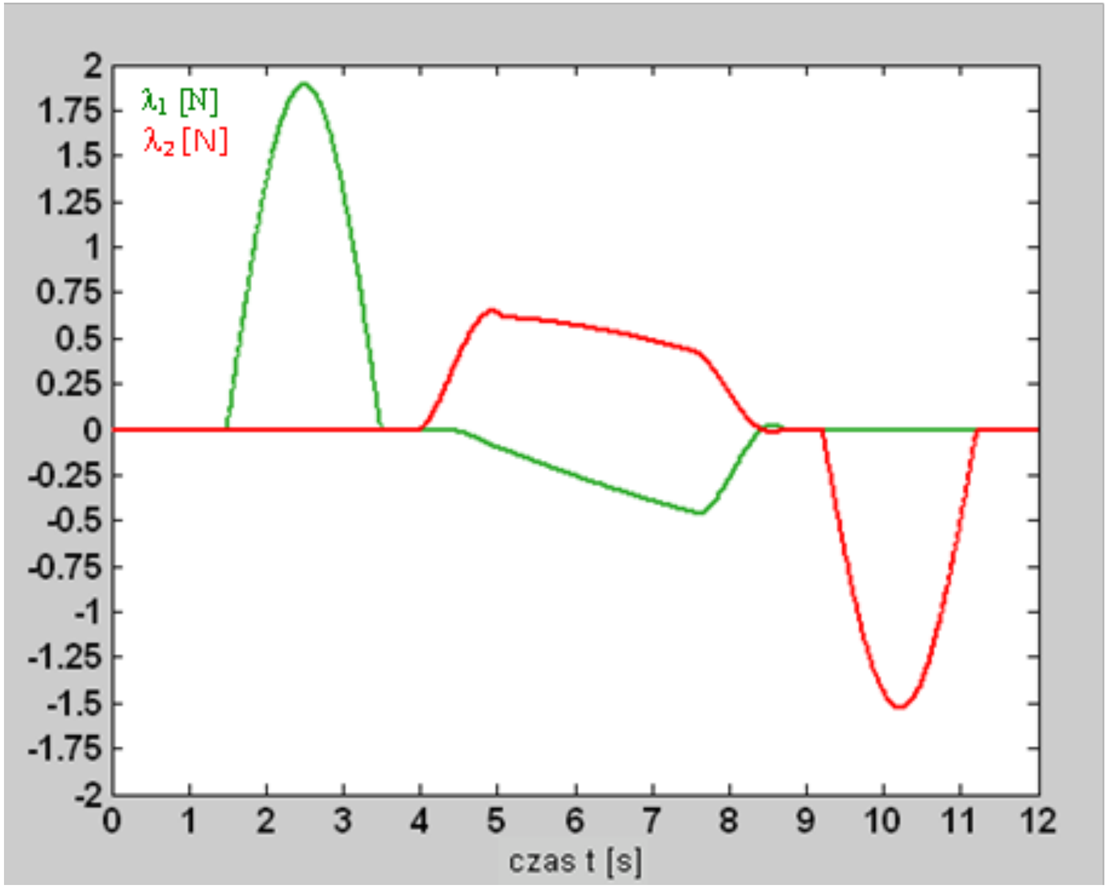
Lagrange II – dynamics simulation

Driving torques output (w/o idler wheel)



