DIFFERENTIAL KINEMATICS

- relationship between joint velocities and end-effector velocities

Geometric Jacobian

Analytical Jacobian

Kinematic singularities

Kinematic redundancy

Inverse differential kinematics

Inverse kinematics algorithms

STATICS

- relationship between end-effector forces and joint torques

GEOMETRIC JACOBIAN

$$\boldsymbol{T}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}(\boldsymbol{q}) & \boldsymbol{p}(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

- Goal

$$\dot{\boldsymbol{p}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$
$$\boldsymbol{\omega} = \boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\boldsymbol{v} = \begin{bmatrix} \dot{\boldsymbol{p}} \\ \boldsymbol{\omega} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Derivative of a rotation matrix

$$\boldsymbol{R}(t)\boldsymbol{R}^T(t) = \boldsymbol{I}$$

$$\dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) + \boldsymbol{R}(t)\dot{\boldsymbol{R}}^T(t) = \boldsymbol{O}$$

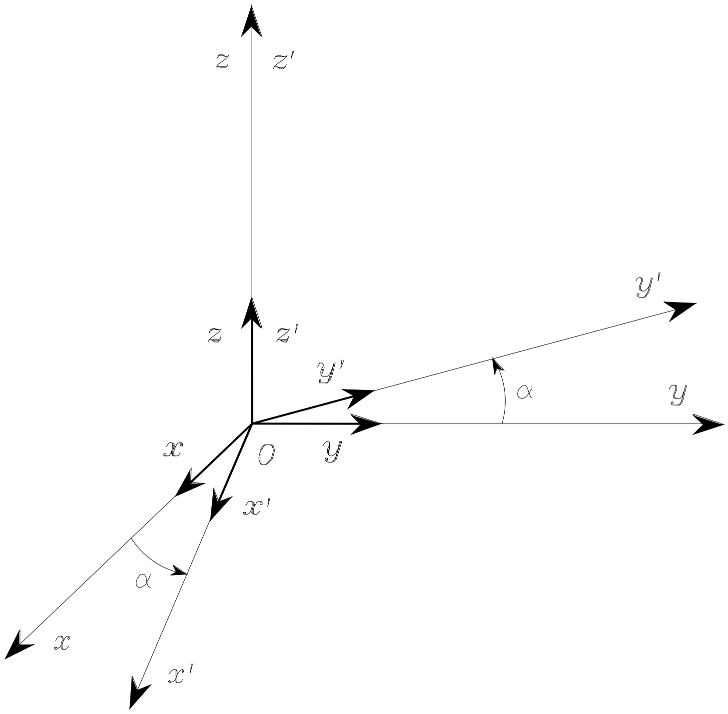
- Given $\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t)$

$$\boldsymbol{S}(t) + \boldsymbol{S}^T(t) = \boldsymbol{O}$$

$$\dot{\boldsymbol{R}}(t) = \boldsymbol{S}(\boldsymbol{\omega}(t))\boldsymbol{R}(t)$$

$$\boldsymbol{S} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

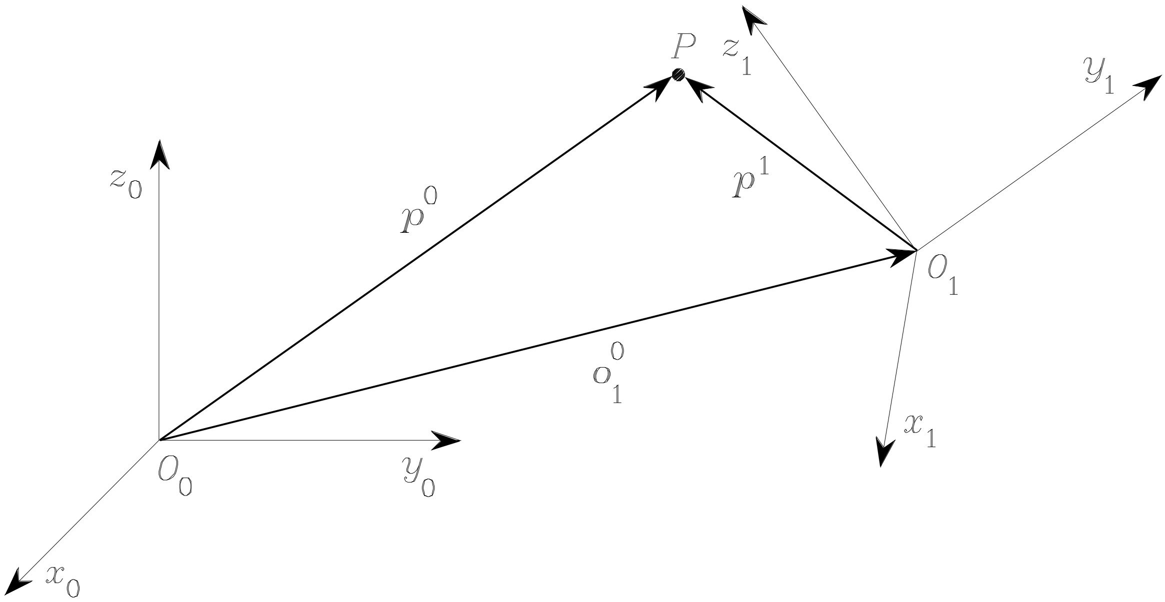
- Example



$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{S}(t) = \begin{bmatrix} -\dot{\alpha}\sin\alpha & -\dot{\alpha}\cos\alpha & 0 \\ \dot{\alpha}\cos\alpha & -\dot{\alpha}\sin\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\dot{\alpha} & 0 \\ \dot{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \boldsymbol{S}(\boldsymbol{\omega}(t))$$

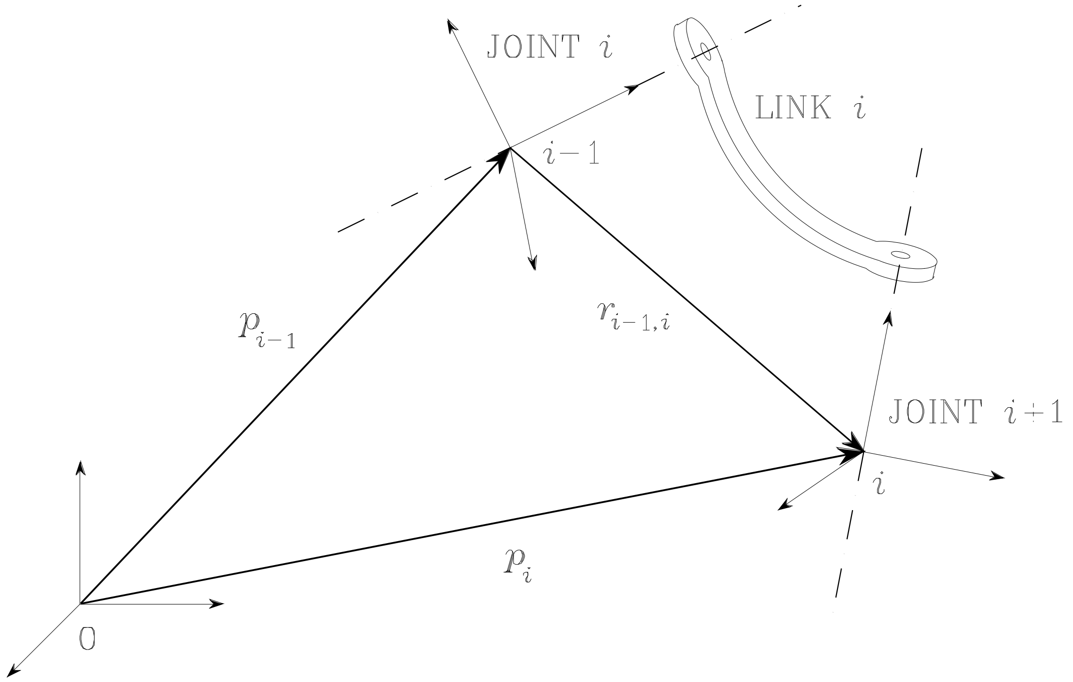


$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1$$

$$\begin{aligned}
\dot{\boldsymbol{p}}^0 &= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \dot{\boldsymbol{R}}_1^0 \boldsymbol{p}^1 \\
&= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{S}(\boldsymbol{\omega}_1^0) \boldsymbol{R}_1^0 \boldsymbol{p}^1 \\
&= \dot{\boldsymbol{o}}_1^0 + \boldsymbol{R}_1^0 \dot{\boldsymbol{p}}^1 + \boldsymbol{\omega}_1^0 \times \boldsymbol{r}_1^0
\end{aligned}$$

Link velocity

- Linear velocity



$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1}$$

$$\begin{aligned}\dot{\boldsymbol{p}}_i &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{R}_{i-1}\dot{\boldsymbol{r}}_{i-1,i}^{i-1} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{R}_{i-1}\boldsymbol{r}_{i-1,i}^{i-1} \\ &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i}\end{aligned}$$

- Angular velocity

$$\boldsymbol{R}_i = \boldsymbol{R}_{i-1}\boldsymbol{R}_i^{i-1}$$

$$\boldsymbol{S}(\boldsymbol{\omega}_i)\boldsymbol{R}_i = \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{R}_{i-1}\boldsymbol{S}(\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i^{i-1}$$

$$= \boldsymbol{S}(\boldsymbol{\omega}_{i-1})\boldsymbol{R}_i + \boldsymbol{S}(\boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1})\boldsymbol{R}_i$$

$$\begin{aligned}\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} + \boldsymbol{R}_{i-1}\boldsymbol{\omega}_{i-1,i}^{i-1}\\ &= \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i}\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} + \boldsymbol{\omega}_{i-1,i} \\
\dot{\boldsymbol{p}}_i &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{v}_{i-1,i} + \boldsymbol{\omega}_{i-1} \times \boldsymbol{r}_{i-1,i}
\end{aligned}
$$

- Prismatic joint

$$
\begin{aligned}
\boldsymbol{\omega}_{i-1,i} &= \mathbf{0} \\
\boldsymbol{v}_{i-1,i} &= \dot{d}_i \boldsymbol{z}_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} \\
\dot{\boldsymbol{p}}_i &= \dot{\boldsymbol{p}}_{i-1} + \dot{d}_i \boldsymbol{z}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}
\end{aligned}
$$

- Revolute joint

$$
\begin{aligned}
\boldsymbol{\omega}_{i-1,i} &= \dot{\vartheta}_i \boldsymbol{z}_{i-1} \\
\boldsymbol{v}_{i-1,i} &= \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,i}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\omega}_i &= \boldsymbol{\omega}_{i-1} + \dot{\vartheta}_i \boldsymbol{z}_{i-1} \\
\dot{\boldsymbol{p}}_i &= \dot{\boldsymbol{p}}_{i-1} + \boldsymbol{\omega}_i \times \boldsymbol{r}_{i-1,i}
\end{aligned}
$$

Jacobian computation

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{\jmath}_{P1} & & \boldsymbol{\jmath}_{Pn} \\ & \dots & \\ \boldsymbol{\jmath}_{O1} & & \boldsymbol{\jmath}_{On} \end{bmatrix}$$

- Angular velocity
  - ⋆ Joint i *prismatic*

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \mathbf{0} \qquad \Longrightarrow \qquad \boldsymbol{\jmath}_{Oi} = \mathbf{0}$$

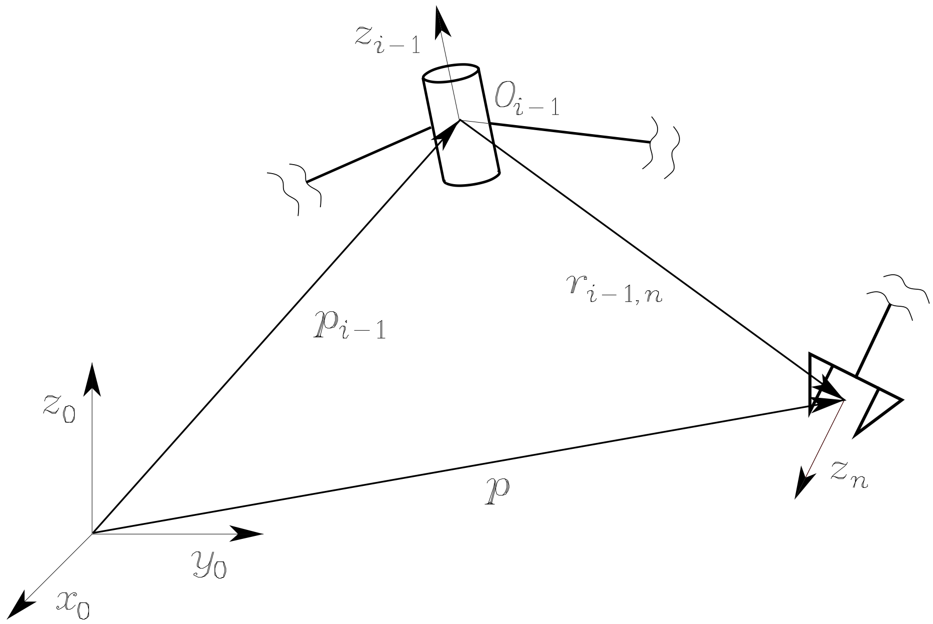
  - ⋆ Joint i *revolute*

$$\dot{q}_i \boldsymbol{\jmath}_{Oi} = \dot{\vartheta}_i \boldsymbol{z}_{i-1} \qquad \Longrightarrow \qquad \boldsymbol{\jmath}_{Oi} = \boldsymbol{z}_{i-1}$$

- Linear velocity
  - ⋆ Joint i *prismatic*

$$\dot{q}_i \boldsymbol{\jmath}_{Pi} = \dot{d}_i \boldsymbol{z}_{i-1} \qquad \Longrightarrow \qquad \boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1}$$

