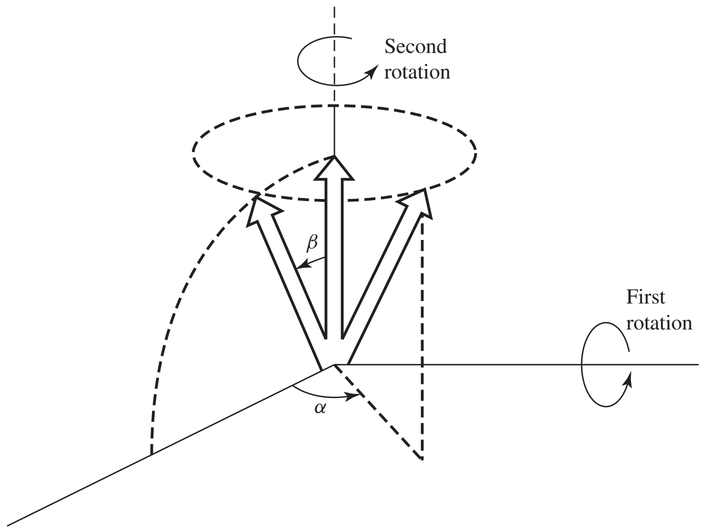

Fig. 3.3 Construction of the $\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}$ eigenspinor.

spinor language this sequence of operations is equivalent to applying $\exp(-i\sigma_2\beta/2)$ to $\begin{pmatrix}1\\0\end{pmatrix}$ followed by an application of $\exp(-i\sigma_3\alpha/2)$. The net result is

$$\begin{aligned}\chi &= \left[\cos\left(\frac{\alpha}{2}\right) - i\sigma_3\sin\left(\frac{\alpha}{2}\right)\right]\left[\cos\left(\frac{\beta}{2}\right) - i\sigma_2\sin\left(\frac{\beta}{2}\right)\right]\begin{pmatrix}1\\0\end{pmatrix}\\ &= \begin{pmatrix}\cos\left(\frac{\alpha}{2}\right) - i\sin\left(\frac{\alpha}{2}\right) & 0\\ 0 & \cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\end{pmatrix}\begin{pmatrix}\cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right)\\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right)\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ &= \begin{pmatrix}\cos\left(\frac{\beta}{2}\right)e^{-i\alpha/2}\\ \sin\left(\frac{\beta}{2}\right)e^{i\alpha/2}\end{pmatrix},\end{aligned} \tag{3.70}$$

in complete agreement with Problem 1.11 of Chapter 1 if we realize that a phase common to both the upper and lower components is devoid of physical significance.

3.3 SO(3), SU(2), and Euler Rotations

3.3.1 Orthogonal Group

We will now study a little more systematically the group properties of the operations with which we have been concerned in the previous two sections.

The most elementary approach to rotations is based on specifying the axis of rotation and the angle of rotation. It is clear that we need three real numbers to characterize a general rotation: the polar and the azimuthal angles of the unit vector $\hat{\mathbf{n}}$ taken in the direction of the rotation axis and the rotation angle ϕ itself. Equivalently, the same rotation can be specified by the three Cartesian components of the vector $\hat{\mathbf{n}}\phi$. However, these ways of characterizing rotation are not so convenient from the point of view of studying the group properties of rotations. For one thing, unless ϕ is infinitesimal or $\hat{\mathbf{n}}$ is always in the same direction, we cannot add vectors of the form $\hat{\mathbf{n}}\phi$ to characterize a succession of rotations. It is much easier to work with a 3×3 orthogonal matrix R because the effect of successive rotations can be obtained just by multiplying the appropriate orthogonal matrices.

How many independent parameters are there in a 3×3 orthogonal matrix? A real 3×3 matrix has 9 entries, but we have the orthogonality constraint

$$RR^T = 1 \tag{3.71}$$

which corresponds to 6 independent equations because the product RR^T, being the same as R^TR, is a symmetrical matrix with 6 independent entries. As a result, there are 3 (that is, 9–6) independent numbers in R, the same number previously obtained by a more elementary method.

The set of all multiplication operations with orthogonal matrices forms a group. By this we mean that the following four requirements are satisfied.

1. The product of any two orthogonal matrices is another orthogonal matrix, which is satisfied because

$$(R_1R_2)(R_1R_2)^T = R_1R_2R_2^TR_1^T = 1. \tag{3.72}$$

2. The associative law holds:

$$R_1(R_2R_3) = (R_1R_2)R_3. \tag{3.73}$$

3. The identity matrix 1, physically corresponding to no rotation, defined by

$$R1 = 1R = R \tag{3.74}$$

is a member of the class of all orthogonal matrices.