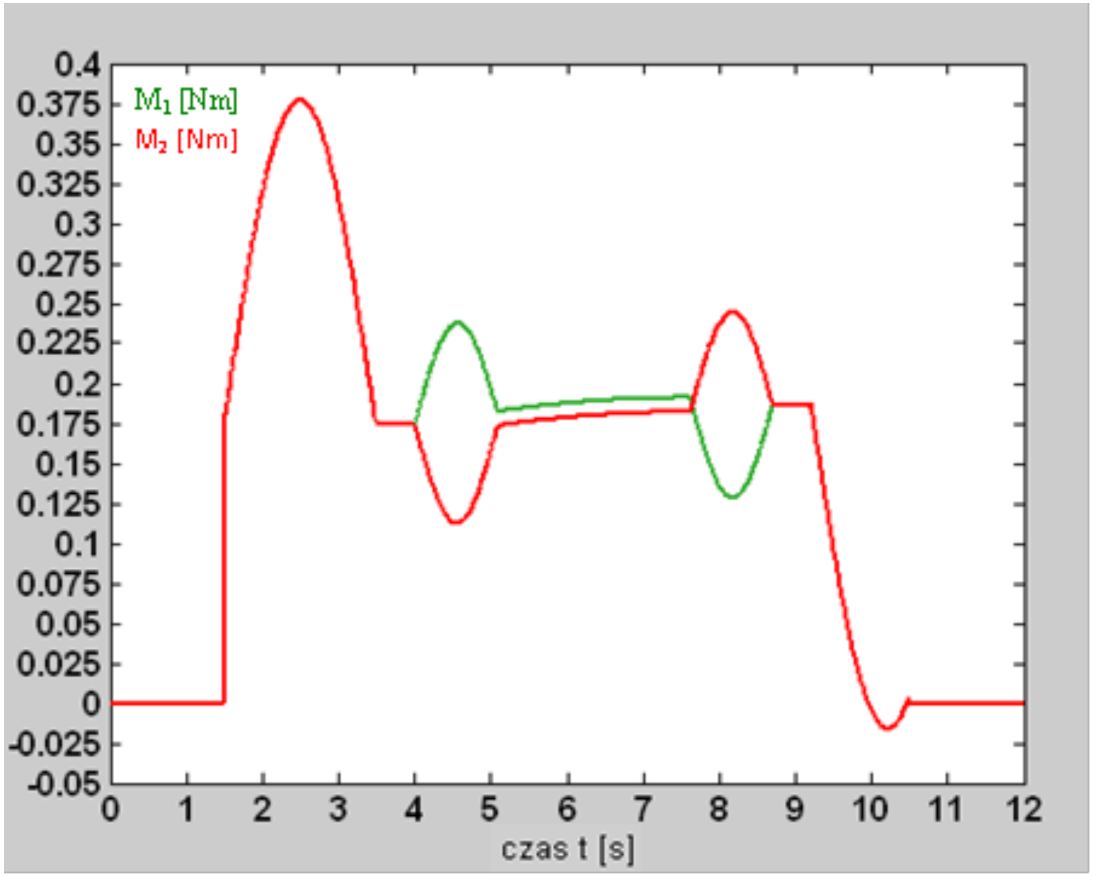
Lagrange II – dynamics simulation

Lagrange's multipliers output (w/o idler wheel)



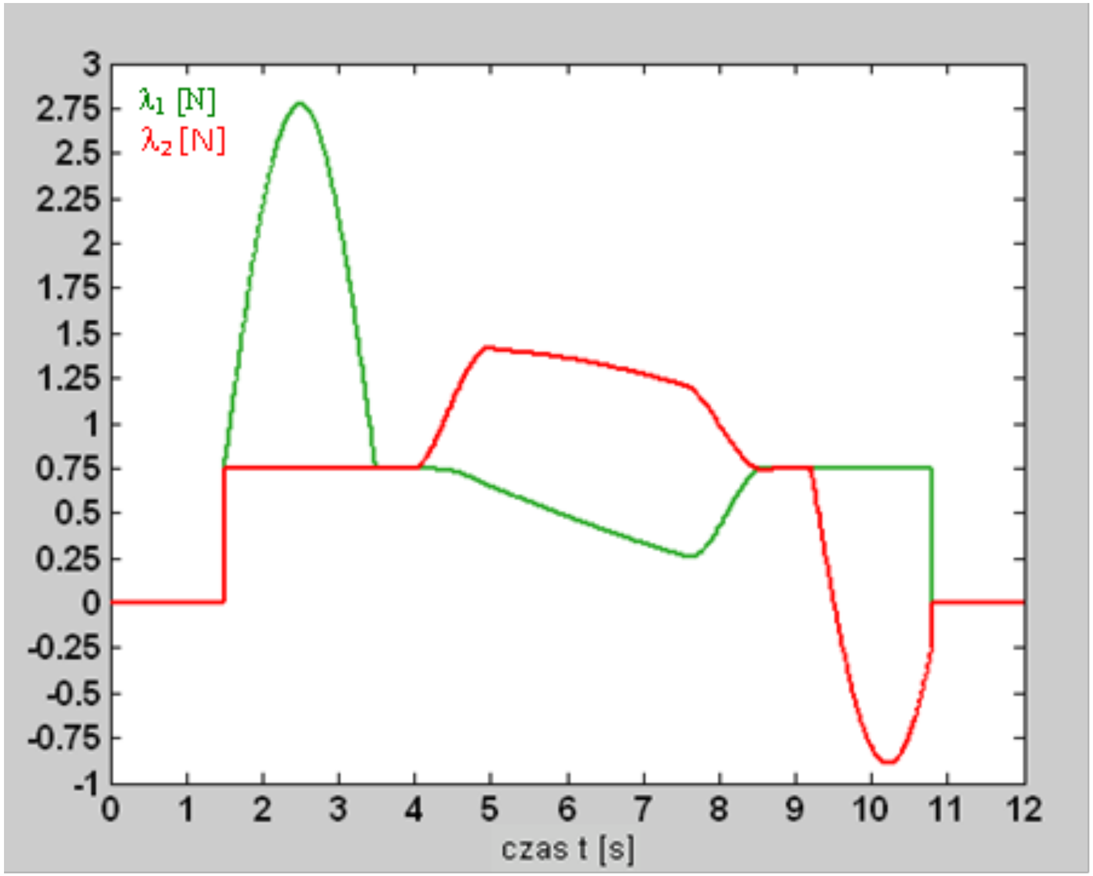
Lagrange II – dynamics simulation

Driving torques output (with idler wheel)



Lagrange II – dynamics simulation

Lagrange's multipliers output (with idler wheel)



THANK YOU