A Formal Framework for Integer and Real Number Functions

Introduction

This document introduces a formal framework for representing arithmetic operations through a family of integer and real number functions. The objective is to describe addition, subtraction, multiplication, and division by means of systematically defined symbolic functions on positive and negative numbers. These definitions aim to mirror standard arithmetic while providing a functional, structured perspective for symbolic manipulation or theoretical modeling.

Definitions

Positive Integer Function (PIF)

Define the Positive Integer Function (PIF) as the identity on the set of positive integers:

$$\mathrm{PIF} : \mathbb{Z}^+ \to \mathbb{Z}^+, \quad \mathrm{PIF}(n) = n$$

for all $n \in \mathbb{Z}^+$.

Negative Integer Function (NIF)

Define the Negative Integer Function (NIF) as the identity on the set of negative integers:

$$\mathrm{NIF} : \mathbb{Z}^- \to \mathbb{Z}^-, \quad \mathrm{NIF}(n) = n$$

for all $n \in \mathbb{Z}^-$.

If a reference offset k is needed to enumerate negative numbers, one may define

$$\mathrm{NIF}_k(n) = -\big(k + (n - 1)\big)$$

for a chosen base k.

Positive Real Number Function (PRNF)

Define the Positive Real Number Function:

$$\mathrm{PRNF} : \mathbb{R}^+ \to \mathbb{R}^+, \quad \mathrm{PRNF}(x) = x$$

for all $x \in \mathbb{R}^+$.

Negative Real Number Function (NRNF)

Define the Negative Real Number Function:

$$\mathrm{NRNF} : \mathbb{R}^- \to \mathbb{R}^-, \quad \mathrm{NRNF}(x) = x$$

for all $x \in \mathbb{R}^-$.

Arithmetic Operations

Addition

For any $a, b \in \mathbb{Z}$:

$$+(a, b) = a + b$$

expressed functionally:

$$+(a, b) = \begin{cases} \mathrm{PIF}(a) + \mathrm{PIF}(b), & a, b > 0 \\ \mathrm{NIF}(a) + \mathrm{NIF}(b), & a, b < 0 \end{cases}$$

recovering standard addition.

Subtraction

For any $a, b \in \mathbb{Z}$:

$$-(a, b) = a - b$$

expressed as

$$-(a, b) = \begin{cases} \mathrm{PIF}(a) - \mathrm{PIF}(b), & a, b > 0 \\ \mathrm{NIF}(a) - \mathrm{NIF}(b), & a, b < 0 \end{cases}$$

recovering standard subtraction.

Multiplication

Define multiplication by repeated addition:

$$*(a, k) = \underbrace{a + a + \cdots + a}_{k \text{ times}} = a \times k$$

or using PIF:

$$*(a, k) = \mathrm{PIF}(a) \times \mathrm{PIF}(k)$$

if $a, k > 0$.

Division

Using the standard division algorithm:

$$a = b \times q + r$$

with

$$q = \left\lfloor \frac{a}{b} \right\rfloor, \quad r = a - b \times q, \quad 0 \leq r < b.$$

Define

$$\mathrm{DIV}(a, b) = (q, r)$$

with

$$q = \left\lfloor \frac{\mathrm{PIF}(a)}{\mathrm{PIF}(b)} \right\rfloor, \quad r = \mathrm{PIF}(a) - \mathrm{PIF}(b) \times q.$$

Worked-Out Examples

1. **Addition:** $3 + 4 = 7$, via PIF:

$$\mathrm{PIF}(3) + \mathrm{PIF}(4) = 7.$$

2. **Negative Addition:** $-2 + (-5) = -7$, via NIF:

$$\mathrm{NIF}(-2) + \mathrm{NIF}(-5) = -7.$$

3. **Subtraction:** $8 - 5 = 3$, via PIF:

$$\mathrm{PIF}(8) - \mathrm{PIF}(5) = 3.$$

4. **Multiplication:** $3 \times 4 = 12$, as repeated addition.

5. **Division:** $17 \div 5 = 3$ remainder 2:

$$\mathrm{DIV}(17, 5) = (3, 2).$$

Conclusion

This framework of PIF, NIF, PRNF, and NRNF serves to model standard arithmetic operations on integers and real numbers with a consistent symbolic functional style. The consistency checks confirm these functions align with standard definitions. Such a structured perspective may be valuable for symbolic mathematics, theorem proving, or educational systems, where explicitly tracking function application is desirable.