  - ⋆ Joint i *revolute*



$$\begin{aligned}\dot{q}_i \boldsymbol{\jmath}_{Pi} &= \boldsymbol{\omega}_{i-1,i} \times \boldsymbol{r}_{i-1,n} \\ &= \dot{\vartheta}_i \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1})\end{aligned}$$

$$\Downarrow$$

$$\boldsymbol{\jmath}_{Pi} = \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1})$$

- Column of geometric Jacobian

$$\begin{bmatrix} \boldsymbol{j}_{Pi} \\ \boldsymbol{j}_{Oi} \end{bmatrix} = \begin{cases} \begin{bmatrix} \boldsymbol{z}_{i-1} \\ \boldsymbol{0} \end{bmatrix} & \textit{prismatic} \text{ joint} \\ \begin{bmatrix} \boldsymbol{z}_{i-1} \times (\boldsymbol{p} - \boldsymbol{p}_{i-1}) \\ \boldsymbol{z}_{i-1} \end{bmatrix} & \textit{revolute} \text{ joint} \end{cases}$$

⋆ $\boldsymbol{z}_{i-1} = \boldsymbol{R}_1^0(q_1) \ldots \boldsymbol{R}_{i-1}^{i-2}(q_{i-1}) \boldsymbol{z}_0$

⋆ $\tilde{\boldsymbol{p}} = \boldsymbol{A}_1^0(q_1) \ldots \boldsymbol{A}_n^{n-1}(q_n) \tilde{\boldsymbol{p}}_0$

⋆ $\tilde{\boldsymbol{p}}_{i-1} = \boldsymbol{A}_1^0(q_1) \ldots \boldsymbol{A}_{i-1}^{i-2}(q_{i-1}) \tilde{\boldsymbol{p}}_0$

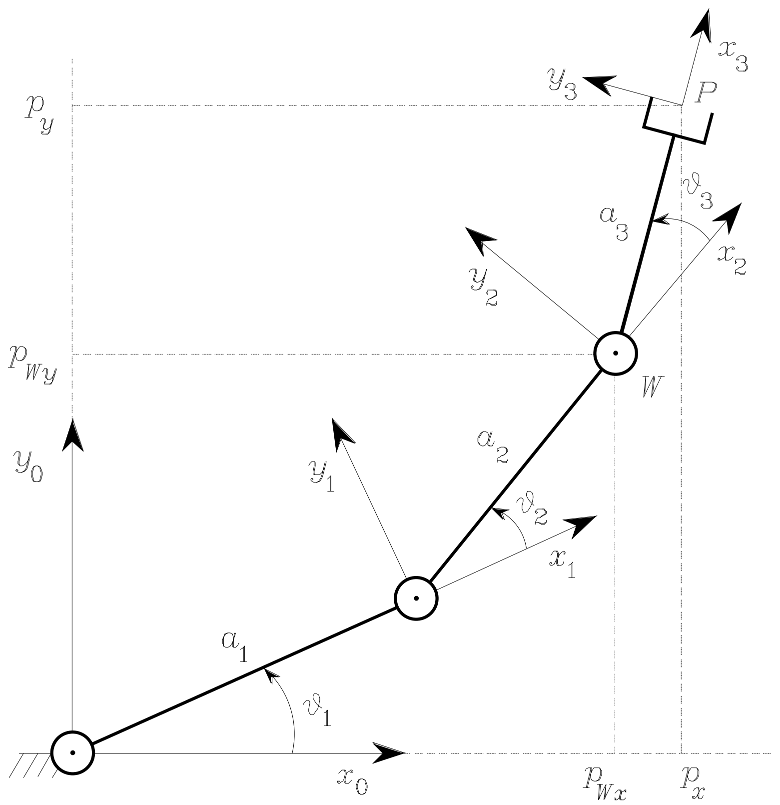
- Representation in different frame

$$\begin{bmatrix} \dot{\boldsymbol{p}}^t \\ \boldsymbol{\omega}^t \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}^t & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^t \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}} \\ \boldsymbol{\omega} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{R}^t & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^t \end{bmatrix} \boldsymbol{J}\dot{\boldsymbol{q}}$$

$$\boldsymbol{J}^t = \begin{bmatrix} \boldsymbol{R}^t & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{R}^t \end{bmatrix} \boldsymbol{J}$$

Three–link planar aram



$$\boldsymbol{J}(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p} - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

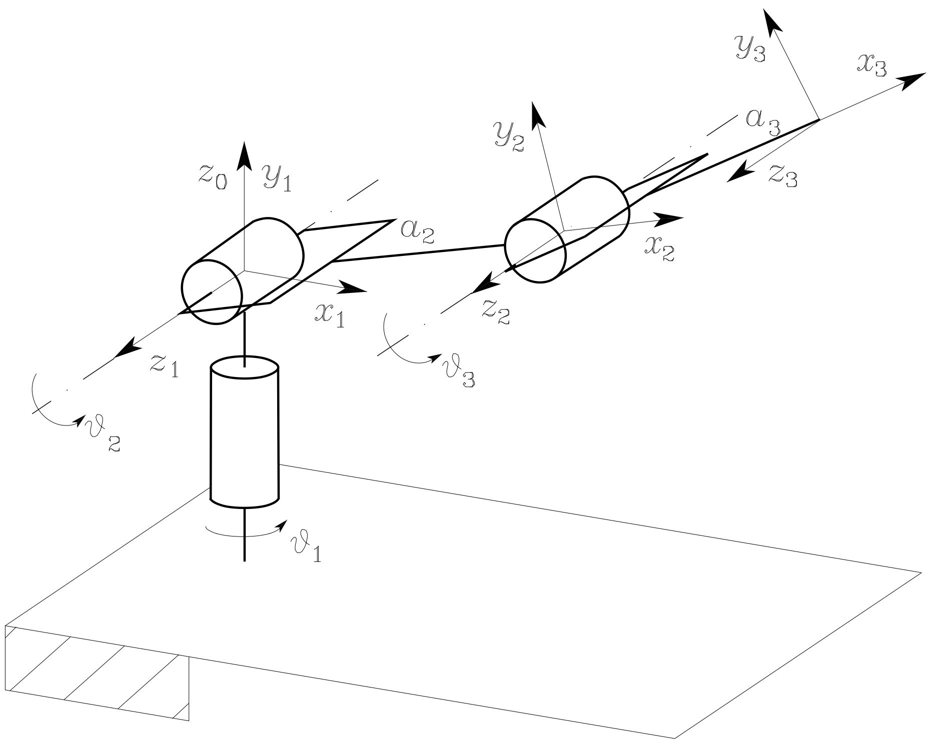
$$\boldsymbol{p} = \begin{bmatrix} a_1 c_1 + a_2 c_{12} + a_3 c_{123} \\ a_1 s_1 + a_2 s_{12} + a_3 s_{123} \\ 0 \end{bmatrix}$$

$$\boldsymbol{z}_0 = \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boldsymbol{J}_P = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} - a_3 s_{123} & -a_2 s_{12} - a_3 s_{123} & -a_3 s_{123} \\ a_1 c_1 + a_2 c_{12} + a_3 c_{123} & a_2 c_{12} + a_3 c_{123} & a_3 c_{123} \end{bmatrix}$$

Anthropomorphic arm



$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 \times (\boldsymbol{p} - \boldsymbol{p}_2) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{p}_2 = \begin{bmatrix} a_2 c_1 c_2 \\ a_2 s_1 c_2 \\ a_2 s_2 \end{bmatrix}$$

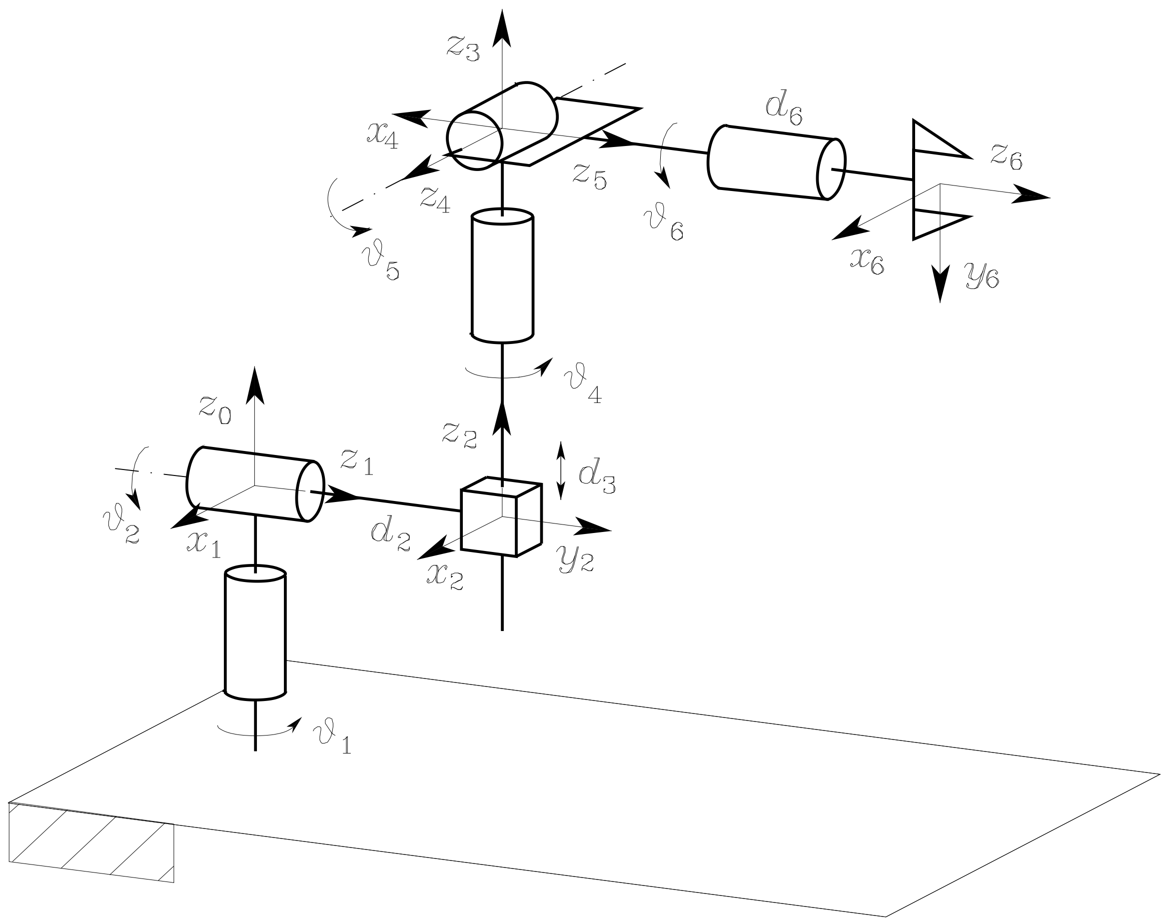
$$\boldsymbol{p} = \begin{bmatrix} c_1(a_2 c_2 + a_3 c_{23}) \\ s_1(a_2 c_2 + a_3 c_{23}) \\ a_2 s_2 + a_3 s_{23} \end{bmatrix}$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{z}_1 = \boldsymbol{z}_2 = \begin{bmatrix} s_1 \\ -c_1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{J} = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \\ 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{J}_P = \begin{bmatrix} -s_1(a_2c_2 + a_3c_{23}) & -c_1(a_2s_2 + a_3s_{23}) & -a_3c_1s_{23} \\ c_1(a_2c_2 + a_3c_{23}) & -s_1(a_2s_2 + a_3s_{23}) & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix}$$

Stanford manipulator



$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{z}_0 \times (\boldsymbol{p} - \boldsymbol{p}_0) & \boldsymbol{z}_1 \times (\boldsymbol{p} - \boldsymbol{p}_1) & \boldsymbol{z}_2 & \boldsymbol{z}_3 \times (\boldsymbol{p} - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p} - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p} - \boldsymbol{p}_5) \\ \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{0} & \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$$

$$\boldsymbol{p}_0 = \boldsymbol{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \boldsymbol{p}_3 = \boldsymbol{p}_4 = \boldsymbol{p}_5 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6 (c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 + c_1 d_2 + d_6 (c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6 (c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \boldsymbol{z}_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad \boldsymbol{z}_2 = \boldsymbol{z}_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$\boldsymbol{z}_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \quad \boldsymbol{z}_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

KINEMATIC SINGULARITIES

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

- if $\boldsymbol{J}$ is rank-deficient $\quad\Longrightarrow\quad$ *kinematic singularities*
  - **(a)** reduced mobility
  - **(b)** infinite solutions to inverse kinematics problem
  - **(c)** large joint velocities (in the neighbourhood of singularity)

- Classification
  - ⋆ *Boundary* singularities
  - ⋆ *Internal* singularities

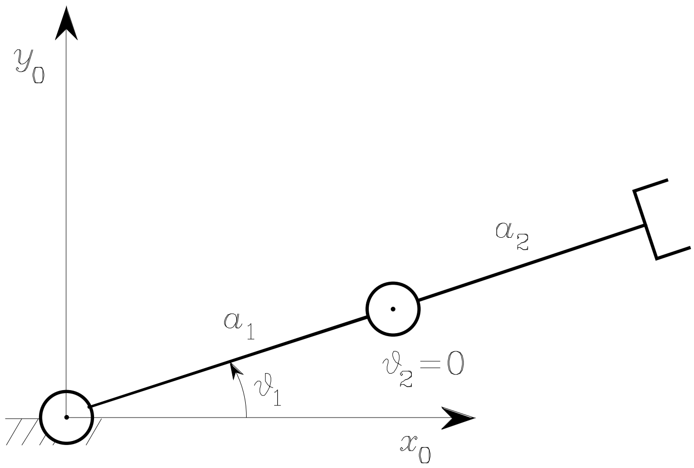
- Two–link planar arm

$$\boldsymbol{J} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\det(\boldsymbol{J}) = a_1 a_2 s_2$$

