Physics Foundations: Vectors

Part 1: Scalars *vs.* Vectors, Basic Operations

A proper measurement needs both a numerical *value* and a *unit*. The unit serves as necessary context for readers of your work, and the value conveys information about the relative scale and significant digits of the measurement. This value-unit combination is called the **magnitude** of a measurement; *the value of a measurement without associated units is meaningless!*

Some physical properties can be adequately described using *only* a magnitude; such quantities are called **scalars**. You already have some experience measuring various scalar quantities in the lab, including lengths [cm,*m], times [*s], masses [g,*kg], temperatures [°C,*K], volumes [cm^3,*m^3], and densities [g/cm^3,*kg/m^3]. Addition and subtraction of scalar quantities is trivial when all measurements have the same units: just add the magnitudes! In fact, all scalars that represent the same type of physical quantity (e.g., masses) can be arithmetically combined, provided you convert all of them to the same units.

*: International Standard System of Units, (a.k.a., **SI units** / standard units)

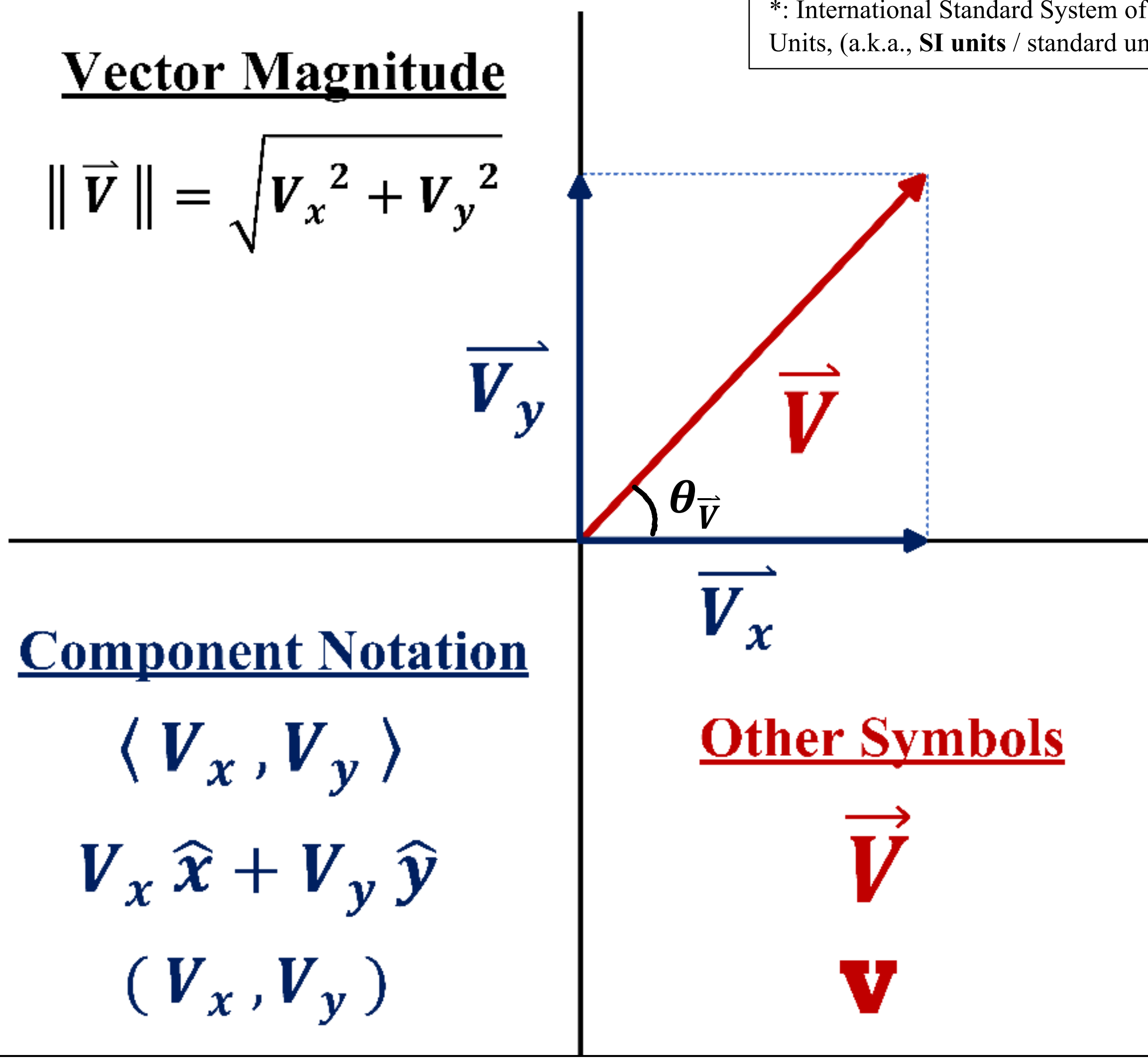


Figure 1: An arbitrary **vector $\vec{V}$** with direction $\theta_{\vec{V}}$, along with its **x- and y-components**. Also shown are various other ways that vectors are represented symbolically, including with its components; $\hat{x}$ and $\hat{y}$ are **unit vectors**, which specify *1-D direction*. A vector's length indicates its magnitude (denoted $\|\vec{V}\|$), which can either be measured or calculated using its components in the 2-D distance formula (see **Equation 1**).

