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DYNAMIC CHOICES OF HYPERBOLIC CONSUMERS

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Laboratory and field studies of time preference find that discount rates are much greater in the short-run than in the long-run. Hyperbolic discount functions capture this property. This paper solves the decision problem of a hyperbolic consumer who faces stochastic income and a borrowing constraint. The paper uses the bounded variation calculus to derive the Hyperbolic Euler Relation, a natural generalization of the standard Exponential Euler Relation. The Hyperbolic Euler Relation implies that consumers act as if they have endogenous rates of time preference that rise and fall with the future marginal propensity to consume (e.g., discount rates that endogenously range from 5% to 41% for the example discussed in the paper).

KEYWORDS: Hyperbolic discounting, time preference, dynamic inconsistency, consumption, savings, buffer stock, Euler Relation, dynamic games, bounded-variation calculus, altruistic growth.

1. INTRODUCTION

LABORATORY AND FIELD STUDIES of time preference find that discount rates are much greater in the short-run than in the long-run.[2] To model this phenomena, psychologists have adopted discount functions from the class of generalized hyperbolas,[3] and economists have used the discrete-time quasi-hyperbolic discount function: $1, \beta\delta, \beta\delta^2, \ldots, \beta\delta^t, \ldots$.[4] When $\beta < 1$, this quasi-hyperbolic function captures the qualitative property that discount rates decline (weakly) with horizon length. The short-run discount rate, $-\ln(\beta\delta)$, is greater than the long-run discount rate, $-\ln(\delta)$.

Our paper shows how these preferences affect the savings decisions of a consumer who faces stochastic income and a borrowing constraint. We adopt the standard incomplete markets assumptions of the buffer stock consumption

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[2] See Ainslie (1992) and Loewenstein and Thaler (1989).

[3] Events τ periods away are discounted with factor $(1+\alpha\tau)^{-(\gamma/\alpha)}$, with $\alpha, \gamma > 0$. See Ainslie (1992) and Loewenstein and Prelec (1992).

[4] This function was first used by Phelps and Pollak (1968) and Zeckhauser and Fels (1968) to discount intergenerational utility flows. Laibson (1997a) adopted the structure to approximate hyperbolic discounting of intra-personal utility flows. Applications include Barro (1999), Laibson, Repetto, and Tobacman (1998, 1999), and O'Donoghue and Rabin (1999). Akerlof (1991) used a similar model in which $\delta = 1$. Strotz (1956) was the first economist to analyze nonexponential discount functions. Thaler and Shefrin (1981) present a model of self-control in which an internal "planner" and "doer" interact, representing the conflict between long- and short-run interests.

literature.[5] We deviate from the standard buffer stock model only by assuming that the discount function is qualitatively hyperbolic.

In the standard exponential discounting model (i.e., $\beta = 1$ above) the equilibrium path satisfies the well known Exponential Euler Relation

$$U'(C(x_t)) \geq E_t R\, \delta U'(C(x_{t+1})),$$

where U is an instantaneous utility function, C is the consumption function, x is cash-on-hand, R is the interest rate, and δ is the exponential discount factor. We show that this relationship has a natural generalization in the hyperbolic economy. This generalization is easy to interpret when the consumption function is Lipschitz continuous (a property that holds in a neighborhood of $\beta = 1$). In this case,

$$U'(C(x_t)) \geq E_t R[C'(x_{t+1})\beta\delta + (1 - C'(x_{t+1}))\delta]U'(C(x_{t+1})).$$

We call this the (Strong) Hyperbolic Euler Relation. The difference between the Exponential Euler Relation and the Hyperbolic Euler Relation is that the latter replaces the constant exponential discount factor, δ, by the bracketed term above,

$$[C'(x_{t+1})\beta\delta + (1 - C'(x_{t+1}))\delta].$$

We call this the effective discount factor; it is a weighted average of the short-run discount factor $\beta\delta$, and the long-run discount factor δ. The respective weights are $C'(x_{t+1})$, the marginal propensity to consume out of liquid wealth x, and $(1 - C'(x_{t+1}))$. The effective discount factor is stochastic and endogenous to the model.