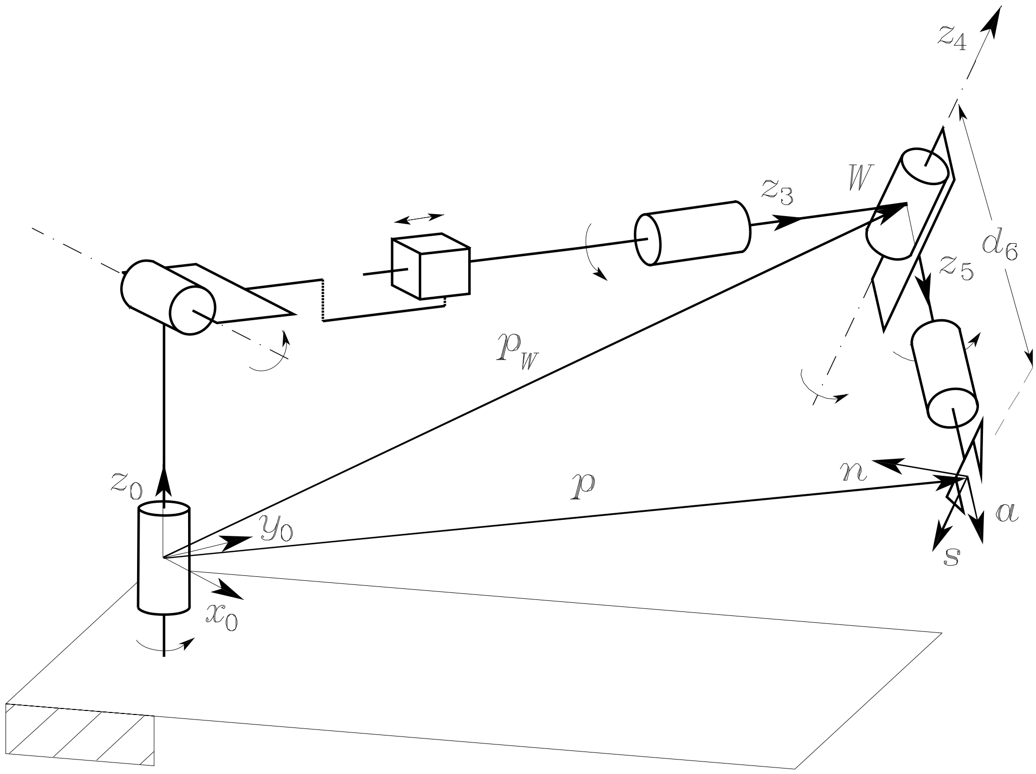
$$\Downarrow$$

$$\vartheta_2 = 0 \qquad \vartheta_2 = \pi$$



⋆ $[\,-(a_1+a_2)s_1 \quad (a_1+a_2)c_1\,]^T$ parallel to $[\,-a_2 s_1 \quad a_2 c_1\,]^T$ (components of end-effector velocity non independent)

Singularity decoupling



- computation of *arm singularities*
- computation of *wrist singularities*

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} \end{bmatrix}$$

$$\boldsymbol{J}_{12} = \begin{bmatrix} \boldsymbol{z}_3 \times (\boldsymbol{p} - \boldsymbol{p}_3) & \boldsymbol{z}_4 \times (\boldsymbol{p} - \boldsymbol{p}_4) & \boldsymbol{z}_5 \times (\boldsymbol{p} - \boldsymbol{p}_5) \end{bmatrix}$$

$$\boldsymbol{J}_{22} = \begin{bmatrix} \boldsymbol{z}_3 & \boldsymbol{z}_4 & \boldsymbol{z}_5 \end{bmatrix}$$

- $\boldsymbol{p} = \boldsymbol{p}_W \quad \Longrightarrow \quad \boldsymbol{p}_W - \boldsymbol{p}_i$ parallel to $\boldsymbol{z}_i$, $i = 3, 4, 5$

$$\boldsymbol{J}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

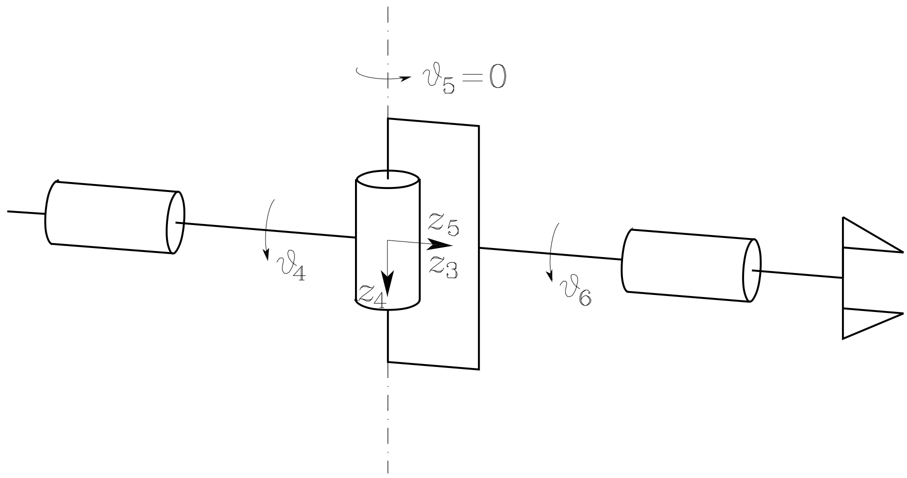
$$\det(\boldsymbol{J}) = \det(\boldsymbol{J}_{11})\det(\boldsymbol{J}_{22})$$

$$\det(\boldsymbol{J}_{11}) = 0 \qquad \det(\boldsymbol{J}_{22}) = 0$$

Wrist singularities

- $\boldsymbol{z}_3$ parallel to $\boldsymbol{z}_5$

$$\vartheta_5 = 0 \qquad \vartheta_5 = \pi$$



⋆ rotations of equal magnitude about opposite directions on ϑ_4 and ϑ_6 do not produce any rotation at the end-effector

Arm singularities

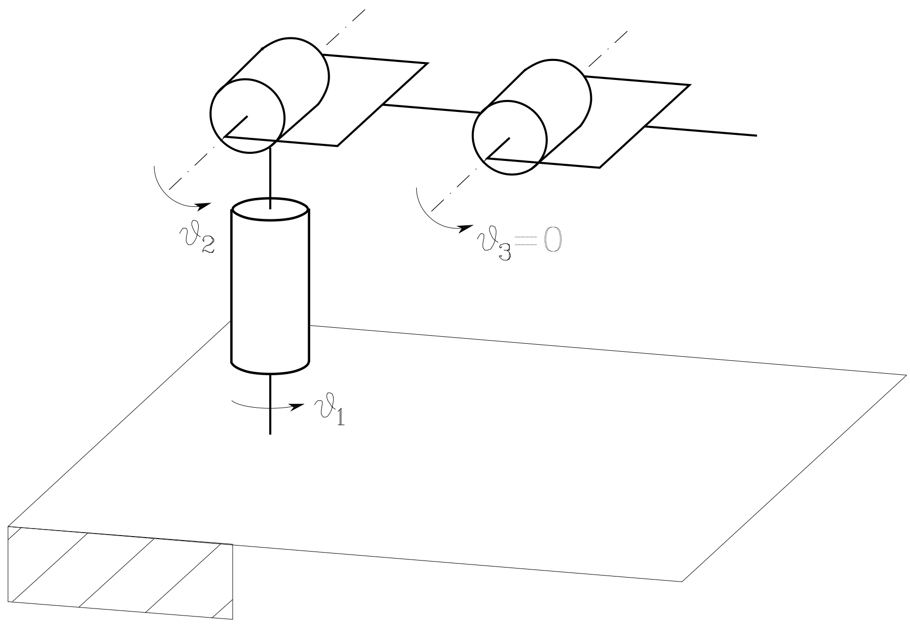
- Anthropomorphic arm

$$\det(\boldsymbol{J}_P) = -a_2 a_3 s_3 (a_2 c_2 + a_3 c_{23})$$

$$s_3 = 0 \qquad\qquad a_2 c_2 + a_3 c_{23} = 0$$

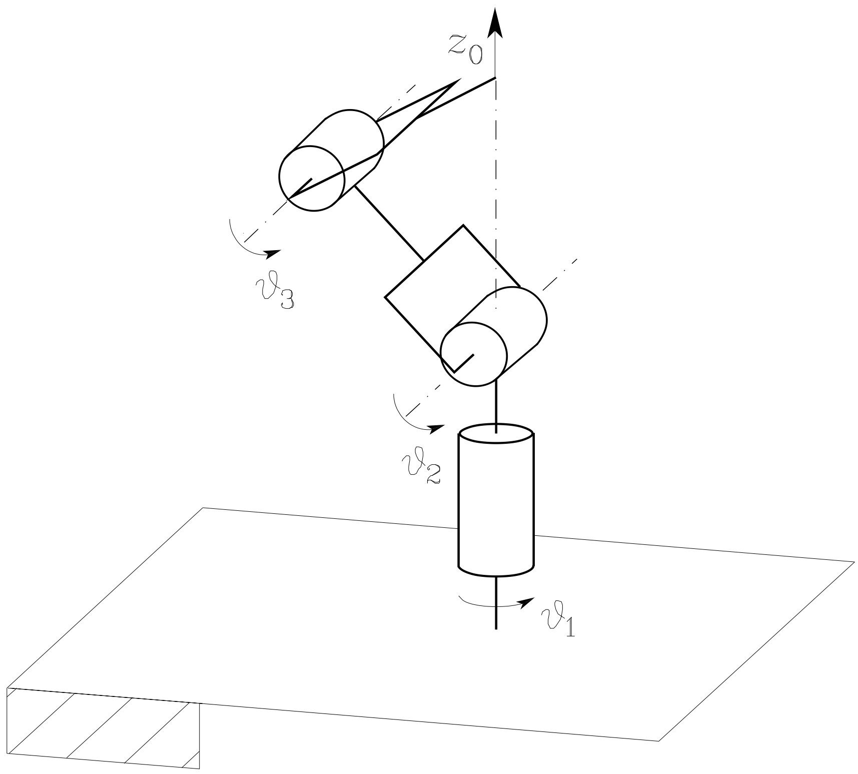
  ⋆ *Elbow* singularity

$$\vartheta_3 = 0 \qquad\qquad \vartheta_3 = \pi$$



⋆ *Shoulder* singularity

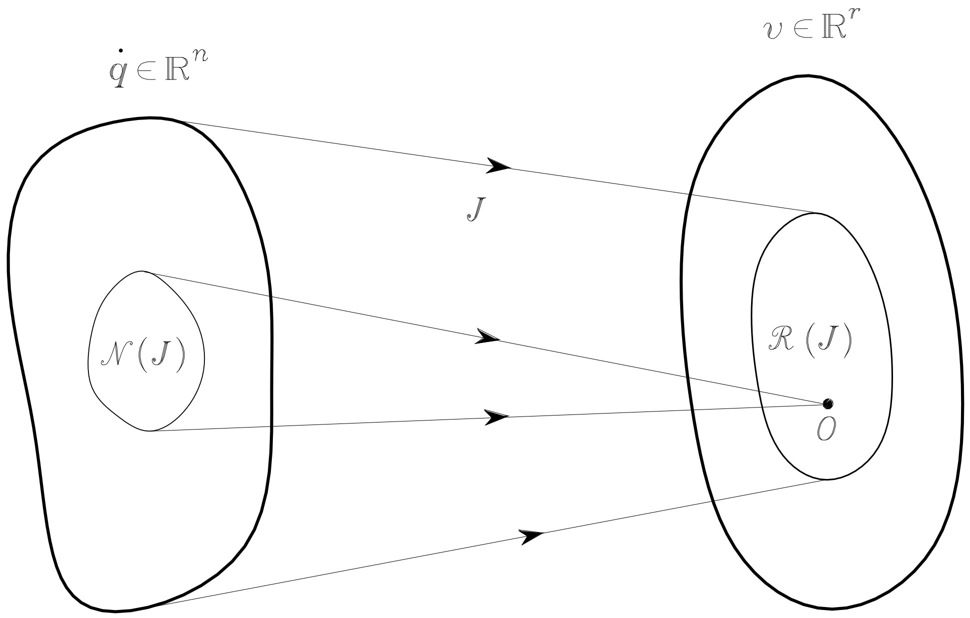
$$p_x = p_y = 0$$



ANALYSIS OF REDUNDANCY

- Differential kinematics

$$\boldsymbol{v} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$



⋆ if $\varrho(\boldsymbol{J}) = r$

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) = r \qquad \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n - r$$

⋆ in general

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) + \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n$$

- If $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_a$$

  where

$$\mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{J})$$

  ⋆ check:

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}^* + \boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_a = \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}$$

- $\dot{\boldsymbol{q}}_a$ generates *internal motions* of the structure

INVERSE DIFFERENTIAL KINEMATICS

- Nonlinear kinematics equation
- Differential kinematics equation linear in the velocities
- Given $\boldsymbol{v}(t)$ + initial conditions $\Longrightarrow$ $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$
  - ⋆ if $n = r$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}$$

$$\boldsymbol{q}(t) = \int_0^t \dot{\boldsymbol{q}}(\varsigma)d\varsigma + \boldsymbol{q}(0)$$

  - ⋆ (Euler) numerical integration

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \dot{\boldsymbol{q}}(t_k)\Delta t$$

Redundant manipulators

- For a given configuration $\boldsymbol{q}$, find the solutions $\dot{\boldsymbol{q}}$ that satisfy

$$\boldsymbol{v} = \boldsymbol{J}\dot{\boldsymbol{q}}$$

and minimize

$$g(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W}\dot{\boldsymbol{q}}$$

⋆ *method of Lagrange multipliers*

$$g(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W}\dot{\boldsymbol{q}} + \boldsymbol{\lambda}^T(\boldsymbol{v} - \boldsymbol{J}\dot{\boldsymbol{q}})$$

$$\left(\frac{\partial g}{\partial \dot{\boldsymbol{q}}}\right)^T = \boldsymbol{0} \qquad \left(\frac{\partial g}{\partial \boldsymbol{\lambda}}\right)^T = \boldsymbol{0}$$