Other physical properties *cannot* be fully described with magnitude alone. Life in 3-dimensional space leads physicists to seek *directional* information as well, which is often a relevant factor when analyzing physical systems. Measurements that include both a *magnitude* and a *direction* are called **vectors**, which are used extensively throughout the field of physics. Vectors are represented graphically by arrows, and symbolically with a variety of notation, examples of which are summarized in **Figure 1**. The direction is quantified by the angle $\boldsymbol{\theta_{\vec{V}}}$, which is **always measured from the positive x-direction**.

The classic example of a scalar quantity is **speed**, which is how we describe changes in distance with respect to time (e.g., 100 km/hour) in terms of magnitude alone. If we also have *direction* information associated with a speed, the combined magnitude and direction is called a **velocity** (e.g., 100 km/hour *Eastward*), which is a vector quantity. While speeds may occasionally be useful in everyday situations, physics problems will almost always involve comparison of and operations with velocities, not to mention forces, momenta, and other vector quantities.

Addition and subtraction of vector quantities isn't nearly as simple as addition and subtraction of scalar quantities, considering the directional information that must also be combined. Graphically, vectors can be combined using the **head-to-tail method**, which is illustrated in **Figure 2** below for three arbitrary vectors: $\vec{A}$, $\vec{B}$, and $\vec{C}$.

Step 1: Draw a coordinate system, then draw each vector with its **tail starting at the origin**.

Step 2: Place the tail of each successive vector at the **head** of the previous vector.

Step 3: Draw a vector from the origin to the head of the final vector being added. This new vector is called the **resultant vector**.

