Consistent Linearization for Compressible Stokes System with Plastic Dilation

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1 The Compressible Stokes System

The compressible Stokes system (without thermal expansion) is given by

$$-\nabla\cdot\boldsymbol{\tau}+\nabla p=\boldsymbol{f}, \tag{1}$$
$$-\nabla\cdot\boldsymbol{u}=\beta\dot{p}, \tag{2}$$

where $\boldsymbol{\tau}$ is the deviatoric stress tensor, p is pressure, $\beta := \frac{1}{\rho}\frac{\partial\rho}{\partial p}$ characterizes the compressiblity of the material, and $\boldsymbol{f}$ represents a body force. Approximating the time derivative of p with a backward Euler scheme, we can rewrite Eq. (2) as

$$\nabla\cdot\boldsymbol{u}+\frac{\beta p}{\Delta t}=\frac{\beta p^0}{\Delta t}, \tag{3}$$

where Δt is the time step length, and p^0 denotes the pressure in the previous time step.

We derive the weak for of the momentum conservative equation by integrating the inner product of Eq. (1) and a virtual velocity $\boldsymbol{v}$ across the computational domain Ω, which yields after integrating by part

$$\int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\tau}\mathrm{d}\Omega-\int_\Omega\nabla\cdot\boldsymbol{v}p\mathrm{d}\Omega=\int_\Omega\boldsymbol{v}\cdot\boldsymbol{f}\mathrm{d}\Omega, \tag{4}$$

where $\boldsymbol{\varepsilon}(\cdot) := \frac{1}{2}[\nabla(\cdot)+\nabla^T(\cdot)]-\frac{1}{3}\nabla\cdot(\cdot)$ is the deviatoric symmetric gradient operator. Similarly, the weak form of mass conservative equation is obtained by multiplying Eq. (3) by a virtual pressure q and integrating across Ω, which gives

$$\int_\Omega q\nabla\cdot\boldsymbol{u}\mathrm{d}\Omega+\int_\Omega q\frac{\beta p}{\Delta t}\mathrm{d}\Omega=\int_\Omega q\frac{\beta p^0}{\Delta t}\mathrm{d}\Omega. \tag{5}$$

2 Constitutive Relation

Here we consider a Maxwell-type viscoelastic plastic model, which is based on the additive decomposition of the deviatoric strain rate $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}^v+\boldsymbol{\varepsilon}^e+\boldsymbol{\varepsilon}^p. \tag{6}$$

The constitutive relationship between $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$ can then be expressed as

$$\boldsymbol{\varepsilon}=\frac{\boldsymbol{\tau}}{2\eta}+\frac{\mathring{\boldsymbol{\tau}}}{2G}+\gamma\frac{\partial\Psi}{\partial\boldsymbol{\tau}}, \tag{7}$$

where $\eta = \eta(\boldsymbol{u}, p)$ is the viscosity, G is the shear modulus, γ is the plastic multiplyer, Ψ is the plastic potential, and $\mathring{\boldsymbol{\tau}}$ denotes the co-rotational derivative of $\boldsymbol{\tau}$. We assume that the plastic flow is governed by the Drucker-Prager model:

$$\Phi = \tau_{\mathrm{II}} - \xi p - \zeta, \tag{8}$$

$$\Psi = \tau_{\mathrm{II}} - \hat{\xi} p, \tag{9}$$

where $\tau_{\mathrm{II}} := \sqrt{\frac{1}{2}\boldsymbol{\tau} : \boldsymbol{\tau}}$ stands for the second invariant of $\boldsymbol{\tau}$, ξ, $\hat{\xi}$ and ζ are material parameters related with frictional angle ϕ, dilatancy angle ψ and cohesion c. Integrating the stress rate with a first-order difference scheme, i.e. $\boldsymbol{\tau} = \boldsymbol{\tau}^0 + \mathring{\boldsymbol{\tau}}\Delta t$, we can rewrite Eq. (7) as

