Elliptical wavelets in the level-set method

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Abstract

This document provides some formulas for wildfire simulations algorithms. Specifically, the context is that the level-set method is used to track the fire front, and that the spread rate normal to the fire front is obtained from a model of elliptical wavelets. Emphasis is put on capturing the effect of slope correctly, which a surprisingly error-prone problem. This document does not concern itself with how the elliptical dimensions are computed; we assume they are given as input.

Table 1: Main symbols used in this document.

Symbol	Units	Meaning
S	m	Spatial location (point).
t	s	Instant in time.
$\phi(S,t)$		Front-tracking field ($\phi(S,t) = 0$ is the fire front at t).
$\mathcal{H}$		Horizontal plane.
$\mathcal{S}$		Slope-tangential plane.
$\vec{U}$	m/s	Spread rate vector.
$U_\perp$	m/s	Front-normal spread rate.
$\vec{V}_H$	m/s	Heading spread rate vector. $\vec{V}_H \in \mathcal{S}$.
V_H	m/s	Heading spread rate.
V_B	m/s	Backing spread rate.
V_F	m/s	Flanking spread rate.
L/W		Length-to-Width ratio of the elliptical wavelet.
α	rad	Aspect angle.
γ	rad	Slope angle (incline).
ω	rad	Angle between $\vec{V}_H$ and the front-normal direction.

1 Context: the level-set method for wildfire simulation

1.1 The level-set PDE

The level-set method is an approach to tracking the fire front in wildfire simulations. See for example the ELMFIRE documentation. The approach consists of describing the fire front at time t as the set of spatial points S such that $\phi(S, t) = 0$, in which ϕ is a smooth real-valued function. Consequently, ϕ follows the **level-set Partial Differential Equation (PDE):**

$$\frac{\partial \phi}{\partial t} = -\nabla_S \phi \cdot \vec{U} \tag{1}$$

in which:

1. $\nabla_S \phi$ is the spatial gradient of ϕ;
2. $\vec{U}(S, t)$ is the[1] **spread rate vector**, which quantifies the speed at which each infinitesimal segment of the fire front advances.

Typically, $\vec{U}(\cdot, t)$ will be computed as a function of $\phi(\cdot, t)$ and of the landscape's properties; it is then a matter of using a numerical solver to integrate equation 1 over time.

1.1.1 Side note: a Euclidean-free formulation

Equation 1 involves a gradient and a dot-product, i.e. a euclidean structure. This euclidean structure is arbitrary (and can be misleading, as we will see when we consider slope). A simplifying insight is to rewrite the level-set PDE without any euclidean structure, such that it now relates a derivative along time to a directional derivative along the spread rate vector:

$$\frac{\partial \phi}{\partial t} = -\frac{d}{d\tau} \left\{ \tau \mapsto \phi(S + \tau \vec{U}, t) \right\}_{\tau = 0} \tag{2}$$

Notes:

1. The above equation won't be used in the rest of this document. We only present it here to clarify that the level-set method is not inherently tied to a particular euclidean structure.
2. The right-hand side is derived simply by applying generic properties of the gradient - nothing specific to the level-set method.

1.2 Elliptical wavelets

Elliptical wavelets are one way to model the spread rate vector $\vec{U}$. The underlying principle is that the fire propagates around any point S as an ellipse: if S is on the fire front at time t, then at time $t + dt$ you can picture an ellipse around S, of dimensions proportional to dt. The ellipse around a given point is defined by 3 quantities:

1. the *heading spread rate vector* $\vec{V}_H$, describing the direction and speed that a heading fire would have at point S;

[1] well, more rigorously, $\vec{U}$ is *a* spread rate vector field: $\vec{U}$ is defined up to adding a front-tangential field. Typically, $\vec{U}$ will be chosen so as to be orthogonal *along the slope* to the fire front.