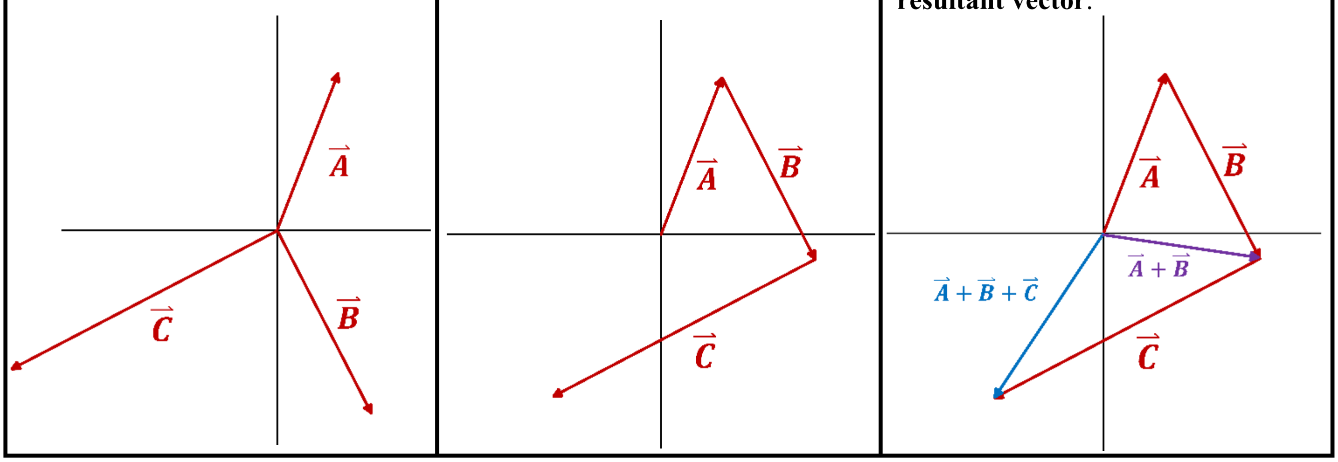
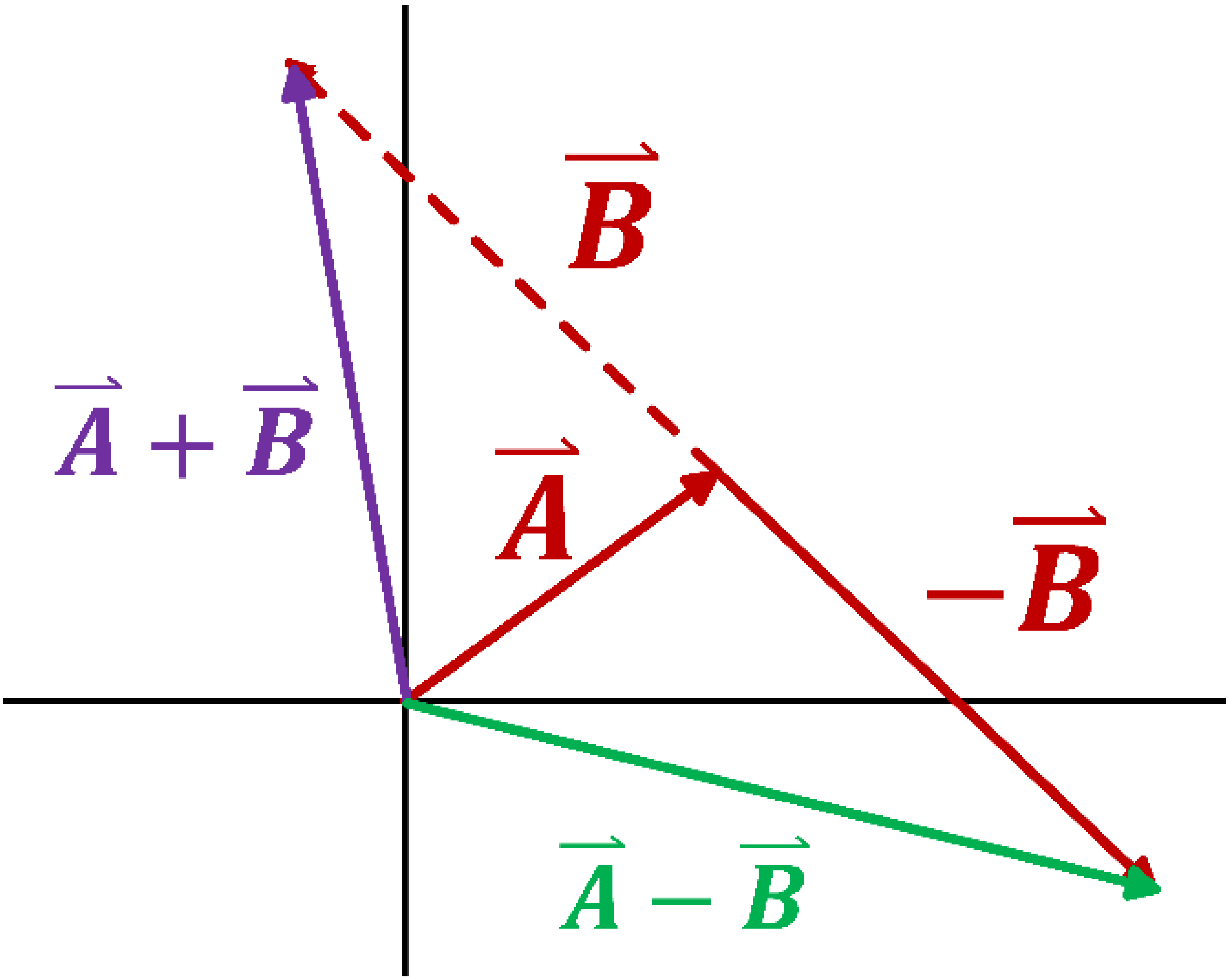


Figure 2: Head-to-tail vector addition, illustrated step-by-step. Any number of vectors can be added, provided you always follow the steps above. When adding vectors, the result is *always* a vector; **vectors and scalars cannot be added together**.

Vector addition isn't the only possible vector operation; we can also multiply vectors by *dimensionless scalars* (i.e., by numbers with no units)! The vector's magnitude (length) is what will be multiplied by the scalar, with the vector's direction remaining unchanged when the scalar is positive. For example, consider some arbitrary distance vector, given by $\vec{D} = 7\ \text{km}\ \hat{E}$. The operation $2 \times \vec{D}$ would create a resultant vector that **points in the same direction** (East), but with **twice the magnitude** (14 km). Symbolically, $2 \times \vec{D} = 14\ \text{km}\ \hat{E}$.