4. The inverse matrix R^{-1}, physically corresponding to rotation in the opposite sense, defined by

$$RR^{-1} = R^{-1}R = 1 \tag{3.75}$$

is also a member.

This group has the name SO(3), where S stands for special, O stands for orthogonal, 3 for three dimensions. Note only rotational operations are considered here, hence we have SO(3) rather than O(3) (which can include the inversion operation of Chapter 4 later).

3.3.2 Unitary Unimodular Group

In the previous section we learned yet another way to characterize an arbitrary rotation, that is, to look at the 2×2 matrix (3.63) that acts on the two-component spinor χ. Clearly, (3.63) is unitary. As a result, for the c_+ and c_-, defined in (3.46),

$$|c_+|^2 + |c_-|^2 = 1 \tag{3.76}$$

is left invariant. Furthermore, matrix (3.63) is unimodular; that is, its determinant is 1, as will be shown explicitly below.

We can write the most general unitary unimodular matrix as

$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \tag{3.77}$$

where a and b are *complex* numbers satisfying the unimodular condition

$$|a|^2 + |b|^2 = 1. \tag{3.78}$$

We can easily establish the unitary property of (3.77) as follows:

$$U(a,b)^\dagger U(a,b) = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = 1. \tag{3.79}$$

We can readily see that the 2×2 matrix (3.63) that characterizes a rotation of a spin $\frac{1}{2}$ system can be written as $U(a,b)$. Comparing (3.63) with (3.77), we identify

$$\begin{aligned} \mathrm{Re}(a) &= \cos\left(\frac{\phi}{2}\right), & \mathrm{Im}(a) &= -n_z \sin\left(\frac{\phi}{2}\right), \\ \mathrm{Re}(b) &= -n_y \sin\left(\frac{\phi}{2}\right), & \mathrm{Im}(b) &= -n_x \sin\left(\frac{\phi}{2}\right), \end{aligned} \tag{3.80}$$

from which the unimodular property of (3.78) is immediate. Conversely, it is clear that the most general unitary unimodular matrix of form (3.77) can be interpreted as representing a rotation.

The two complex numbers a and b are known as **Cayley–Klein parameters**. Historically the connection between a unitary unimodular matrix and a rotation was known long before the birth of quantum mechanics. In fact, the Cayley–Klein parameters were used to characterize complicated motions of gyroscopes in rigid-body kinematics.

Without appealing to the interpretations of unitary unimodular matrices in terms of rotations, we can directly check the group properties of multiplication operations with unitary unimodular matrices. Note in particular that

$$U(a_1,b_1)U(a_2,b_2) = U(a_1a_2 - b_1b_2^*, a_1b_2 + a_2^*b_1), \tag{3.81}$$

where the unimodular condition for the product matrix is

$$|a_1a_2 - b_1b_2^*|^2 + |a_1b_2 + a_2^*b_1|^2 = 1. \tag{3.82}$$

For the inverse of U we have

$$U^{-1}(a,b) = U(a^*, -b). \tag{3.83}$$

This group is known as SU(2), where S stands for special, U for unitary, and 2 for dimensionality 2. In contrast, the group defined by multiplication operations with general 2×2 unitary matrices (not necessarily constrained to be unimodular) is known as U(2). The most general unitary matrix in two dimensions has four independent parameters and can be written as $e^{i\gamma}$ (with γ real) times a unitary unimodular matrix:

$$U = e^{i\gamma} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1, \qquad \gamma^* = \gamma. \tag{3.84}$$

SU(2) is called a **subgroup** of U(2).

Because we can characterize rotations using both the SO(3) language and the SU(2) language, we may be tempted to conclude that the *groups* SO(3) and SU(2) are isomorphic, that is, that there is a one-to-one correspondence between an element of SO(3) and an element of SU(2). This inference is not correct. Consider a rotation by 2π and another one by 4π. In the SO(3) language, the matrices representing a 2π rotation and a 4π rotation are both 3×3 identity matrices; however, in the SU(2) language the corresponding matrices are -1 times the 2×2 identity matrix and the identity matrix itself, respectively. More generally, $U(a, b)$ and $U(-a, -b)$ both correspond to a *single* 3×3 matrix in the SO(3) language. The correspondence therefore is two-to-one; for a given R, the corresponding U is double valued. One can say, however, that the two groups are *locally* isomorphic.