⋆ optimal solution

$$\dot{\boldsymbol{q}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{v}$$

⋆ if $\boldsymbol{W} = \boldsymbol{I}$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v}$$

where

$$\boldsymbol{J}^{\dagger} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}$$

is the *right pseudo-inverse* of $\boldsymbol{J}$

- Use of redundancy

$$g'(\dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{\boldsymbol{q}}^T - \dot{\boldsymbol{q}}_a^T)(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_a)$$

  ⋆ like above . . .

$$g'(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}(\dot{\boldsymbol{q}}^T - \dot{\boldsymbol{q}}_a^T)(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_a) + \boldsymbol{\lambda}^T(\boldsymbol{v} - \boldsymbol{J}\dot{\boldsymbol{q}})$$

  ⋆ optimal solution

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v} + (\boldsymbol{I} - \boldsymbol{J}^{\dagger}\boldsymbol{J})\dot{\boldsymbol{q}}_a$$

- Characterization of internal motions

$$\dot{\boldsymbol{q}}_a = k_a \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T$$

⋆ *manipulability measure*

$$w(\boldsymbol{q}) = \sqrt{\det\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)}$$

⋆ *distance from mechanical joint limits*

$$w(\boldsymbol{q}) = -\frac{1}{2n} \sum_{i=1}^{n} \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right)^2$$

⋆ *distance from an obstacle*

$$w(\boldsymbol{q}) = \min_{\boldsymbol{p},\boldsymbol{o}} \|\boldsymbol{p}(\boldsymbol{q}) - \boldsymbol{o}\|$$

Kinematic singularities

- The previous solutions hold only when $\boldsymbol{J}$ is full-rank
- If $\boldsymbol{J}$ is *not* full-rank (singularity)
  - ⋆ if $\boldsymbol{v} \in \mathcal{R}(\boldsymbol{J}) \quad\Longrightarrow\quad$ solution $\dot{\boldsymbol{q}}$ extracting all linearly independent equations ("physically" executable path)
  - ⋆ if $\boldsymbol{v} \notin \mathcal{R}(\boldsymbol{J}) \quad\Longrightarrow\quad$ the system of equations has no solution (path cannot be executed)
- Inversion in the neighbourhood of singularities
  - ⋆ $\det(\boldsymbol{J})$ small $\quad\Longrightarrow\quad$ $\dot{\boldsymbol{q}}$ large
  - ⋆ *damped least-squares inverse*

$$\boldsymbol{J}^{\star} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + k^2\boldsymbol{I})^{-1}$$

where $\dot{\boldsymbol{q}}$ minimizes

$$g''(\dot{\boldsymbol{q}}) = \|\boldsymbol{v} - \boldsymbol{J}\dot{\boldsymbol{q}}\|^2 + k^2\|\dot{\boldsymbol{q}}\|^2$$

ANALYTICAL JACOBIAN

$$\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{q})$$
$$\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{q})$$

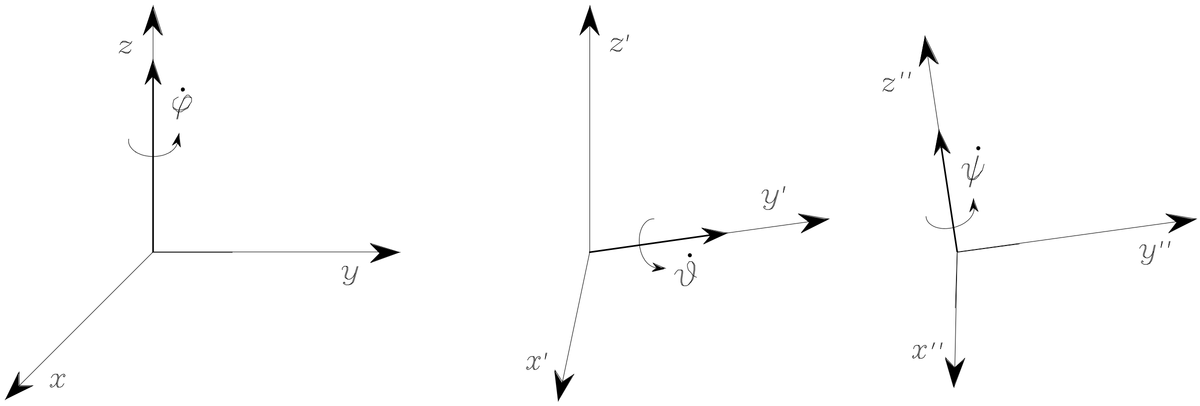
$$\dot{\boldsymbol{p}} = \frac{\partial \boldsymbol{p}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\dot{\boldsymbol{\phi}} = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_\phi(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$$\begin{aligned}\dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\boldsymbol{p}} \\ \dot{\boldsymbol{\phi}}\end{bmatrix} &= \begin{bmatrix}\boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q})\end{bmatrix}\dot{\boldsymbol{q}} \\ &= \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned}$$

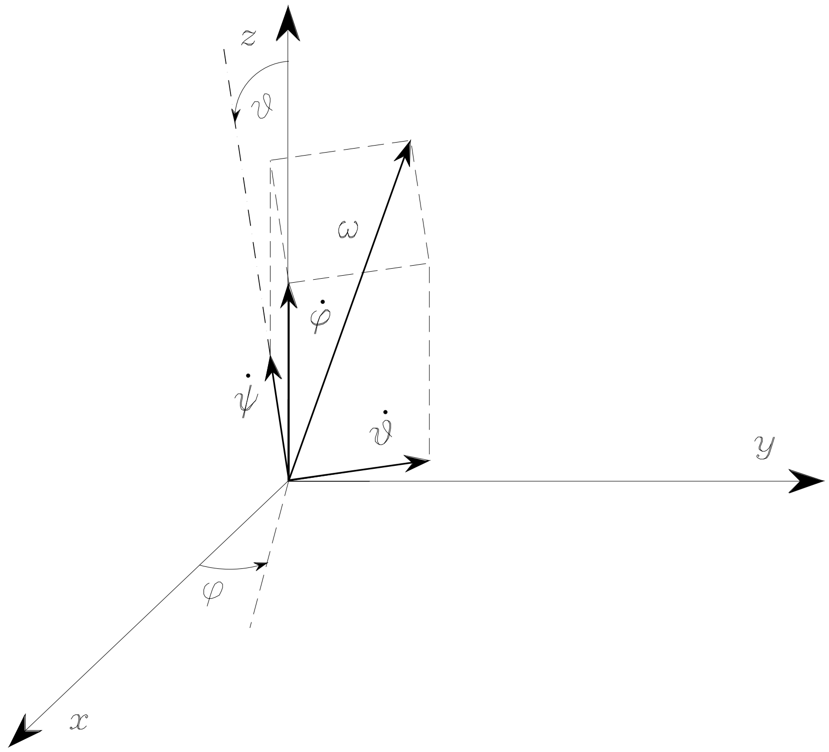
$$\boldsymbol{J}_A(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}}$$

- Angular velocity of Euler angles ZYZ about axes of reference frame



  - ⋆ as a result of $\dot{\varphi}$: $[\,\omega_x \quad \omega_y \quad \omega_z\,]^T = \dot{\varphi}\,[\,0 \quad 0 \quad 1\,]^T$
  - ⋆ as a result of $\dot{\vartheta}$: $[\,\omega_x \quad \omega_y \quad \omega_z\,]^T = \dot{\vartheta}\,[\,-s_\varphi \quad c_\varphi \quad 0\,]^T$
  - ⋆ as a result of $\dot{\psi}$: $[\,\omega_x \quad \omega_y \quad \omega_z\,]^T = \dot{\psi}\,[\,c_\varphi s_\vartheta \quad s_\varphi s_\vartheta \quad c_\vartheta\,]^T$

- Composition of elementary angular velocities

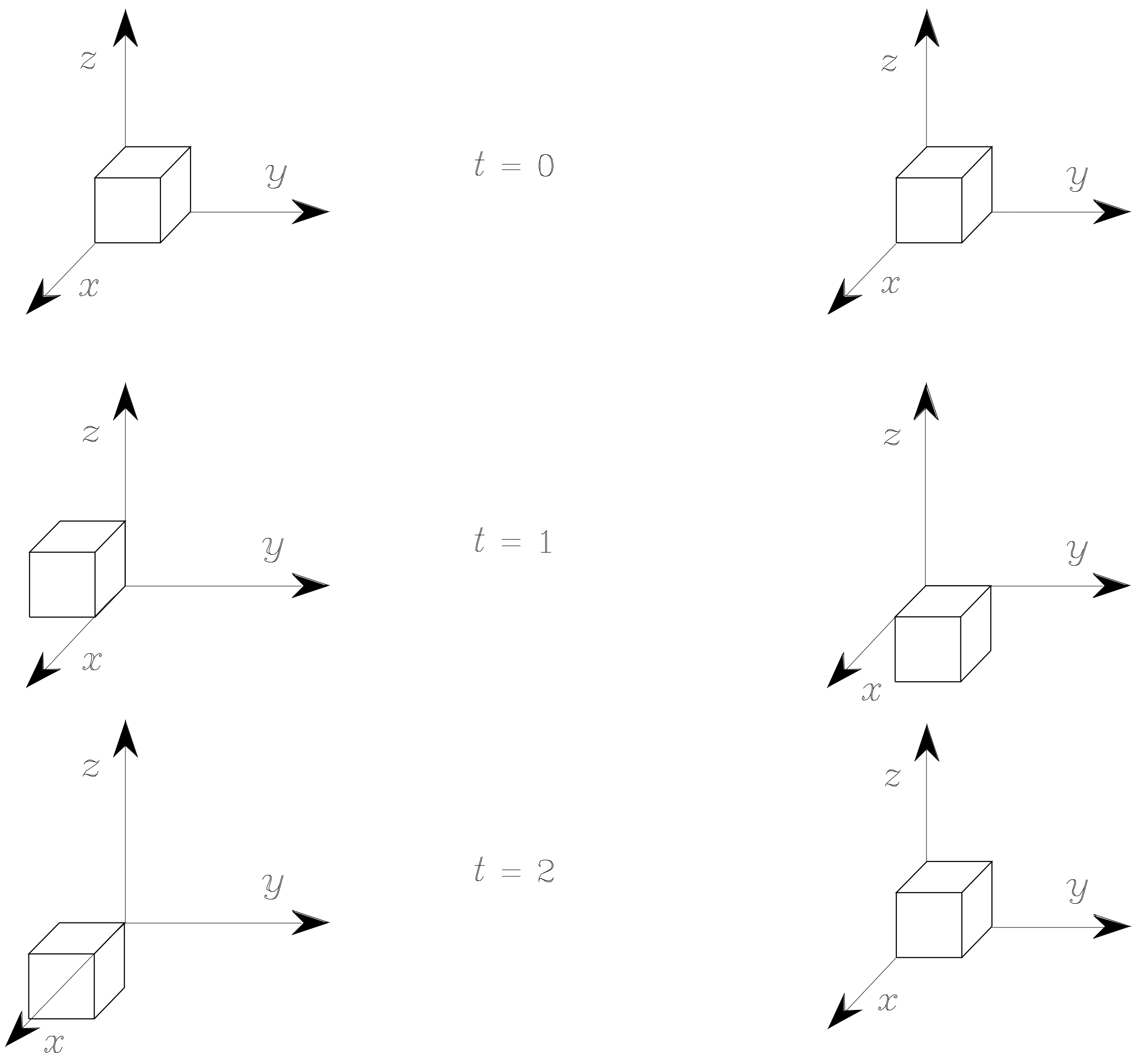


$$\boldsymbol{\omega} = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix} \dot{\boldsymbol{\phi}} = \boldsymbol{T}(\boldsymbol{\phi})\dot{\boldsymbol{\phi}}$$

- Physical meaning of $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = [\,\pi/2 \quad 0 \quad 0\,]^T \quad 0 \le t \le 1$$
$$\boldsymbol{\omega} = [\,0 \quad \pi/2 \quad 0\,]^T \quad 1 < t \le 2$$

$$\boldsymbol{\omega} = [\,0 \quad \pi/2 \quad 0\,]^T \quad 0 \le t \le 1$$
$$\boldsymbol{\omega} = [\,\pi/2 \quad 0 \quad 0\,]^T \quad 1 < t \le 2$$



$$\int_0^2 \boldsymbol{\omega} dt = [\,\pi/2 \quad \pi/2 \quad 0\,]^T$$

Relationship between analytical Jacobian and geometric Jacobian

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}(\boldsymbol{\phi}) \end{bmatrix} \dot{\boldsymbol{x}} = \boldsymbol{T}_A(\boldsymbol{\phi})\dot{\boldsymbol{x}}$$

$$\boldsymbol{J} = \boldsymbol{T}_A(\boldsymbol{\phi})\boldsymbol{J}_A$$

- Geometric Jacobian
  - ⋆ quantities of clear physical meaning
- Analytical Jacobian
  - ⋆ differential quantities in the operational space

INVERSE KINEMATICS ALGORITHMS

- Kinematic inversion

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \boldsymbol{J}^{-1}(\boldsymbol{q}(t_k))\boldsymbol{v}(t_k)\Delta t$$

  ⋆ *drift* of solution

- Closed-Loop Inverse Kinematics (CLIK) algorithm

  ⋆ *operational space error*

$$\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}$$

$$\begin{aligned}\dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}}\\ &= \dot{\boldsymbol{x}}_d - \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned}$$

  ⋆ find $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{e})$: $\quad \boldsymbol{e} \rightarrow \boldsymbol{0}$