The same logic applies to multiplying a vector by a *negative* scalar, but the minus sign indicates that the vector should *also* be rotated by 180° (i.e., flipped). For the hypothetical vector $\vec{D}$, the operation $-1 \times \vec{D}$ would create a resultant vector with the same magnitude (7 km), but pointing in the ***opposite*** **direction** (West). Symbolically, $-1 \times \vec{D} = -\vec{D} = 7\text{ km }\widehat{W}$.This idea forms the basis of **vector subtraction**, which is equivalent to vector addition after rotating one vector by 180°. See **Figure 3** for a comparison of vector addition and vector subtraction for two arbitrary vectors.



<u>Figure 3</u>: Vector subtraction vs. vector addition. Notice how $\|-\vec{B}\| = \|\vec{B}\|$, and how $\theta_{-\vec{B}} = \theta_{\vec{B}} + 180°$.

Part 2: Vector Components

The method above is typically only used to illustrate the concept of vector addition, because drawing and measuring vectors is totally impractical for doing actual science. For the head-to-tail method to actually work in the laboratory, the magnitudes and angles of the vectors involved (1) must be drawn and re-drawn *perfectly* to scale (i.e., perfectly representative of the vectors' actual magnitudes and directions), and (2) must be measured with sufficiently high precision.

Instead, vectors are typically added using the **component method**, which involves splitting multi-dimensional vectors into their 1-D components. For simplicity, we'll generally be working in two dimensions, where vectors have horizontal (x) and vertical (y) components. The magnitude of *any* 2-D vector—e.g., the vector in **Figure 1**—can be obtained using the trusty 2-D distance formula:'

$$V \equiv \|\vec{V}\| = \sqrt{{V_x}^2 + {V_y}^2} \tag{1}$$

The magnitudes of a vector's *components* (V_x and V_y) and are mathematically related to the *vector's* magnitude (V) via trigonometry:

$$V_x = V\cos\theta_V \tag{2.X}$$
$$V_y = V\sin\theta_V \tag{2.Y}$$

Note that **Equation 2** is <u>not universally applicable</u> for *any* angle given in a physics problem; sometimes the given angle is <u>**not**</u> $\boldsymbol{\theta_V}$—i.e., is not the vector's direction as measured from the $+\hat{x}$ direction—but some *other* angle. This is why it's critical to <u>always</u> draw a detailed diagram!

Additionally, for directions with $\theta_V > 90°$, using **Equation 2** may result in negative values! A summary of component signs for each quadrant is given in **Table 1** below:

<u>Table 1</u>: Vector Components, Their Signs, and Direction

Quadrant	Angular Range	x-component	y-component	Direction (θ_V)
I	0°–90°	(+)	(+)	θ^*
II	90°–180°	(−)	(+)	$180° - \theta^*$
III	180°–270°	(−)	(−)	$180° + \theta^*$
IV	270°–360°	(+)	(−)	$360° - \theta^*$

***:** θ is the angle from **Equation 4**, and θ_V is the vector's angle with respect to $+\hat{x}$ (i.e., its direction).

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Let's say we have n vectors ($\vec{V_i}$) that we want to add together using the component method. After splitting the vectors into their x- and y-components using **Equation 2**, we can **sum all components in the same dimension** to obtain the x- and y-components of the *resultant vector*. Mathematically,

$$R_x = \sum_{i=1}^{n} V_{i,x} \tag{3.X}$$

$$R_y = \sum_{i=1}^{n} V_{i,y} \tag{3.Y}$$

An example of adding the components of two arbitrary vectors is shown in **Figure 4** below. The resultant vector $\vec{R}$ is in Quadrant I, so no need to worry about component signs…yet.

