FERM 506 HW I

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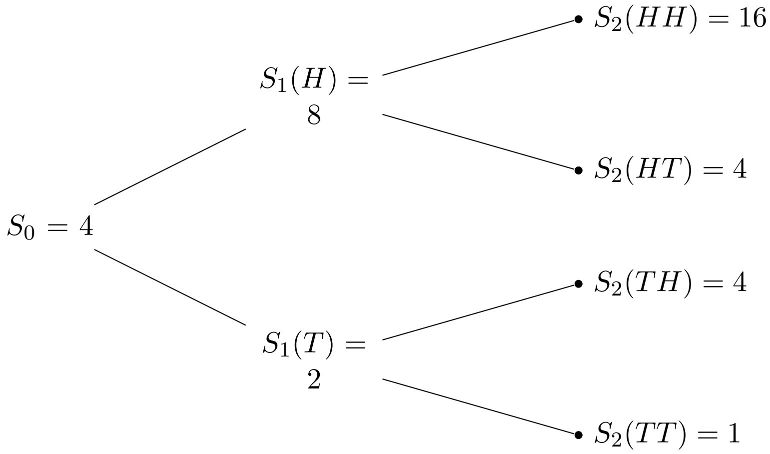
1 Problem I

We will have the decision tree as follows:



Assuming we have waited until $t = 2$, values of the American put option with strike $K = 6$ will be $V_2(HH) = 0, V_2(HT) = V_2(TH) = 2, V_2(TT) = 5$.Then, we have risk neutral probabilities $\hat{p} = \hat{q} = 0.5$ and

$$V_1(H) = \frac{1}{1 + 0.25}(0.5 \times 0 + 0.5 \times 2) = 0.8 \text{ and } V_1(T) = \frac{1}{1 + 0.25}(0.5 \times 2 + 0.5 \times 5) = 2.8$$

However, if we choose to exercise at $t = 1$ and at probability T, value of the option will be $V_1(T) = K - S_1(T) = 4$. Since this value is greater than 2.8 based on expected value, it means investor should exercise option at this stage without waiting further. I will use this value $V_1(T)$ and $V_1(H) = 0$ for the calculation of V_0 .

$$V_0 = \frac{1}{1 + 0.25}(0.5 \times 0 + 0.5 \times 4) = 1.6$$

Number of stocks at replicating portfolio can be found at $t = 0$ as

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{0 - 4}{8 - 2} = -\frac{2}{3}$$

, investor should short approximately 0.67 stocks.
If stock goes up at $t = 1$, number of stocks in replicating portdolio should be

$$\Delta_1 = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{0 - 2}{16 - 4} = -\frac{1}{6}$$

, investor should decrease amount of shorted stocks to approximately 0.17 stocks.

2 Problem II

a) To have this series of random variable a martingale, we must have

$$M_n = \mathbb{E}_n[M_{n+1}] \text{ for } n = 0, 1, ...$$

Let's prove this by induction. We have $\mathbb{E}_0[M_1] = 0.5\times1+0.5\times-1 = 0 = M_0$. Assume $M_n = \mathbb{E}_n[M_{n+1}]$. Then

$$M_{n+2} = X_{n+2} + \sum_{j=1}^{n+1} X_j = X_{n+2} + M_{n+1}$$

Taking expected value of both sides at $t = n+1$ and taking into account X_j's are independent, we have

$$\mathbb{E}_{n+1}[M_{n+2}] = \mathbb{E}_{n+1}[X_{n+2}] + \mathbb{E}_{n+1}[M_{n+1}]$$

At $t = n+1$, since there is no probability of change for M_{n+1}, $\mathbb{E}_{n+1}[M_{n+1}] = M_{n+1}$, and $\mathbb{E}_{n+1}[X_{n+2}] = 0.5 \times 1 + 0.5 \times -1 = 0$, we have

$$\mathbb{E}_{n+1}[M_{n+2}] = M_{n+1}$$

This concludes the proof.

b) We have $M_0 = 0$, then

$$S_0 = e^{\sigma M_0}(\frac{2}{e^{\sigma} + e^{-\sigma}})^0 = 1$$

Let's again prove this by induction. We have

$$\begin{aligned}
\mathbb{E}_0[S_1] &= 0.5 \times S_{1,(M_1=1)} + 0.5 \times S_{1,(M_1=-1)} \\
&= \frac{1}{2}(e^{\sigma}(\frac{2}{e^{\sigma} + e^{-\sigma}}) + e^{-\sigma}(\frac{2}{e^{\sigma} + e^{-\sigma}})) \\
&= \frac{1}{2}\frac{2(e^{\sigma} + e^{-\sigma})}{e^{\sigma} + e^{-\sigma}}) \\
&= 1
\end{aligned}$$

Assume $S_n = \mathbb{E}_n[S_{n+1}]$. Then ,as $M_{n+2} = M_{n+1} + X_{n+2}$,

$$\begin{aligned}
S_{n+2} &= e^{\sigma M_{n+2}}(\frac{2}{e^{\sigma} + e^{-\sigma}})^{n+2} \\
&= e^{\sigma M_{n+1}} e^{\sigma X_{n+2}}(\frac{2}{e^{\sigma} + e^{-\sigma}})^{n+2} \\
&= e^{\sigma M_{n+1}}(\frac{2}{e^{\sigma} + e^{-\sigma}})^{n+1} e^{\sigma X_{n+2}}(\frac{2}{e^{\sigma} + e^{-\sigma}}) \\
&= S_{n+1} e^{\sigma X_{n+2}}(\frac{2}{e^{\sigma} + e^{-\sigma}})
\end{aligned}$$

Taking expected value of both sides at $t = n+1$ and taking into account X_j's are independent we know S_n's are independent too. Since expected value of multiplication of independent variables is multiplication of expected values(for independent random variables X and Y, $\mathbb{E}(XY) = \mathbb{E}(X)\times\mathbb{E}(Y)$), we have

$$\begin{aligned}
\mathbb{E}_{n+1}[S_{n+2}] &= \mathbb{E}_{n+1}[S_{n+1}] \times \mathbb{E}_{n+1}(e^{\sigma X_{n+2}}(\frac{2}{e^{\sigma} + e^{-\sigma}})) \\
&= S_{n+1} \times \mathbb{E}_{n+1}(e^{\sigma X_{n+2}}(\frac{2}{e^{\sigma} + e^{-\sigma}})) \\
&= S_{n+1} \times \frac{1}{2}(e^{\sigma}(\frac{2}{e^{\sigma} + e^{-\sigma}}) + e^{-\sigma}(\frac{2}{e^{\sigma} + e^{-\sigma}})) \\
&= S_{n+1}
\end{aligned}$$

As in previous section, at $t = n+1$, since there is no probability of change for S_{n+1}, $\mathbb{E}_{n+1}[S_{n+1}] = S_{n+1}$. This concludes the proof.