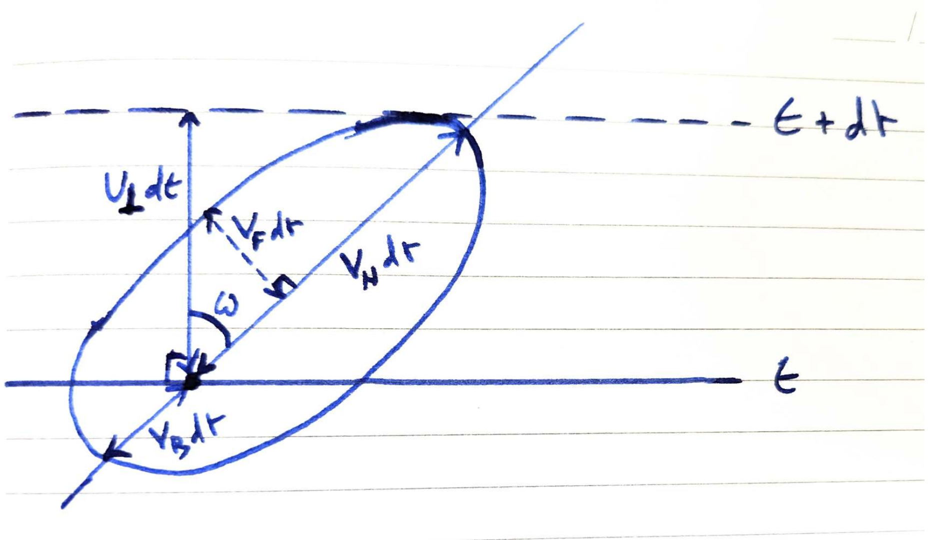

Figure 1: Anatomy of an elliptical wavelet. dt is an infinitesimal time increment. The horizontal lines are the fire fronts at time t (solid) and $t + dt$ (dashed).

2. the *backing spread rate* V_B (a scalar), describing the speed at which a backing fire would have at point S;

3. the Length-to-Width ratio L/W, describing the "thinness" of the ellipse; L/W can be related to the *flanking spread rate* V_F as:

$$V_F = \frac{V_H + V_B}{2(L/W)} \tag{3}$$

The burned region at time $t + dt$ is the union of all these infinitesimal ellipses with the burned region at time t. A non-trivial consequence of this model is that the spread rate orthogonal to the fire front, which we denote $U_\perp$, can be expressed as follows:

$$U_\perp(\omega) = \frac{V_H - V_B}{2}\cos(\omega) + V_F\sqrt{1 + \left(\left(\frac{L}{W}\right)^2 - 1\right)\cos(\omega)^2} \tag{4}$$

in which ω is the angle between the heading spread direction (given by $\vec{V}_H$) and the outward front-normal direction.

I derived equation 4 from scratch[2], but it is also confirmed by the literature:

1. In (Richards, Gwynfor D, 1995), it corresponds to applying equation (14) with substitutions $R := U_\perp$, $\phi - \theta = \omega$, $a = \frac{V_H + V_B}{2}$, $b = V_F$, and $c = \frac{V_H - V_B}{2}$.

[2]Sketch of proof: (a) the ellipse is tangential to the $t + dt$ front at some point T; (b) compute the vector $\vec{ST}$ as a linear combination of $\vec{V}_H$ and $\vec{V}_F$, then (c) apply $U_\perp = |\vec{ST}|\cos(\omega)$. For step (b), I found it helpful to adopt a dot-product operator that turns the ellipse into a circle, e.g. one that makes $(dt\vec{V}_H, dt\vec{V}_F)$ an orthonormal basis.

2. In (Catchpole, EA and Mestre, NJ de and Gill, AM, 1982), it corresponds to applying equation 7 with substitutions $R := U_\perp$, $\phi = \omega$, $g = (V_H - V_B)/2R_\circ$, $f = (V_H + V_B)/2R_\circ$ and $h = V_F/R_\circ$.