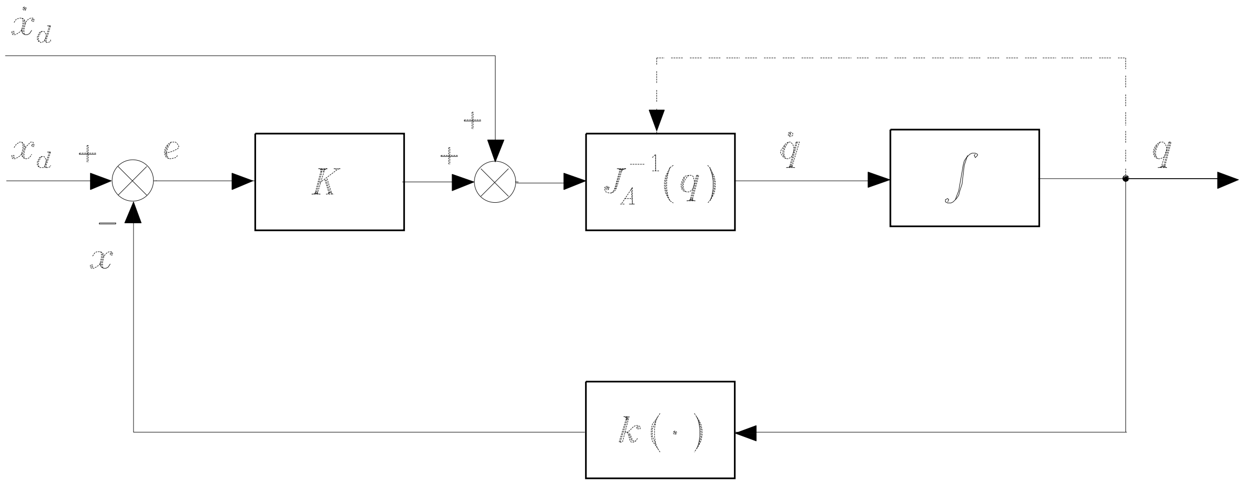
Jacobian (pseudo-)inverse

- Linearization of error dynamics

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q})(\dot{\boldsymbol{x}}_d + \boldsymbol{K}\boldsymbol{e})$$

$$\Downarrow$$

$$\dot{\boldsymbol{e}} + \boldsymbol{K}\boldsymbol{e} = \boldsymbol{0}$$



⋆ For a *redundant manipulator*

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{\dagger}(\dot{\boldsymbol{x}}_d + \boldsymbol{K}\boldsymbol{e}) + (\boldsymbol{I} - \boldsymbol{J}_A^{\dagger}\boldsymbol{J}_A)\dot{\boldsymbol{q}}_a$$

Jacobian transpose

- $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}(\boldsymbol{e})$ without linearizing error dynamics
- Lyapunov method

$$V(\boldsymbol{e}) = \frac{1}{2}\boldsymbol{e}^T\boldsymbol{K}\boldsymbol{e}$$

where

$$V(\boldsymbol{e}) > 0 \quad \forall \boldsymbol{e} \neq \boldsymbol{0} \qquad\qquad V(\boldsymbol{0}) = 0$$

$$\begin{aligned}\dot{V}(\boldsymbol{e}) &= \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}} \\ &= \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned}$$

⋆ the choice

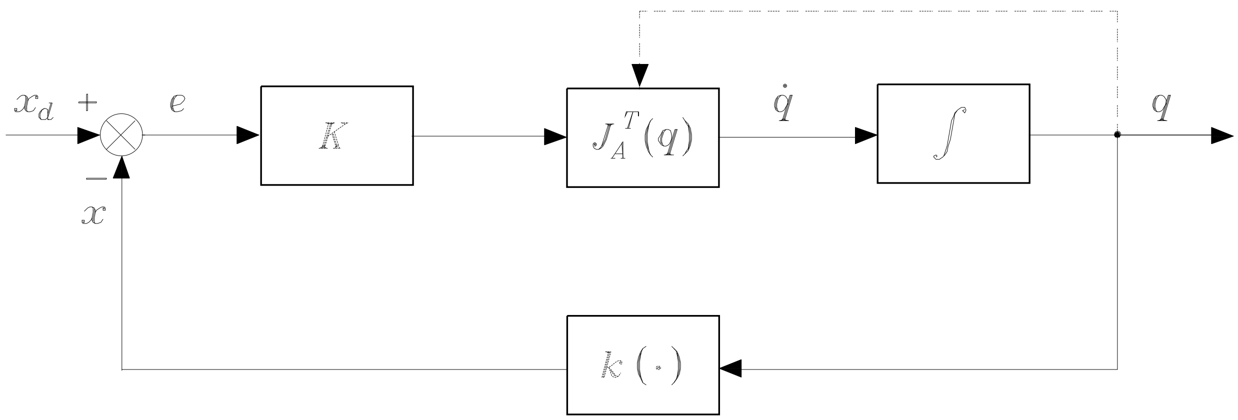
$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}$$

leads to

$$\dot{V}(\boldsymbol{e}) = \boldsymbol{e}^T\boldsymbol{K}\dot{\boldsymbol{x}}_d - \boldsymbol{e}^T\boldsymbol{K}\boldsymbol{J}_A(\boldsymbol{q})\boldsymbol{J}_A^T(\boldsymbol{q})\boldsymbol{K}\boldsymbol{e}$$

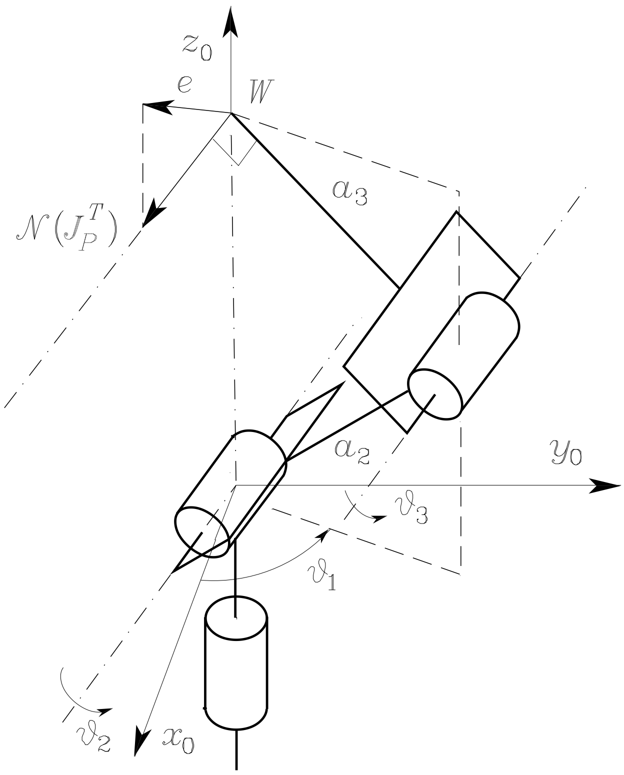
⋆ if $\dot{\boldsymbol{x}}_d = \boldsymbol{0} \quad \Longrightarrow \quad \dot{V} < 0$ with $V > 0$ (*asymptotic stability*)

⋆ if $\mathcal{N}(\boldsymbol{J}_A^T) \neq \emptyset \quad \Longrightarrow \quad \dot{V} = 0$ if $\boldsymbol{K}\boldsymbol{e} \in \mathcal{N}(\boldsymbol{J}_A^T)$
$\dot{\boldsymbol{q}} = \boldsymbol{0}$ with $\boldsymbol{e} \neq \boldsymbol{0}$ (stuck?)



- If $\dot{\boldsymbol{x}}_d \neq \boldsymbol{0}$
  - ⋆ $\boldsymbol{e}(t)$ bounded (worth increasing norm of $\boldsymbol{K}$)
  - ⋆ $\boldsymbol{e}(\infty) \rightarrow \boldsymbol{0}$

- Example



$$\boldsymbol{J}_P^T = \begin{bmatrix} 0 & 0 & 0 \\ -c_1(a_2 s_2 + a_3 s_{23}) & -s_1(a_2 s_2 + a_3 s_{23}) & 0 \\ -a_3 c_1 s_{23} & -a_3 s_1 s_{23} & a_3 c_{23} \end{bmatrix}$$

⋆ null of $\boldsymbol{J}_P^T$

$$\frac{\nu_y}{\nu_x} = -\frac{1}{\tan \vartheta_1} \qquad \nu_z = 0$$

Orientation error

- Position error

$$\boldsymbol{e}_P = \boldsymbol{p}_d - \boldsymbol{p}(\boldsymbol{q})$$

$$\dot{\boldsymbol{e}}_P = \dot{\boldsymbol{p}}_d - \dot{\boldsymbol{p}}$$

- Euler angles

$$\boldsymbol{e}_O = \boldsymbol{\phi}_d - \boldsymbol{\phi}(\boldsymbol{q})$$

$$\dot{\boldsymbol{e}}_O = \dot{\boldsymbol{\phi}}_d - \dot{\boldsymbol{\phi}}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q}) \begin{bmatrix} \dot{\boldsymbol{p}}_d + \boldsymbol{K}_P \boldsymbol{e}_P \\ \dot{\boldsymbol{\phi}}_d + \boldsymbol{K}_O \boldsymbol{e}_O \end{bmatrix}$$

  - ⋆ handy to assign the time history $\boldsymbol{\phi}_d(t)$
  - ⋆ anyhow requires computation of $\boldsymbol{R} = [\,\boldsymbol{n} \quad \boldsymbol{s} \quad \boldsymbol{a}\,]$

- Manipulator with spherical wrist
  - ⋆ compute $\boldsymbol{q}_P \quad \Longrightarrow \quad \boldsymbol{R}_W$
  - ⋆ compute $\boldsymbol{R}_W^T \boldsymbol{R}_d \quad \Longrightarrow \quad \boldsymbol{q}_O$ (Euler angles ZYZ)

- Angle and axis

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_d \boldsymbol{R}^T$$

  - ⋆ orientation error

$$\begin{aligned}\boldsymbol{e}_O &= \boldsymbol{r} \sin\vartheta \\ &= \frac{1}{2}(\boldsymbol{n} \times \boldsymbol{n}_d + \boldsymbol{s} \times \boldsymbol{s}_d + \boldsymbol{a} \times \boldsymbol{a}_d)\end{aligned}$$

$$\dot{\boldsymbol{e}}_O = \boldsymbol{L}^T \boldsymbol{\omega}_d - \boldsymbol{L}\boldsymbol{\omega}$$

where

$$\boldsymbol{L} = -\frac{1}{2}\big(\boldsymbol{S}(\boldsymbol{n}_d)\boldsymbol{S}(\boldsymbol{n}) + \boldsymbol{S}(\boldsymbol{s}_d)\boldsymbol{S}(\boldsymbol{s}) + \boldsymbol{S}(\boldsymbol{a}_d)\boldsymbol{S}(\boldsymbol{a})\big)$$

$$\begin{aligned}\dot{\boldsymbol{e}} = \begin{bmatrix}\dot{\boldsymbol{e}}_P \\ \dot{\boldsymbol{e}}_O\end{bmatrix} &= \begin{bmatrix}\dot{\boldsymbol{p}}_d - \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}} \\ \boldsymbol{L}^T\boldsymbol{\omega}_d - \boldsymbol{L}\boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}}\end{bmatrix} \\ &= \begin{bmatrix}\dot{\boldsymbol{p}}_d \\ \boldsymbol{L}^T\boldsymbol{\omega}_d\end{bmatrix} - \begin{bmatrix}\boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{L}\end{bmatrix}\boldsymbol{J}\dot{\boldsymbol{q}}\end{aligned}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\begin{bmatrix}\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\boldsymbol{e}_P \\ \boldsymbol{L}^{-1}\left(\boldsymbol{L}^T\boldsymbol{\omega}_d + \boldsymbol{K}_O\boldsymbol{e}_O\right)\end{bmatrix}$$

- Unit quaternion

$$\Delta\mathcal{Q} = \mathcal{Q}_d * \mathcal{Q}^{-1}$$

- ⋆ orientation error

$$\boldsymbol{e}_O = \Delta\boldsymbol{\epsilon} = \eta(\boldsymbol{q})\boldsymbol{\epsilon}_d - \eta_d\boldsymbol{\epsilon}(\boldsymbol{q}) - \boldsymbol{S}(\boldsymbol{\epsilon}_d)\boldsymbol{\epsilon}(\boldsymbol{q})$$

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\begin{bmatrix}\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P\boldsymbol{e}_P \\ \boldsymbol{\omega}_d + \boldsymbol{K}_O\boldsymbol{e}_O\end{bmatrix}$$

$$\boldsymbol{\omega}_d - \boldsymbol{\omega} + \boldsymbol{K}_O\boldsymbol{e}_O = \boldsymbol{0}$$

- ⋆ quaternion propagation

$$\dot{\eta} = -\frac{1}{2}\boldsymbol{\epsilon}^T\boldsymbol{\omega}$$

$$\dot{\boldsymbol{\epsilon}} = \frac{1}{2}\left(\eta\boldsymbol{I} - \boldsymbol{S}(\boldsymbol{\epsilon})\right)\boldsymbol{\omega}$$