3.3.3 Euler Rotations

From classical mechanics the reader may be familiar with the fact that an arbitrary rotation of a rigid body can be accomplished in three steps, known as **Euler rotations**. The Euler rotation language, specified by three Euler angles, provides yet another way to characterize the most general rotation in three dimensions.

The three steps of Euler rotations are as follows. First, rotate the rigid body counter-clockwise (as seen from the positive z-side) about the z-axis by angle α. Imagine now that there is a body y-axis embedded, so to speak, in the rigid body such that before the z-axis rotation is carried out, the body y-axis coincides with the usual y-axis, referred to as the **space-fixed y-axis**. Obviously, after the rotation about the z-axis, the body y-axis no longer coincides with the space-fixed y-axis; let us call the former the y'-axis. To see how all this may appear for a thin disk, refer to Figure 3.4a. We now perform a second rotation, this time about the y'-axis by angle β. As a result, the body z-axis no longer points in the space-fixed z-axis direction. We call the body-fixed z-axis after the second rotation the z'-axis; see Figure 3.4b. The third and final rotation is about the z'-axis by angle γ. The body y-axis now becomes the y''-axis of Figure 3.4c. In terms of 3×3 orthogonal matrices the product of the three operations can be written as

$$R(\alpha,\beta,\gamma) \equiv R_{z'}(\gamma)R_{y'}(\beta)R_z(\alpha). \tag{3.85}$$

A cautionary remark is in order here. Most textbooks in classical mechanics prefer to perform the second rotation (the middle rotation) about the body x-axis rather than about the body y-axis (see, for example, Goldstein et al. (2002)). This convention is to be avoided in quantum mechanics for a reason that will become apparent in a moment.

In (3.85) there appear $R_{y'}$ and $R_{z'}$, which are matrices for rotations about body axes. This approach to Euler rotations is rather inconvenient in quantum mechanics because we earlier obtained simple expressions for the space-fixed (unprimed) axis components of the **S** operator, but not for the body-axis components. It is therefore desirable to express the body-axis rotations we considered in terms of space-fixed axis rotations. Fortunately there is a very simple relation, namely,

$$R_{y'}(\beta) = R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha). \tag{3.86}$$

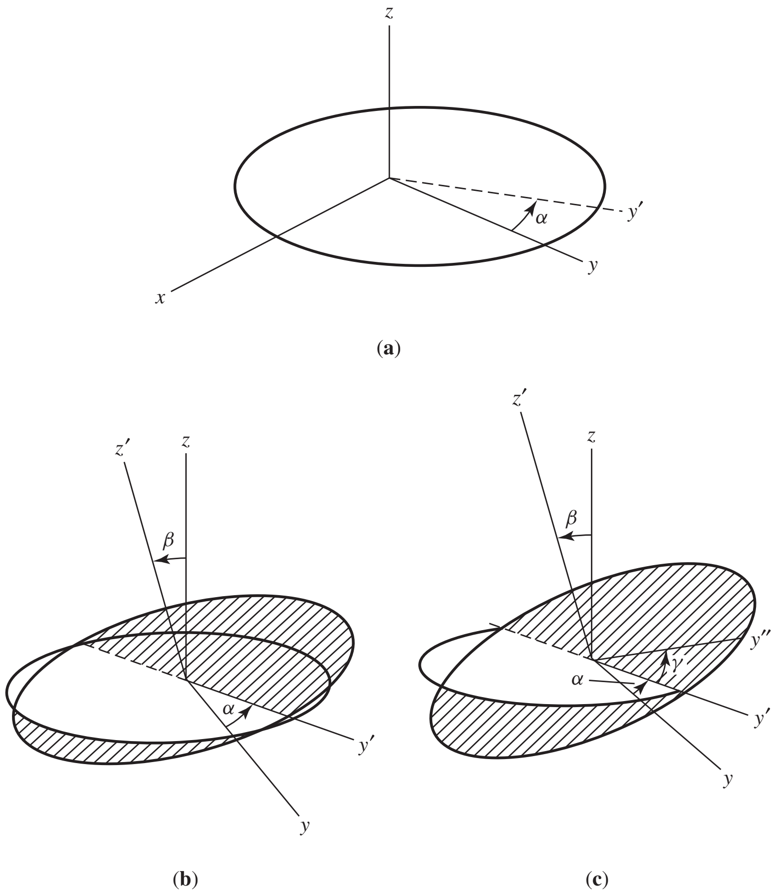

Fig. 3.4 Euler rotations.