Since $\beta < 1$, the effective discount factor is negatively related to the future marginal propensity to consume (MPC). To gain intuition for this effect, consider a consumer at time 0 who is thinking about saving a marginal dollar for the future. We assume that this consumer acts strategically in an *intra*personal game where the players are temporally situated "selves." The consumer at time zero—'self 0'—expects future selves to overconsume relative to the consumption rate that self 0 prefers those future selves to implement. Hence, on the equilibrium path, self 0 values marginal saving more than marginal consumption at any future time period. From self 0's perspective, therefore, it matters how a marginal unit of wealth at time period 1 will be divided between savings and consumption by self 1. Self 1's MPC determines this division. Since self 0 values marginal saving more than marginal consumption at time period 1, self 0 values the future less the higher the expected MPC at time period 1.[6]

[5] For example, see Deaton (1991) and Carroll (1992, 1997).

[6] Impatience and the MPC are not linked in the exponential model if impatience is defined as

$$\frac{U'(C(x_{t+1}))}{E_t RU'(C(x_{t+1}))}.$$

If patience were instead defined as $E_t[1 + \rho\Delta \ln c_{t+1} - r]$ (where ρ is the coefficient of relative risk aversion), then the exponential model would predict that patience covaries *positively* with the MPC; in the hyperbolic world this positive covariation is mitigated.

The effective discount factor in the Hyperbolic Euler Relation varies significantly over time. Consumers who expect to have low levels of future cash-on-hand will expect $C'(x_{t+1})$ to be close to one,[7] implying that the effective discount factor will approximately equal $\beta\delta$. Assuming that periods are annual with a standard calibration of $\beta = .7$ and $\delta = .95$, the effective discount rate would be $-\ln(.7 \times .95) = .41$.[8] By contrast, consumers with high levels of future cash-on-hand will expect $C'(x_{t+1})$ to be close to zero,[9] implying that the effective discount factor will approximately equal δ. In this case, the effective discount rate will be $-\ln(.95) = .05$.

The Hyperbolic Euler Relation explains numerous consumption anomalies. For example, the model's implied variation in effective impatience predicts the observed pattern of widespread credit card borrowing among young and middle-aged households while also being able to match the high level of observed preretirement wealth accumulation. Standard exponential models do not explain these joint phenomena. In addition, the Hyperbolic Euler Relation explains comovement between income and consumption, missing precautionary savings effects, pro-savings incentives like 401(k)'s, disproportionately low holdings of liquid assets, and the anomalous drop in consumption around retirement.[10]

The current paper derives the Hyperbolic Euler Relation in two complementary ways. First, we present a heuristic derivation, which relies on an ad hoc assumption of smoothness of the consumption function. This assumption is not valid since hyperbolic consumption functions may be downward discontinuous.

Second, we present the first rigorous derivation of the Hyperbolic Euler Relation and provide a general characterization of the hyperbolic problem. This analysis does not rely on smoothness restrictions. Instead, we apply the bounded-variation calculus, a subfield of analysis that has been little-used in economics. This general framework enables us to prove three sets of results. First, we show that pure-strategy, stationary, Markov equilibria exist in hyperbolic problems with unbounded wealth and a class of unbounded instantaneous utility functions.[11] Second, we show that the Strong Hyperbolic Euler Relation

[7] Low levels of cash-on-hand imply that the agent is liquidity constrained. Hence, low levels of cash-on-hand imply a high MPC. See Harris and Laibson (2000) for simulated hyperbolic consumption functions in buffer stock models.

[8] These parameter values generate simulated wealth and debt profiles that match observed data. See Laibson (1997a) and Laibson, Repetto, and Tobacman (2000) for a discussion of calibration issues.

[9] When the agent is not liquidity constrained, marginal consumption is approximately equal to the annuity value of marginal increments of wealth. Hence, the local slope of the consumption function is close to the real interest rate.

[10] See Laibson (1997b), Laibson, Repetto, and Tobacman (1998, 2000), and Angeletos, Laibson, Repetto, Tobacman, and Weinberg (2000).