- ⋆ study of stability

$$V = (\eta_d - \eta)^2 + (\boldsymbol{\epsilon}_d - \boldsymbol{\epsilon})^T(\boldsymbol{\epsilon}_d - \boldsymbol{\epsilon})$$

$$\dot{V} = -\boldsymbol{e}_O^T\boldsymbol{K}_O\boldsymbol{e}_O$$

- Second-order algorithms
  - ⋆ time differentiation of differential kinematics

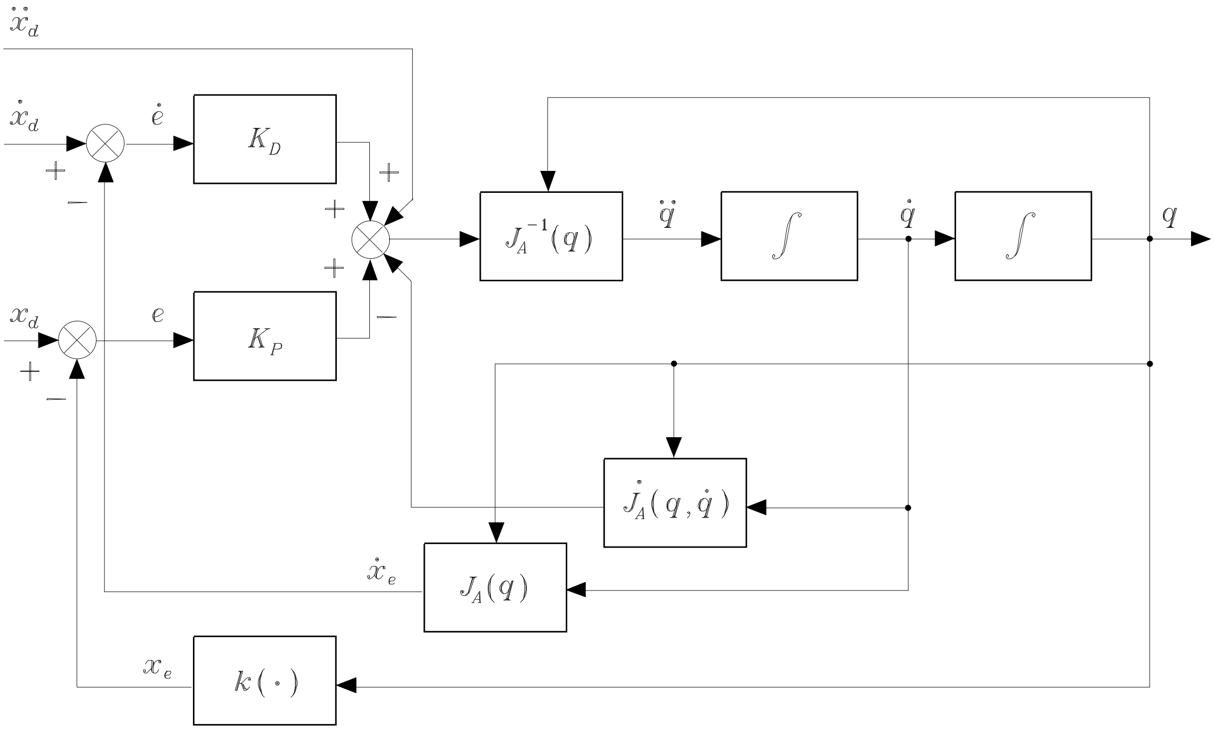
$$\ddot{\boldsymbol{x}}_e = \boldsymbol{J}_A(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_A(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

  - ⋆ joint accelerations solution

$$\ddot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q}) \left(\ddot{\boldsymbol{x}}_e - \dot{\boldsymbol{J}}_A(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} \right)$$

$$\Downarrow$$

$$\ddot{\boldsymbol{e}} + \boldsymbol{K}_D \dot{\boldsymbol{e}} + \boldsymbol{K}_P \boldsymbol{e} = \boldsymbol{0}$$


$\ddot{x}_d$
$\dot{x}_d$
$\dot{e}$
K_D
x_d
e
K_P
$J_A^{-1}(q)$
$\ddot{q}$
$\int$
$\dot{q}$
$\int$
q
$\dot{J}_A(q,\dot{q})$
$\dot{x}_e$
$J_A(q)$
x_e
$k(\cdot)$

Comparison among inverse kinematics algorithms

- Three–link planar arm

$$\boldsymbol{x} = \boldsymbol{k}(\boldsymbol{q})$$

$$\begin{bmatrix} p_x \\ p_y \\ \phi \end{bmatrix} = \begin{bmatrix} a_1c_1 + a_2c_{12} + a_3c_{123} \\ a_1s_1 + a_2s_{12} + a_3s_{123} \\ \vartheta_1 + \vartheta_2 + \vartheta_3 \end{bmatrix}$$

⋆ $a_1 = a_2 = a_3 = 0.5\,\mathrm{m}$

$$\boldsymbol{J}_A = \begin{bmatrix} -a_1s_1 - a_2s_{12} - a_3s_{123} & -a_2s_{12} - a_3s_{123} & -a_3s_{123} \\ a_1c_1 + a_2c_{12} + a_3c_{123} & a_2c_{12} + a_3c_{123} & a_3c_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

- $\boldsymbol{q}_i = [\,\pi \quad -\pi/2 \quad -\pi/2\,]^T$ rad

$$\Downarrow$$

- $\boldsymbol{p}_{di} = [\,0 \quad 0.5\,]^T$ m $\qquad \phi = 0\,$rad

- desired trajectory

$$\boldsymbol{p}_d(t) = \begin{bmatrix} 0.25(1 - \cos \pi t) \\ 0.25(2 + \sin \pi t) \end{bmatrix} \qquad 0 \le t \le 4$$

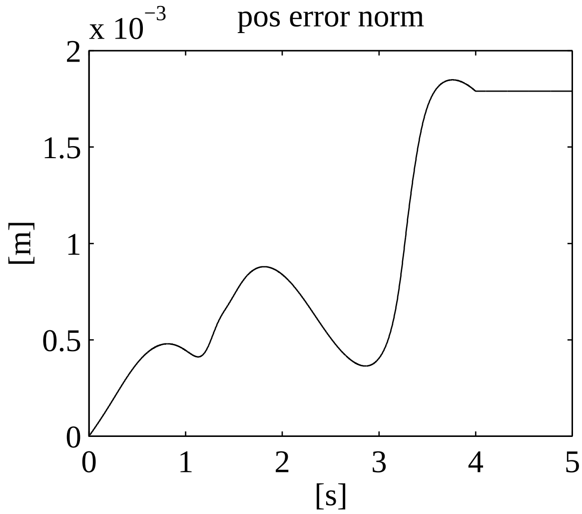
$$\phi_d(t) = \sin \frac{\pi}{24} t \qquad 0 \le t \le 4$$

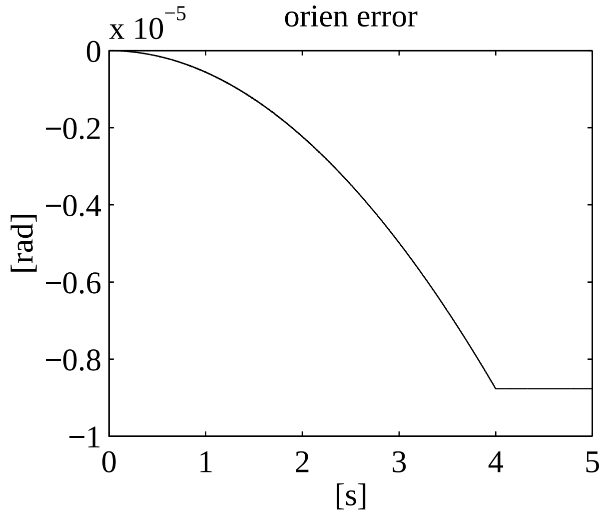
- MATLAB simulation with Euler numerical integration

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \dot{\boldsymbol{q}}(t_k)\Delta t$$

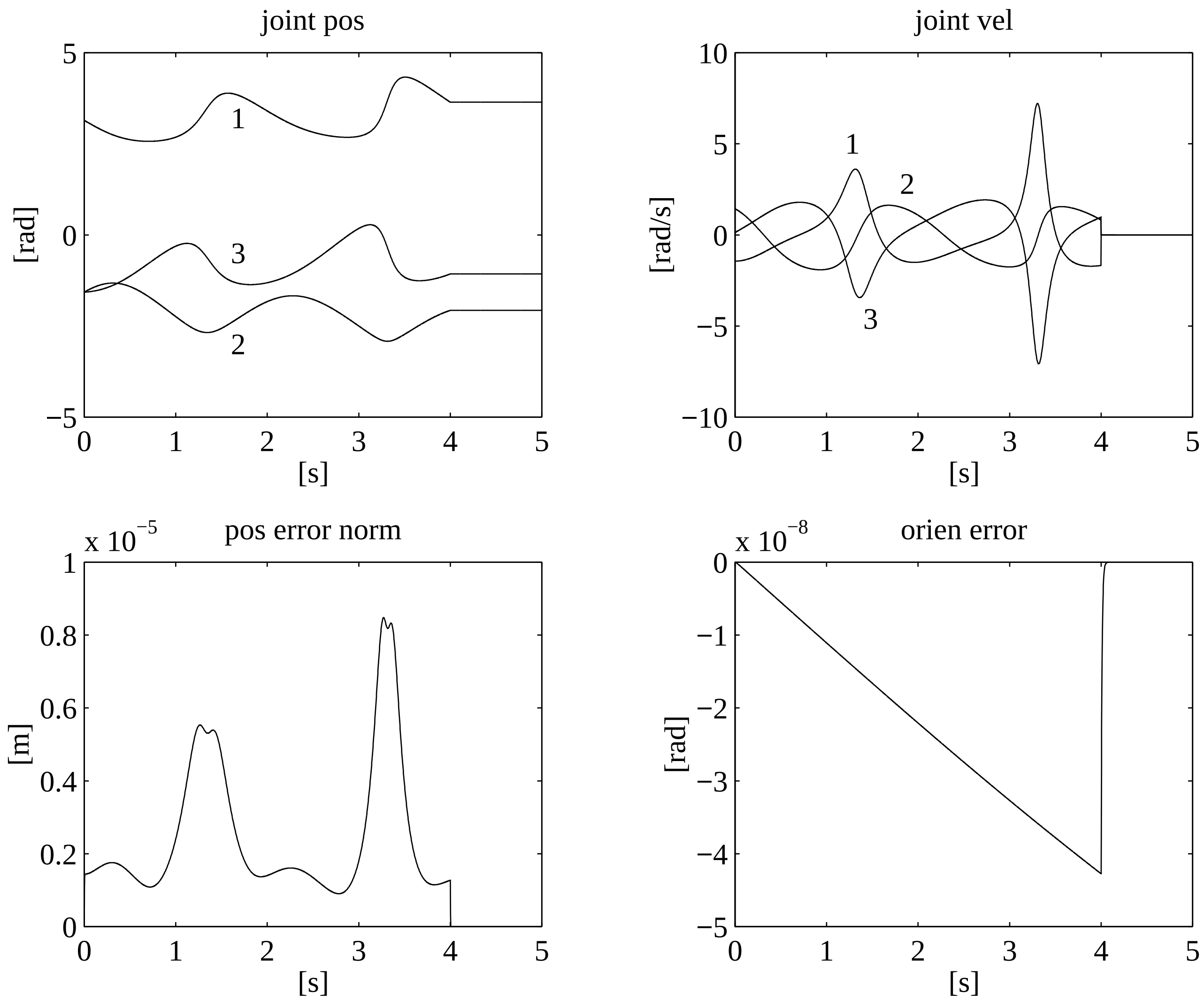
and $\Delta t = 1\,$ms

- $\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q})\dot{\boldsymbol{x}}$



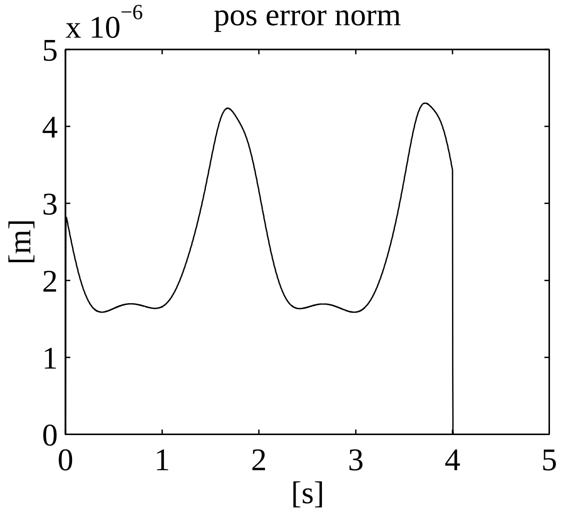


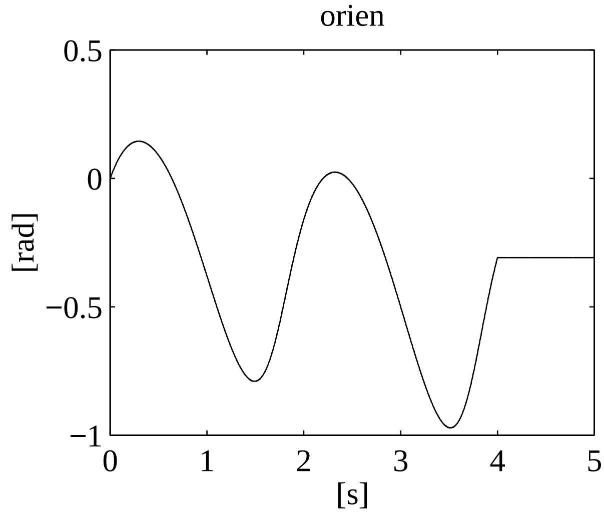
- $\dot{\boldsymbol{q}} = \boldsymbol{J}_A^{-1}(\boldsymbol{q})(\dot{\boldsymbol{x}}_d + \boldsymbol{K}\boldsymbol{e})$ $\qquad \boldsymbol{K} = \mathrm{diag}\{500, 500, 100\}$



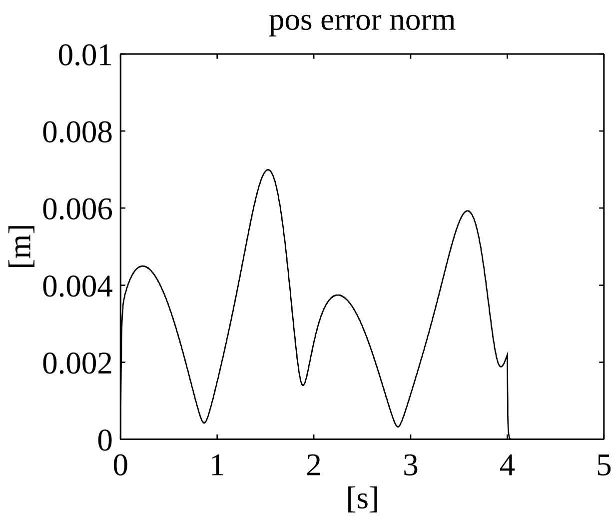
- free ϕ ($r = 2$, $n = 3$)