The meaning of the right-hand side is as follows. First, bring the body y-axis of Figure 3.4a (that is, the y'-axis) back to the original fixed-space y-direction by rotating *clockwise* (as seen from the positive z-side) about the z-axis by angle α then rotate about the y-axis by angle β. Finally, return the body y-axis to the direction of the y'-axis by rotating about the fixed-space z-axis (*not* about the z'-axis!) by angle α. Equation (3.86) tells us that the net effect of these rotations is a single rotation about the y'-axis by angle β.

To prove this assertion, let us look more closely at the effect of both sides of (3.86) on the circular disk of Figure 3.4a. Clearly, the orientation of the body y-axis is unchanged in both cases, namely, in the y'-direction. Furthermore, the orientation of the final body z-axis is the same whether we apply $R_{y'}(\beta)$ or $R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha)$. In both cases the final body z-axis makes a polar angle β with the fixed z-axis (the same as the initial z-axis), and its azimuthal angle, as measured in the fixed-coordinate system, is just α. In other words, the final body z-axis is the same as the z'-axis of Figure 3.4b. Similarly, we can prove

$$R_{z'}(\gamma) = R_{y'}(\beta)R_z(\gamma)R_{y'}^{-1}(\beta). \tag{3.87}$$

Using (3.86) and (3.87), we can now rewrite (3.85). We obtain

$$\begin{aligned} R_{z'}(\gamma)R_{y'}(\beta)R_z(\alpha) &= R_{y'}(\beta)R_z(\gamma)R_{y'}^{-1}(\beta)R_{y'}(\beta)R_z(\alpha) \\ &= R_z(\alpha)R_y(\beta)R_z^{-1}(\alpha)R_z(\gamma)R_z(\alpha) \\ &= R_z(\alpha)R_y(\beta)R_z(\gamma), \end{aligned} \tag{3.88}$$

where in the final step we used the fact that $R_z(\gamma)$ and $R_z(\alpha)$ commute. To summarize,

$$R(\alpha,\beta,\gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma), \tag{3.89}$$

where all three matrices on the right-hand side refer to *fixed*-axis rotations.

Now let us apply this set of operations to spin $\frac{1}{2}$ systems in quantum mechanics. Corresponding to the product of orthogonal matrices in (3.89) there exists a product of rotation operators in the ket space of the spin $\frac{1}{2}$ system under consideration:

$$\mathscr{D}(\alpha,\beta,\gamma) = \mathscr{D}_z(\alpha)\mathscr{D}_y(\beta)\mathscr{D}_z(\gamma). \tag{3.90}$$

The 2×2 matrix representation of this product is

$$\begin{aligned} &\exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) \\ &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}\sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}\sin(\beta/2) & e^{i(\alpha+\gamma)/2}\cos(\beta/2) \end{pmatrix}, \end{aligned} \tag{3.91}$$

where (3.62) was used. This matrix is clearly of the unitary unimodular form. Conversely, the most general 2×2 unitary unimodular matrix can be written in this Euler angle form.

Notice that the matrix elements of the second (middle) rotation $\exp(-i\sigma_y\phi/2)$ are purely real. This would not have been the case had we chosen to rotate about the x-axis rather than the y-axis, as done in most textbooks in classical mechanics. In quantum mechanics it pays to stick to our convention because we prefer the matrix elements of the second rotation, which is the only rotation matrix containing off-diagonal elements, to be purely real.[4]

The 2×2 matrix in (3.91) is called the $j = \frac{1}{2}$ irreducible representation of the rotation operator $\mathscr{D}(\alpha,\beta,\gamma)$. Its matrix elements are denoted by $\mathscr{D}^{(1/2)}_{m'm}(\alpha,\beta,\gamma)$. In terms of the angular-momentum operators we have

$$\mathscr{D}^{(1/2)}_{m'm}(\alpha,\beta,\gamma) = \left\langle j=\frac{1}{2},m'\left|\exp\left(\frac{-iJ_z\alpha}{\hbar}\right)\times\exp\left(\frac{-iJ_y\beta}{\hbar}\right)\exp\left(\frac{-iJ_z\gamma}{\hbar}\right)\right|j=\frac{1}{2},m\right\rangle. \tag{3.92}$$

In Section 3.5 we will extensively study higher j-analogues of (3.91).

[4] This, of course, depends on our convention that the matrix elements of S_y (or, more generally, J_y) are taken to be purely imaginary.