[11] Our approach differs from that of Bernheim and Ray (1989), who assume bounded instantaneous utility and wealth. Our results can be adapted to admit their assumptions on intertemporal separability and nonadditive production. Our approach yields higher-order smoothness for the value and policy functions and crucially our approach works in the multidimensional case. Our approach also differs from Harris (1990), who proves the existence of mixed-strategy equilibria.

derived heuristically at the beginning of the paper has a natural generalization that arises when the consumption function is not Lipschitz continuous; we call this generalization the Weak Hyperbolic Euler Relation. Third, we show that the Strong Hyperbolic Euler Relation is satisfied when the hyperbolic model is parameterized in a neighborhood of the exponential model.[12] In such neighborhoods the equilibrium consumption function is Lipschitz continuous.[13]

The rest of the paper demonstrates these claims. Section 2 describes our basic model. Section 3 presents a heuristic derivation of the Hyperbolic Euler Relation. Section 4 discusses regularity assumptions for our formal derivations. Section 5 defines functions of locally bounded variation. Section 6 characterizes the relationship between equilibrium value functions and equilibrium Markov policy functions. Section 7 summarizes the technical results needed for the formal derivation of the Weak Hyperbolic Euler Relation. Section 8 derives the Weak Hyperbolic Euler Relation. Section 9 summarizes the technical results needed for the formal derivation of the Strong Hyperbolic Euler Relation. Section 10 describes conditions under which the Weak Hyperbolic Euler Relation reduces to the Strong Hyperbolic Euler Relation.

2. MODEL

Our modeling assumptions divide naturally into four parts: the standard assumptions from the buffer-stock literature; the assumptions that make our model qualitatively hyperbolic; our equilibrium concept; and the technical assumptions that allow us to derive the Hyperbolic Euler Relation. We discuss the first three sets of assumptions in this section. Discussion of the technical assumptions is deferred to Section 4 below.

2.1. *Buffer-Stock Assumptions*

During period t, the consumer has cash-on-hand $x_t \geq 0$. She chooses a consumption level $c_t \in [0, x_t]$, which rules out borrowing. Whatever the consumer does not spend is saved, $s_t = x_t - c_t \in [0, x_t]$. The gross return on her savings is fixed, $R \geq 0$, and next period she receives labor income $y_{t+1} \geq 0$. Cash-on-hand during period $t+1$ is therefore $x_{t+1} = R(x_t - c_t) + y_{t+1}$. Labor income is independently and identically distributed over time with density f.

[12] Such neighborhoods are empirically relevant: simulations with CRRA of 3, $\delta = .95$, $.7 < \beta < 1$, and $R = 1.04$, yield (Lipschitz continuous) concave monotonic consumption functions. Harris and Laibson (1999) present sufficient conditions for concavity and monotonicity.

[13] Numerous papers have evaluated the smoothness of policy and value functions in deterministic dynamic optimization problems: e.g., Araujo (1991), Benveniste and Scheinkman (1979), Clarke et al. (1998), and Montrucchio (1987). Our approach differs because our model is a stochastic game. Our work is most closely related to that of Blume, Easley, and O'Hara (1982) who link the stochastic properties of their model to smoothness of the policy and value functions. Like the other papers cited above, they analyze a class of dynamic optimization problems and not a dynamic game.

The consumer cannot sell her uncertain stream of future labor income payments, because of moral hazard and adverse selection, or because of prohibitions against indenturing. In other words, there is no asset market for labor.

2.2. *Hyperbolic Preferences*

We model an individual as a sequence of autonomous temporal selves. These selves are indexed by the respective periods, $t = 0, 1, 2, \ldots,$ in which they control the consumption choice. Self t receives payoff

$$E_t\left[U(c_t) + \beta \sum_{i=1}^{\infty} \delta^i U(c_{t+i})\right], \tag{1}$$

where $\beta \geq 0$, $\delta \geq 0$, and $U: [0, +\infty) \to [-\infty, +\infty)$.

2.3. *Equilibrium*

We analyze the set of perfect equilibria in stationary Markov strategies of the intrapersonal game with players (or selves) indexed by the nonnegative integers. Because income is i.i.d., the only state variable is cash-on-hand x_t. We therefore restrict attention to consumption strategies C that depend only on x_t.