$$\boldsymbol{\tau} = 2\eta^{ve}\left(\boldsymbol{\varepsilon}^{\mathrm{tr}} - \gamma \frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right), \tag{10}$$

where η^{ve} and $\boldsymbol{\varepsilon}^{\mathrm{tr}}$ are defined as

$$\eta^{ve} := \left(\frac{1}{\eta(\boldsymbol{u}, p)} + \frac{1}{G\Delta t}\right)^{-1}, \qquad \boldsymbol{\varepsilon}^{\mathrm{tr}} := \boldsymbol{\varepsilon} + \frac{\boldsymbol{\tau}^0}{2G\Delta t}. \tag{11}$$

The volumetric constitutive relation is based on an additive decomposition of the divergence of velocity (notice the negative sign for the plastic component)

$$\nabla \cdot \boldsymbol{u} = -\frac{\beta(p - p^0)}{\Delta t} - \gamma \frac{\partial \Psi}{\partial p}. \tag{12}$$

Similar to the deviatoric part, we can also define the trial pressure as

$$p^{\mathrm{tr}} := p^0 - \frac{\Delta t}{\beta} \nabla \cdot \boldsymbol{u}, \tag{13}$$

then from Eq. (12) and Eq. (9) we get

$$p = p^{\mathrm{tr}} - \frac{\gamma \Delta t}{\beta} \frac{\partial \Psi}{\partial p} = p^{\mathrm{tr}} + \frac{\gamma \hat{\xi} \Delta t}{\beta}. \tag{14}$$

3 Newton Linearization

Replacing the independent variables $(\boldsymbol{u}, p)$ in Eq. (4) and Eq. (5) by $(\boldsymbol{u}^{\mathrm{tr}}, p^{\mathrm{tr}})$, we obtain the following nonlinear system (assuming that the volumetric parts of $\boldsymbol{u}$ and $\boldsymbol{u}^{\mathrm{tr}}$ are identical):

$$F_u := \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\tau} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} p^{\mathrm{tr}} \mathrm{d}\Omega - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{f} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} \frac{\gamma \hat{\xi} \Delta t}{\beta} \mathrm{d}\Omega = 0, \tag{15}$$

$$F_p := -\int_\Omega q \nabla \cdot \boldsymbol{u}^{\mathrm{tr}} \mathrm{d}\Omega - \int_\Omega \frac{q \beta p^{\mathrm{tr}}}{\Delta t} \mathrm{d}\Omega + \int_\Omega q \left(\frac{\beta p^0}{\Delta t} - \gamma \hat{\xi}\right) \mathrm{d}\Omega = 0. \tag{16}$$

If we apply the Newton-Raphson method to solve Eqs. (15) and (16), then in each iteration we need to solve a linearized system

$$\frac{\partial F_u}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} + \frac{\partial F_u}{\partial p^{\mathrm{tr}}} : \mathrm{d}p^{\mathrm{tr}} = -F_u, \tag{17}$$

$$\frac{\partial F_p}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} + \frac{\partial F_p}{\partial p^{\mathrm{tr}}} : \mathrm{d}p^{\mathrm{tr}} = -F_p. \tag{18}$$

The differentiations of F_u and F_p are given by

$$\frac{\partial F_u}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} \frac{\Delta t}{\beta} \frac{\partial (\gamma \hat{\xi})}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} \mathrm{d}\Omega, \tag{19}$$

$$\frac{\partial F_u}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{v}) : \frac{\partial \boldsymbol{\tau}}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} \mathrm{d}p^{\mathrm{tr}} \mathrm{d}\Omega - \int_\Omega \nabla \cdot \boldsymbol{v} \frac{\Delta t}{\beta} \frac{\partial (\gamma \hat{\xi})}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} \mathrm{d}\Omega, \tag{20}$$

$$\frac{\partial F_p}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} = -\int_\Omega q \nabla \cdot \mathrm{d}\boldsymbol{u}^{\mathrm{tr}} \mathrm{d}\Omega - \int_\Omega q \frac{\partial (\gamma \hat{\xi})}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} \mathrm{d}\Omega, \tag{21}$$

$$\frac{\partial F_p}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} = -\int_\Omega \frac{q \beta \mathrm{d}p^{\mathrm{tr}}}{\Delta t} \mathrm{d}\Omega - \int_\Omega q \frac{\partial (\gamma \hat{\xi})}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} \mathrm{d}\Omega. \tag{22}$$