As a sanity check, it is readily verified that this formula yields the expected result for heading fires ($\omega = 0$), backing fires ($\omega = -\pi$) and flanking fires ($\omega = \pm\pi/2$). At this point you should feed very confident that equation 4 is correct.

From basic geometry, note that $\vec{U} \cdot \nabla_S \phi = U_\perp |\nabla_S \phi|$ and that $\cos\omega = \frac{\vec{V}_H \cdot \nabla_S \phi}{V_H |\nabla_S \phi|}$. Applying these observations to equation 4 and rearranging enables us to express $\nabla_S \phi \cdot \vec{U}$:

$$\begin{aligned}\nabla_S \phi \cdot \vec{U} =& \frac{V_H - V_B}{2V_H}\left(\vec{V}_H \cdot \nabla_S \phi\right) \\ &+ \frac{V_F}{V_H}\sqrt{(V_H|\nabla_S \phi|)^2 + \left(\left(\frac{L}{W}\right)^2 - 1\right)\left(\vec{V}_H \cdot \nabla_S \phi\right)^2}\end{aligned} \tag{5}$$

Combining the above with equation 1 enables us to express $\frac{\partial \phi}{\partial t}$ as a function of $\nabla_S \phi$ and the $\vec{V}_H$, V_B and L/W fields, which are typically predicted by physical models like Rothermel's.

2 Accounting for slope

2.1 How not to do it: vertical projection of the x,y axes

Accounting for slope is deceptively tricky:

1. On the one hand, we tend to picture fire as spreading on horizontal map with x (East-West) and y (North-South) axes. It is very tempting (but misguided) to reason in terms of distances and angles in that plane.

2. On the other hand, the notions of angles and distances based on which our elliptical wavelets are modeled are those of the 3-dimensional topography, i.e. those of the slope-tangential plane.

Our visual intuition often deceives us into thinking that we can simply use vertical projections to convert (x, y) coordinates between the horizontal plane and the slope-tangential plane. This approach is in fact not viable, because it distorts angles and distances in ways that we have difficulty perceiving. For example:

1. The X and Y directions are orthogonal in the horizontal plane but *not* orthogonal once projected to the slope-tangential plane. In plain English, when you walk eastward along a slope and run into someone walking northward along the same slope, your paths will not cross at right angles[3].

[3]That is unless the slope is exactly aligned with one of the X or Y axes.

2. Projecting an elliptical wavelet still yields an ellipse; however, the projection will fail to preserve the heading spread rate vector $\vec{V}_H$. More explicitly: the heading vector of the projected ellipse is not the projection of the heading vector of the original ellipse.

3. Projecting a front-tangential vector will yield a front-tangential vector; however, projecting a front-normal vector will not yield a front-normal vector.

For this reason, we recommend to completely avoid using concepts like the "horizontal spread rate" $\sqrt{U_x^2 + U_y^2}$ or the horizontal "front-normal unit vector" $\vec{n} := ((\partial\phi/\partial x)^2 + (\partial\phi/\partial y)^2)^{-\frac{1}{2}}((\partial\phi/\partial x)\vec{u}_x + (\partial\phi/\partial y)\vec{u}_y)$: these quantities are based on notions of angles and distances which make essentially no physical sense and will prove useless at best.

2.2 How to do it: converting to a slope-tangential coordinates system

We now consider computing our level-set formulas at a point S. We assume that the slope at S is characterized by a slope angle γ and an aspect angle α. We denote $\mathcal{S}$ the slope-tangential plane at S, and $\mathcal{H}$ the horizontal plane. We also consider 3 unit vectors:

1. $\vec{u}_{\mathcal{S}} \in \mathcal{S}$ (u stands for *upslope*) is a unit vector[4] pointing upslope in the slope-tangential plane $\mathcal{S}$;

2. Similarly, $\vec{u}_{\mathcal{H}} \in \mathcal{H}$ is a unit vector in the horizontal plane, pointing towards the upslope direction.