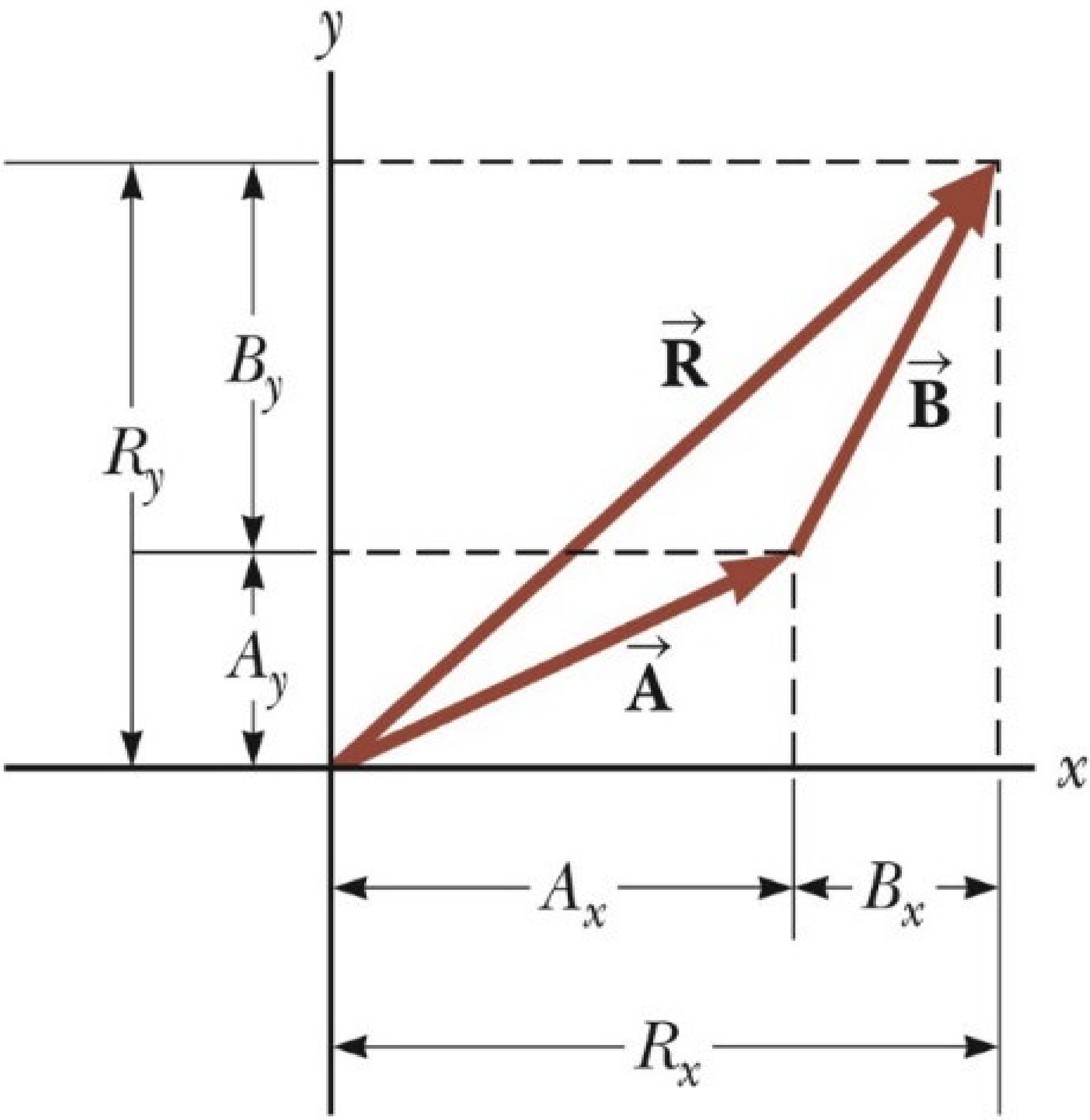

Figure 4: Applying the **component method** to $\vec{A} + \vec{B} = \vec{R}$. To calculate $R \equiv \|\vec{R}\|$, use **Equation 2** to find the vector components, sum those components using **Equation 3**, and use the R_x & R_y in **Equation 1**.

Finally, you can also calculate the resultant vector's *direction* (i.e., its angle with respect to the $+\hat{x}$ direction) by using a little more trigonometry. Any trigonometrically-sound analysis with known quantities will do the trick, but often you'll use **Equations 2** and **3** to obtain the resultant vector's components. The angle that $\vec{R}$ makes with the *x*-axis is then given by **Equation 4**:

$$\theta = \tan^{-1}\left(\frac{|R_y|}{|R_x|}\right) \tag{4}$$

It's important to note that the θ obtained from **Equation 4** is with respect to the negative *x*-axis for vectors in quadrants II and III; compare the signs of your calculated R_x and R_y with **Table 1** if you're unsure which angle you're calculating, or which quadrant your resultant vector occupies. If necessary, use the final column of **Table 1** to help you determine the resultant vector's direction.