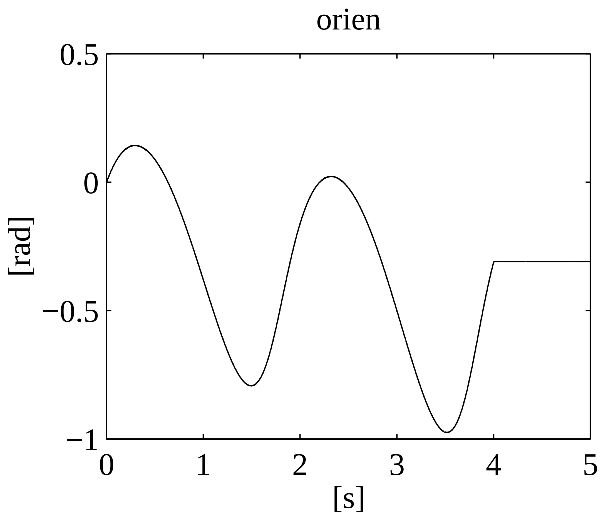
- $\dot{\boldsymbol{q}} = \boldsymbol{J}_P^{\dagger}(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P \boldsymbol{e}_P)$ $\qquad \boldsymbol{K}_P = \mathrm{diag}\{500, 500\}$





- $\dot{\boldsymbol{q}} = \boldsymbol{J}_P^T(\boldsymbol{q})\boldsymbol{K}_P \boldsymbol{e}_P$ $\qquad \boldsymbol{K}_P = \mathrm{diag}\{500, 500\}$





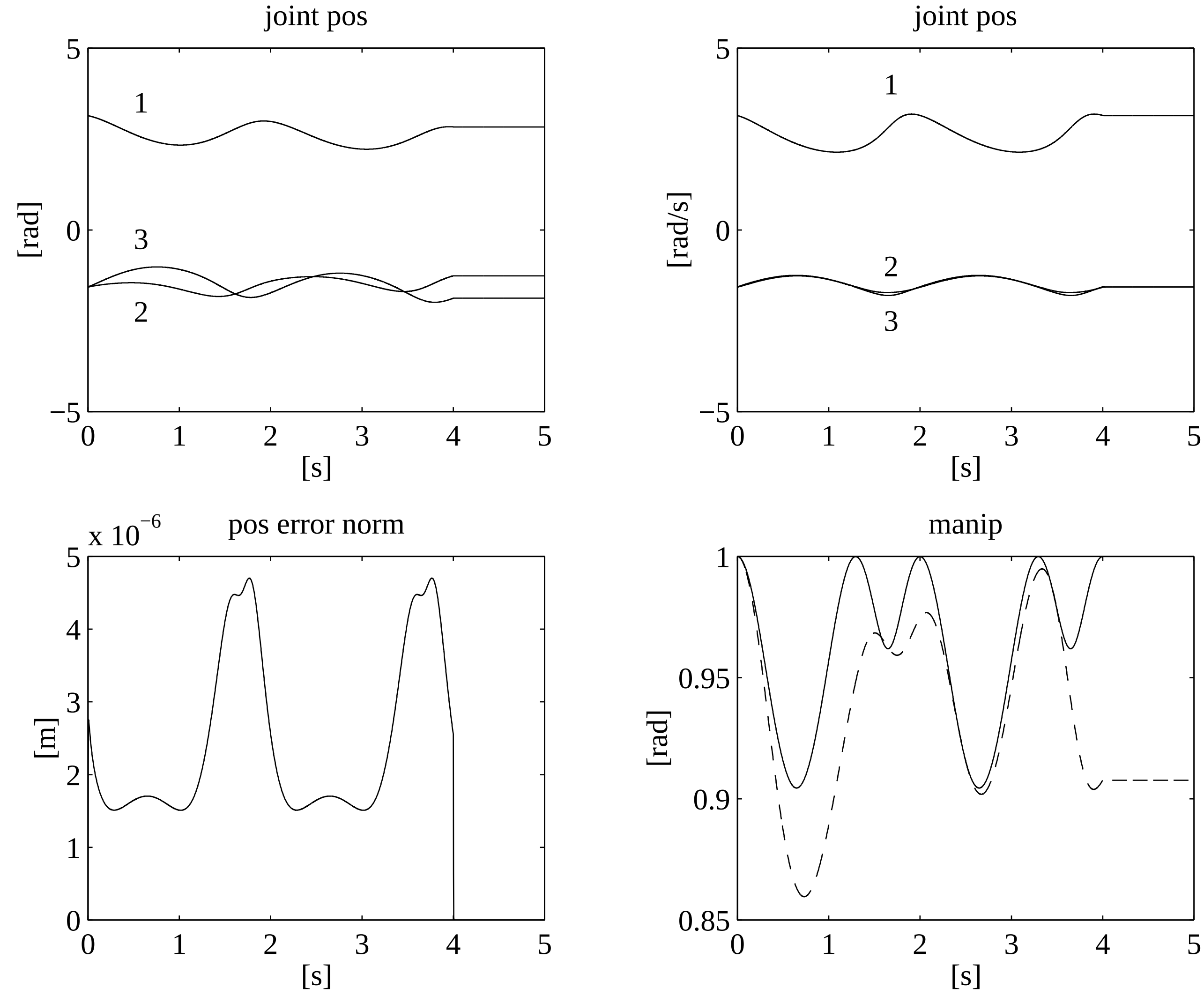
- $\dot{\boldsymbol{q}} = \boldsymbol{J}_P^{\dagger}(\dot{\boldsymbol{p}}_d + \boldsymbol{K}_P \boldsymbol{e}_P) + (\boldsymbol{I} - \boldsymbol{J}_P^{\dagger}\boldsymbol{J}_P)\dot{\boldsymbol{q}}_a$

$$\boldsymbol{K}_P = \mathrm{diag}\{500, 500\}$$

⋆ manipulability measure

$$w(\vartheta_2, \vartheta_3) = \frac{1}{2}(s_2^2 + s_3^2)$$

⋆ $\dot{\boldsymbol{q}}_a = k_a \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T \qquad k_a = 50$

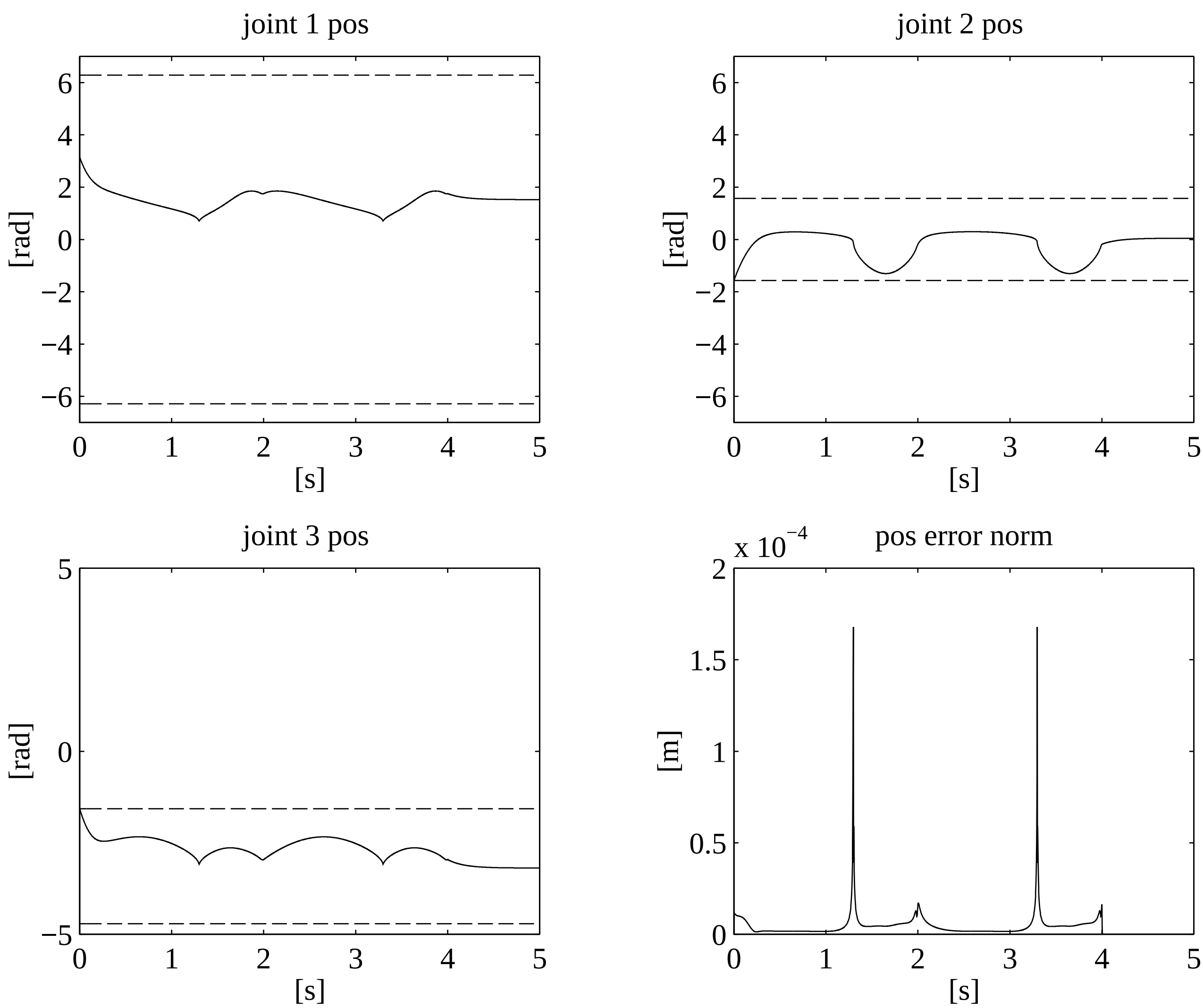


⋆ distance from mechanical joint limits

$$w(\boldsymbol{q}) = -\frac{1}{6}\sum_{i=1}^{3}\left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}}\right)^2$$

$$-2\pi \leq q_1 \leq 2\pi \qquad -\pi/2 \leq q_2 \leq \pi/2 \qquad -3\pi/2 \leq q_3 \leq -\pi/2$$

⋆ $\dot{\boldsymbol{q}}_a = k_a \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}}\right)^T \qquad k_a = 250$



STATICS

- Relationship between end-effector forces and moments (*forces*) $\boldsymbol{\gamma}$ and joint forces and/or torques (*torques*) $\boldsymbol{\tau}$ with manipulator at equilibrium configuration

  ⋆ elementary work associated with torques

$$dW_\tau = \boldsymbol{\tau}^T d\boldsymbol{q}$$

  ⋆ elementary work associated with forces

$$\begin{aligned} dW_\gamma &= \boldsymbol{f}^T d\boldsymbol{p} + \boldsymbol{\mu}^T \boldsymbol{\omega} dt \\ &= \boldsymbol{f}^T \boldsymbol{J}_P(\boldsymbol{q}) d\boldsymbol{q} + \boldsymbol{\mu}^T \boldsymbol{J}_O(\boldsymbol{q}) d\boldsymbol{q} \\ &= \boldsymbol{\gamma}^T \boldsymbol{J}(\boldsymbol{q}) d\boldsymbol{q} \end{aligned}$$

  ⋆ elementary displacements $\equiv$ virtual displacements

$$\begin{aligned} \delta W_\tau &= \boldsymbol{\tau}^T \delta\boldsymbol{q} \\ \delta W_\gamma &= \boldsymbol{\gamma}^T \boldsymbol{J}(\boldsymbol{q}) \delta\boldsymbol{q} \end{aligned}$$

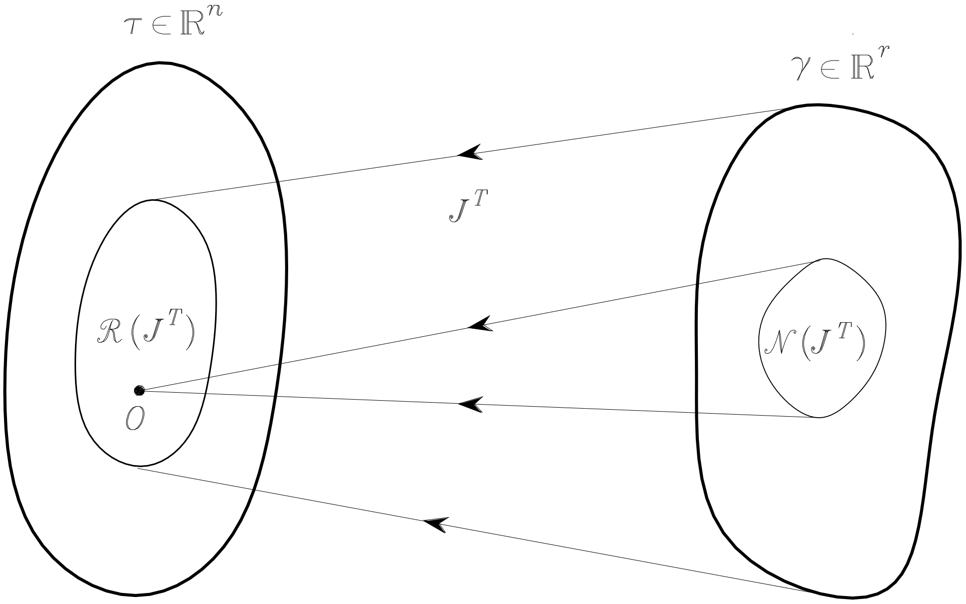
- Principle of virtual work
  - ⋆ the manipulator is at *static equilibrium* if and only if

$$\delta W_{\tau} = \delta W_{\gamma} \qquad \forall \delta \boldsymbol{q}$$

$$\Downarrow$$

$$\boldsymbol{\tau} = \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{\gamma}$$