3. $\vec{v} \in \mathcal{S} \cap \mathcal{H}$ is a unit vector orthogonal to the above, i.e. pointing along the elevation isopleth.

$\vec{u}_{\mathcal{S}}$ and $\vec{v}$ define a (u, v) coordinate system in the slope-tangential plane $\mathcal{S}$. We will thus conceive of a “tangential” front-tracking field $\phi_t(u, v)$ defined over (u, v) coordinates:

$$\phi_t(u, v) := \phi(S + u\cos(\gamma)\vec{u}_{\mathcal{H}} + v\vec{v}, t) \tag{6}$$

The heading spread rate vector $\vec{V}_H$ is then expressed with V_u and V_v components:

$$\vec{V}_H =: V_u\vec{u}_{\mathcal{S}} + V_v\vec{v} \tag{7}$$

2.3 Slope-aware fire spread formulas

ϕ_t satisfies the level-set equation (equation 1) for some spread rate vector $\vec{U} \in \mathcal{S}$, and the above definitions enable us to concretely compute $U_\perp$ and $\frac{\partial\phi}{\partial t}$, through the following formulas:

[4] "Unit vector" means that it has unit length: $|\vec{u}_{\mathcal{S}}| = 1$.

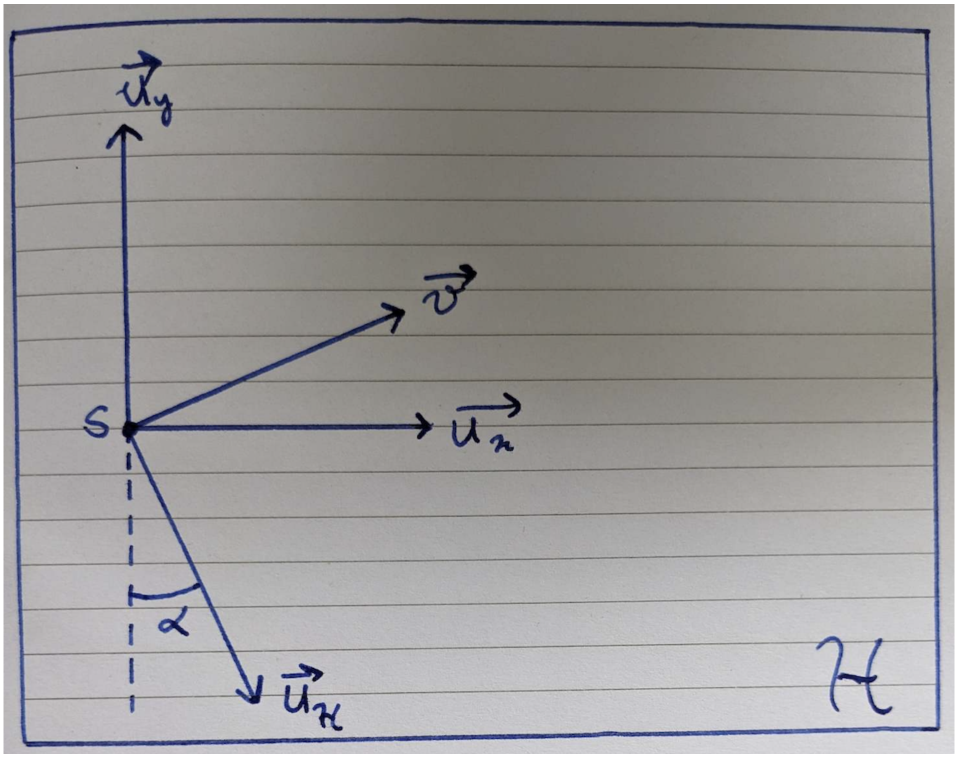

Figure 2: The horizontal plane $\mathcal{S}$, which contains the x (West-East) and y (South-North) axes. α is the aspect angle.