Comparing to the linearized system of incompressible Stokes equations, the additional terms related with plastic dilation are (terms with β and/or $\hat{\xi}$):

$$\text{top left:} \quad -\int_\Omega \nabla \cdot \boldsymbol{v} \frac{\Delta t}{\beta} \frac{\partial (\gamma \hat{\xi})}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} \mathrm{d}\Omega, \tag{23}$$

$$\text{top right:} \quad -\int_\Omega \nabla \cdot \boldsymbol{v} \frac{\Delta t}{\beta} \frac{\partial (\gamma \hat{\xi})}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} \mathrm{d}\Omega, \tag{24}$$

$$\text{bottom left:} \quad -\int_\Omega q \frac{\partial (\gamma \hat{\xi})}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}} : \mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{tr}} \mathrm{d}\Omega, \tag{25}$$

$$\text{bottom right:} \quad -\int_\Omega \frac{q \beta}{\Delta t} \mathrm{d}\Omega - \int_\Omega q \frac{\partial (\gamma \hat{\xi})}{\partial p^{\mathrm{tr}}} \mathrm{d}p^{\mathrm{tr}} \mathrm{d}\Omega. \tag{26}$$

To calculate the additional terms, we need to know the differentiation of $\vartheta := \gamma \hat{\xi}$. As for the differentiation of viscosity, we use a finite difference approximation to calculate $\mathrm{d}\vartheta$:

$$\begin{aligned} \left.\frac{\partial \vartheta}{\partial \boldsymbol{\varepsilon}^{\mathrm{tr}}}\right|_{(\boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}})} &\approx \frac{\vartheta(\boldsymbol{\varepsilon}^{\mathrm{tr}} + \delta \boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}}) - \vartheta(\boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}})}{\delta \boldsymbol{\varepsilon}^{\mathrm{tr}}}, \\ \left.\frac{\partial \vartheta}{\partial p^{\mathrm{tr}}}\right|_{(\boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}})} &\approx \frac{\vartheta(\boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}} + \delta p^{\mathrm{tr}}) - \vartheta(\boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}})}{\delta p^{\mathrm{tr}}}. \end{aligned} \tag{27}$$

The expression of $\hat{\xi}(\boldsymbol{\varepsilon}^{\mathrm{tr}})$ depends on the strain weakening model; the expression of $\gamma(\boldsymbol{\varepsilon}^{\mathrm{tr}}, p^{\mathrm{tr}})$ can be derived from Eq. (10):

$$\begin{aligned} \gamma &= \left(\boldsymbol{\varepsilon}^{\mathrm{tr}} - \frac{\boldsymbol{\tau}}{2\eta^{ve}}\right) : \left(\frac{\partial \Psi}{\partial \boldsymbol{\tau}}\right)^{-1} \\ &= \left(\boldsymbol{\varepsilon}^{\mathrm{tr}} - \frac{\boldsymbol{\tau}}{2\eta^{ve}}\right) : \frac{2\tau_{\mathrm{II}}}{\boldsymbol{\tau}} \\ &= \left(\frac{1}{\eta^{\mathrm{eff}}} - \frac{1}{\eta^{ve}}\right) \tau_{\mathrm{II}} \\ &= 2\left(1 - \frac{\eta^{\mathrm{eff}}}{\eta^{ve}}\right) \varepsilon_{\mathrm{II}}^{\mathrm{tr}}. \end{aligned} \tag{28}$$