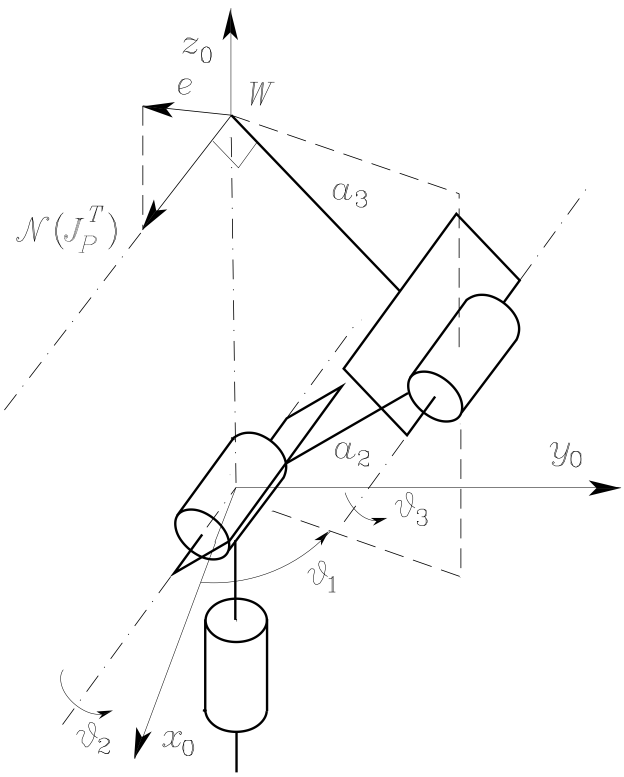
Kineto-statics duality



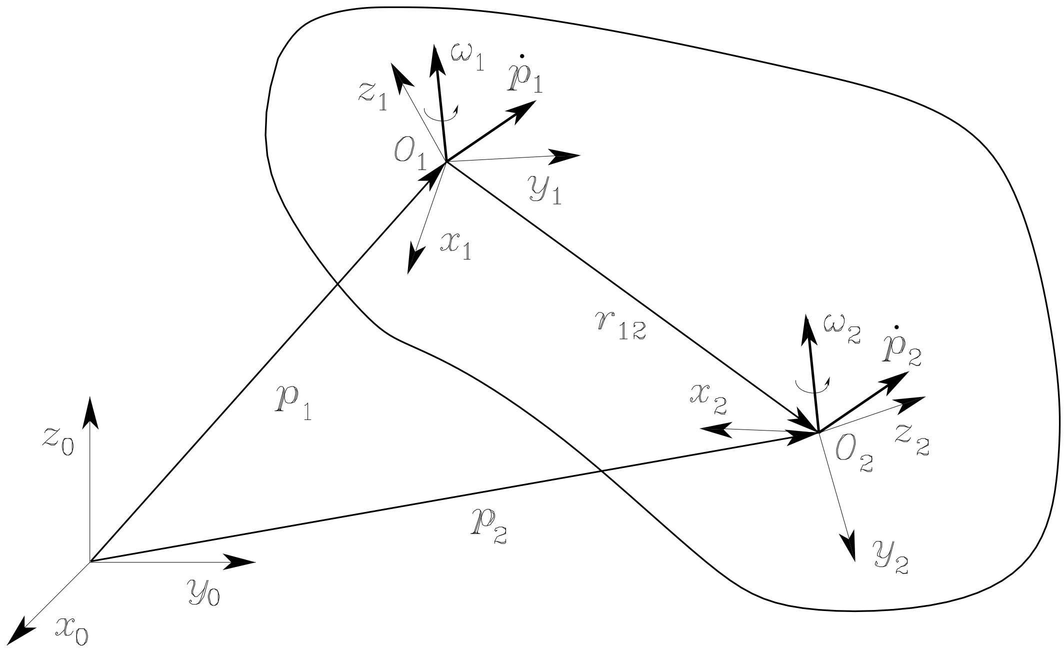
$$\mathcal{N}(\boldsymbol{J}) \equiv \mathcal{R}^{\perp}(\boldsymbol{J}^T) \qquad \mathcal{R}(\boldsymbol{J}) \equiv \mathcal{N}^{\perp}(\boldsymbol{J}^T)$$

- forces $\boldsymbol{\gamma} \in \mathcal{N}(\boldsymbol{J}^T)$ not requiring any balancing torques

- Physical interpretation of Jacobian transpose CLIK algorithm
  - ⋆ ideal dynamics $\boldsymbol{\tau} = \dot{\boldsymbol{q}}$
  - ⋆ elastic force $\boldsymbol{K}\boldsymbol{e}$ pulling end-effector towards desired pose in operational space
  - ⋆ effective only if $\boldsymbol{K}\boldsymbol{e} \notin \mathcal{N}(\boldsymbol{J}^T)$



Velocity and force transformation



$$\begin{bmatrix} \dot{\boldsymbol{p}}_2 \\ \boldsymbol{\omega}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{S}(\boldsymbol{r}_{12}) \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}_1 \\ \boldsymbol{\omega}_1 \end{bmatrix}$$

$$\boldsymbol{r}_{12} = \boldsymbol{R}_1 \boldsymbol{r}_{12}^1$$

$$\dot{\boldsymbol{p}}_1 = \boldsymbol{R}_1 \dot{\boldsymbol{p}}_1^1 \qquad \dot{\boldsymbol{p}}_2 = \boldsymbol{R}_2 \dot{\boldsymbol{p}}_2^2 = \boldsymbol{R}_1 \boldsymbol{R}_2^1 \dot{\boldsymbol{p}}_2^2$$
$$\boldsymbol{\omega}_1 = \boldsymbol{R}_1 \boldsymbol{\omega}_1^1 \qquad \boldsymbol{\omega}_2 = \boldsymbol{R}_2 \boldsymbol{\omega}_2^2 = \boldsymbol{R}_1 \boldsymbol{R}_2^1 \boldsymbol{\omega}_2^2$$

$$\begin{bmatrix} \dot{\boldsymbol{p}}_2^2 \\ \boldsymbol{\omega}_2^2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^2 & -\boldsymbol{R}_1^2 \boldsymbol{S}(\boldsymbol{r}_{12}^1) \\ \boldsymbol{O} & \boldsymbol{R}_1^2 \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}_1^1 \\ \boldsymbol{\omega}_1^1 \end{bmatrix}$$

$$\boldsymbol{v}_2^2 = \boldsymbol{J}_1^2 \boldsymbol{v}_1^1$$

⋆ by virtue of kineto-statics duality

$$\boldsymbol{\gamma}_1^1 = \boldsymbol{J}_1^{2T} \boldsymbol{\gamma}_2^2$$

$$\begin{bmatrix} \boldsymbol{f}_1^1 \\ \boldsymbol{\mu}_1^1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_2^1 & \boldsymbol{O} \\ \boldsymbol{S}(\boldsymbol{r}_{12}^1) \boldsymbol{R}_2^1 & \boldsymbol{R}_2^1 \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_2^2 \\ \boldsymbol{\mu}_2^2 \end{bmatrix}$$

MANIPULABILITY ELLIPSOIDS

- *Velocity manipulability ellipsoid*
  - ⋆ set of joint velocities of constant (unit) norm

$$\dot{\boldsymbol{q}}^T \dot{\boldsymbol{q}} = 1$$

  - ⋆ redundant manipulator

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}(\boldsymbol{q})\boldsymbol{v}$$

$$\Downarrow$$

$$\boldsymbol{v}^T \big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)^{-1} \boldsymbol{v} = 1$$

- Axes
  - ⋆ eigenvectors $\boldsymbol{u}_i$ of $\boldsymbol{J}\boldsymbol{J}^T$ $\Longrightarrow$ directions
  - ⋆ singular values $\sigma_i = \sqrt{\lambda_i(\boldsymbol{J}\boldsymbol{J}^T)}$ $\Longrightarrow$ dimensions
- Volume
  - ⋆ proportional to

$$w(\boldsymbol{q}) = \sqrt{\det\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)}$$

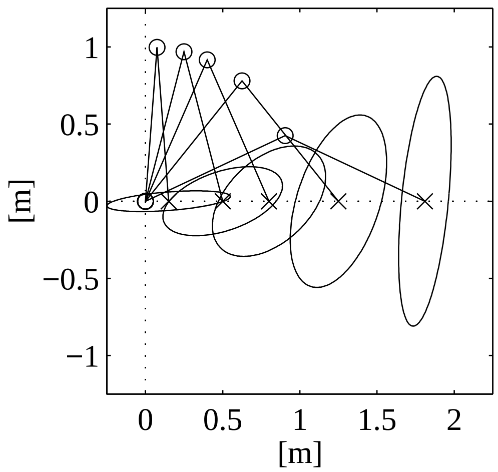
Two–link planar arm

- Manipulability measure

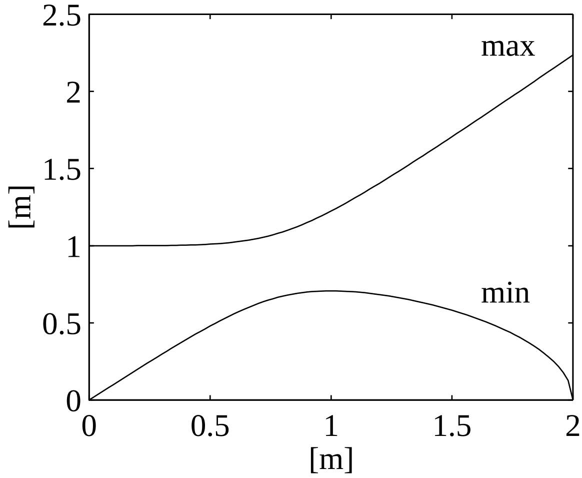
$$w = |\det(\boldsymbol{J})| = a_1 a_2 |s_2|$$

  - ⋆ max at $\vartheta_2 = \pm\pi/2$
  - ⋆ max at $a_1 = a_2$ (for given reach $a_1 + a_2$)

- Velocity manipulability ellipses



- Singular values



- *Force manipulability ellipsoid*
  - ⋆ set of joint torques of constant (unit) norm

$$\boldsymbol{\tau}^T \boldsymbol{\tau} = 1$$

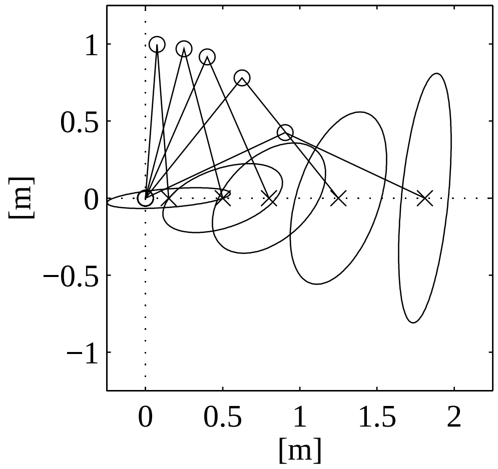
$$\Downarrow$$

$$\boldsymbol{\gamma}^T \big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)\boldsymbol{\gamma} = 1$$

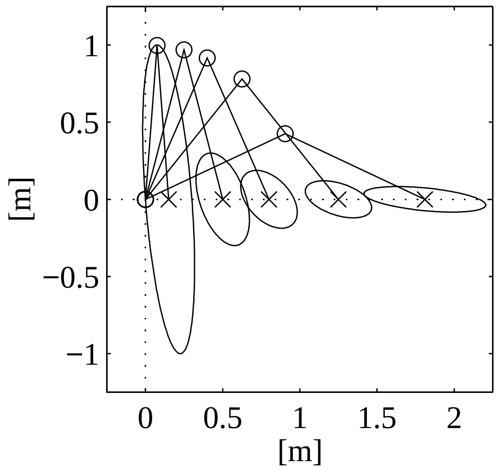
- Kineto-statics duality
  - ⋆ a direction along which good velocity manipulability is obtained is a direction along which poor force manipulability is obtained, and vice versa

Two–link planar arm

- Velocity manipulability ellipses



- Force manipulability ellipses



- Manipulator $\equiv$ *mechanical transformer* of velocities and forces from joint space to operational space
  - ⋆ transformation ratio along a direction for force ellipsoid

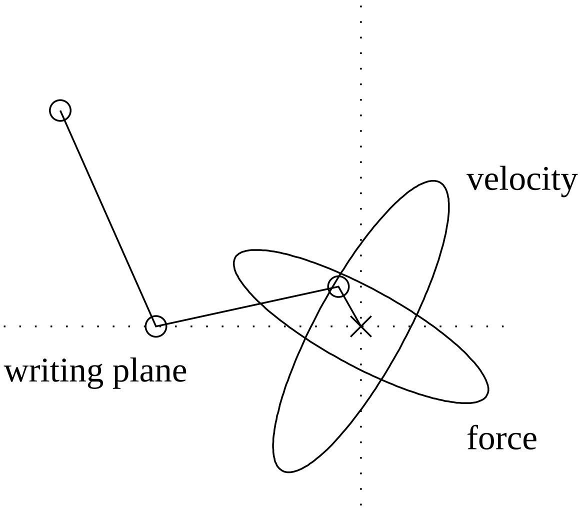
$$\alpha(\boldsymbol{q}) = \left(\boldsymbol{u}^T \boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{u}\right)^{-1/2}$$

  - ⋆ transformation ratio along a direction for velocity ellipsoid

$$\beta(\boldsymbol{q}) = \left(\boldsymbol{u}^T \left(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\right)^{-1}\boldsymbol{u}\right)^{-1/2}$$

  - ⋆ use of redundant degrees of freedom

- Task compatibility of structure along a given direction
  - ⋆ writing



  - ⋆ throwing in bowling game