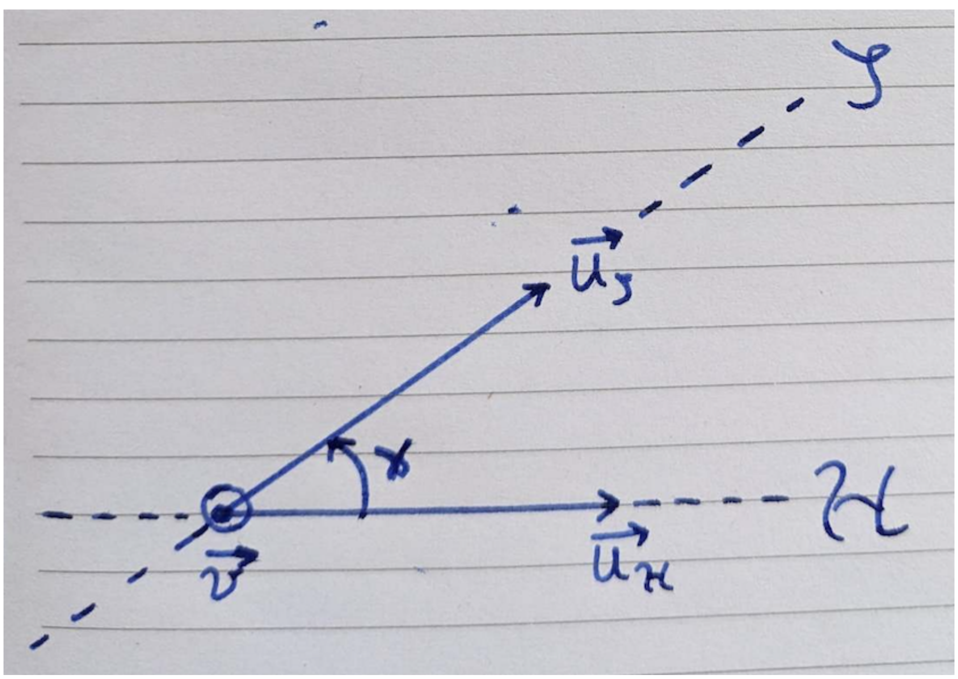

Figure 3: The vertical plane containing the upslope direction. γ is the slope angle.

$$\frac{\partial \phi_t}{\partial u} = \cos(\gamma)\left((\vec{u}_{\mathcal{H}} \cdot \vec{u}_x)\frac{\partial \phi}{\partial x} + (\vec{u}_{\mathcal{H}} \cdot \vec{u}_y)\frac{\partial \phi}{\partial y}\right) \tag{8}$$

$$\frac{\partial \phi_t}{\partial v} = (\vec{v} \cdot \vec{u}_x)\frac{\partial \phi}{\partial x} + (\vec{v} \cdot \vec{u}_y)\frac{\partial \phi}{\partial y} \tag{9}$$

$$|\nabla \phi_t| = \sqrt{\left(\frac{\partial \phi_t}{\partial u}\right)^2 + \left(\frac{\partial \phi_t}{\partial v}\right)^2} \tag{10}$$

$$\Delta := \vec{V}_H \cdot \nabla \phi_t = V_u \frac{\partial \phi_t}{\partial u} + V_v \frac{\partial \phi_t}{\partial v} \tag{11}$$

$$V_H := |\vec{V}_H| = \sqrt{V_u^2 + V_v^2} \tag{12}$$

$$V_F = \frac{V_H + V_B}{2(L/W)} \tag{13}$$

$$-\frac{\partial \phi}{\partial t} = \frac{V_H - V_B}{2V_H}\Delta + \frac{V_F}{V_H}\sqrt{(V_H|\nabla \phi_t|)^2 + \left((L/W)^2 - 1\right)\Delta^2} \tag{14}$$

$$U_\perp = \frac{-1}{|\nabla \phi_t|}\frac{\partial \phi}{\partial t} \tag{15}$$

Notes:

1. In the above, $(\vec{u}_{\mathcal{H}} \cdot \vec{u}_x)$ and $(\vec{u}_{\mathcal{H}} \cdot \vec{u}_y)$ will be computed from the aspect angle α. Typically, $(\vec{u}_{\mathcal{H}} \cdot \vec{u}_y) = -\cos(\alpha)$ and $(\vec{u}_{\mathcal{H}} \cdot \vec{u}_x) = -\sin(\alpha)$. Likewise for $\vec{v}$. The exact formulas depend on choices made in defining the unit vectors.

2. The input quantities of these formulas are: α, γ, V_u, V_v, V_B, L/W, $\frac{\partial \phi}{\partial x}$, $\frac{\partial \phi}{\partial y}$. Typically, V_H will already have been computed from Rothermel equations.

3. $U_\perp$ will typically be useful for computing fireline intensity.

4. $\frac{\partial \phi}{\partial t}$ will typically be fed to a PDE solver to predict the evolution of the fire front.

3 Conclusion

Section 2.3 provides the mathematical formulas required by spread algorithms based on the level-set method: expressions for $\frac{\partial \phi}{\partial t}$ and for the front-normal spread rate $U_\perp$.

Unfortunately, slope makes the elliptical equations less straightforward. AFAICT we have to give up on the intuitive notion of expressing our vectors of interest in terms of (x, y) coordinates, as that's a recipe for error (see section 2.1). The recommended approach is to express the relevant vectors and angles in the slope-tangential plane.

4 Appendix: euclidean geometry refresher

If $\vec{u}_x$ and $\vec{u}_y$ are unit vectors in the x and y directions, they are orthogonal, and the x, y coordinates of any vector $\vec{a}$ can be expressed with dot products $a_x = \vec{u}_x \cdot \vec{a}$ and

$a_y = \vec{u}_y \cdot \vec{a}$. $\vec{a}$ can then be decomposed as $\vec{a} = a_x\vec{u}_x + a_y\vec{u}_y$. $|\vec{a}|$ denotes the *norm* (a.k.a. *length*, a.k.a. *magnitude*) of $\vec{a}$, and can be expressed as $|\vec{a}| = \sqrt{a_x^2 + a_y^2}$.

The dot product between $\vec{a}$ and $\vec{b}$ can be computed from Cartesian coordinates as $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y$, and also using the angle θ between $\vec{a}$ and $\vec{b}$:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta) \tag{16}$$

The above also applies in 3 or more dimensions.

Angles often get people confused and trigonometry is error-prone, so we recommend thinking in terms of vectors and dot-products whenever possible. That said, we will recall the following trigonometric identities:

$$\begin{aligned}
\cos^2(\theta) + \sin^2(\theta) &= 1 \\
\cos(\theta) &= \cos(-\theta) \\
\sin(\theta) &= -\sin(-\theta) \\
\cos(\theta + \frac{\pi}{2}) &= -\sin(\theta) \\
\sin(\theta + \frac{\pi}{2}) &= \cos(\theta) \\
e^{i\theta} &= \cos(\theta) + i\sin(\theta) \\
e^{i\frac{\pi}{2}} &= i \\
e^{i\pi} &= -1 \\
e^{i(\theta_1+\theta_2)} &= e^{i\theta_1} e^{i\theta_2}
\end{aligned}$$

The angles in these formulas are expressed in radians (rad). The conversion between radians and degrees is 2π rad $= 360°$.

5 References

Catchpole, EA and Mestre, NJ de and Gill, AM (1982). *Intensity of fire at its perimeter.*.

Richards, Gwynfor D (1995). *A general mathematical framework for modeling two-dimensional wildland fire spread*, CSIRO Publishing.