3. HEURISTIC DERIVATION OF THE STRONG HYPERBOLIC EULER RELATION

Suppose that C is an equilibrium consumption function. Adopt the perspective of self t. Since all future selves use the consumption function C, and since self t uses the same discount factor δ from period $t+1$ onwards, her continuation-value function V solves the recursive equation

$$V(x_{t+1}) = U(C(x_{t+1})) + E_{t+1}\, \delta V(R(x_{t+1} - C(x_{t+1})) + y_{t+2}). \tag{2}$$

Note that $V(x_{t+1})$ is the expectation, conditional on x_{t+1}, of the present discounted value of the utility stream which starts in period $t+1$.

Self t uses discount factor $\beta\delta$ at time t. Her current-value function W therefore solves the equation

$$W(x_t) = U(C(x_t)) + E_t\, \beta\, \delta V(R(x_t - C(x_t)) + y_{t+1}). \tag{3}$$

Moreover

$$C(x_t) \in \operatorname*{argmax}_{c \in [0, x_t]} U(c) + E_t\, \beta\, \delta V(R(x_t - c) + y_{t+1}), \tag{4}$$

since consumption is chosen by the current self.

The first-order condition associated with (4) implies that

$$U'(C(x_t)) \geq E_t R \beta\, \delta V'(R(x_t - C(x_t)) + y_{t+1}), \tag{5}$$

with equality if $C(x_t) < x_t$. The first-order condition and envelope theorem together imply that the shadow value of cash-on-hand equals the marginal utility of consumption:

$$W'(x_t) = U'(C(x_t)). \tag{6}$$

Finally, note that V and W are linked by the equation

$$\beta V(x_{t+1}) = W(x_{t+1}) - (1-\beta)U(C(x_{t+1})). \tag{7}$$

These expressions can be combined to yield the Strong Hyperbolic Euler Relation. Indeed, we have

$$U'(C(x_t)) \geq E_t R \beta \delta V'(R(x_t - C(x_t)) + y_{t+1})$$

(this is just the first-order condition (5))

$$= E_t R \delta [W'(x_{t+1}) - (1-\beta)U'(C(x_{t+1}))C'(x_{t+1})]$$

(by differentiation and substitution of Equation (7))

$$= E_t R \delta [U'(C(x_{t+1})) - (1-\beta)U'(C(x_{t+1}))C'(x_{t+1})]$$

(from (6), since self $t+1$ also equates the shadow value of cash-on-hand and the marginal utility of consumption). Rearranging yields

$$U'(C(x_t)) \geq E_t R[C'(x_{t+1})\beta\delta + (1 - C'(x_{t+1}))\delta]U'(C(x_{t+1})), \tag{8}$$

with equality if $c_t < x_t$. This is the SHER.

4. BASIC ASSUMPTIONS

In the present section we formulate the technical assumptions on which the remainder of our analysis will be based. They run as follows:

U1: U has domain $[0, +\infty)$ and range $[-\infty, +\infty)$.

U2: U is twice continuously differentiable on $(0, +\infty)$.

U3: $U' > 0$ on $(0, +\infty)$.

U4: There exist $0 < \underline{\rho} \leq \bar{\rho} < +\infty$ such that

$$\underline{\rho} \leq \frac{-cU''(c)}{U'(c)} \leq \bar{\rho} \qquad \text{for all} \quad c \in (0, +\infty).$$

F1: f has domain $(0, +\infty)$ and range $[0, +\infty)$.

F2: f is twice continuously differentiable.

F3: There exist $0 < \underline{y} < \bar{y} < +\infty$ such that $f(y) = 0$ for all $y \notin [\underline{y}, \bar{y}]$.

D1: $\operatorname{Max}\{\delta, \delta R^{1-\underline{\rho}}\} < 1$.

D2: $\beta \in [0,1]$.

Assumption U2 is a straightforward smoothness requirement on U, Assumption U3 ensures that the marginal utility of consumption is strictly positive, and Assumption U4 ensures that the coefficient of relative risk aversion is bounded away from 0 and $+\infty$. These assumptions could be summarized by saying that U belongs to the class of utility functions of bounded relative risk aversion, or BRRA for short. They include the possibility that the coefficient of relative risk aversion is constant as a special case. Assumption F2 is a straightforward smoothness requirement on f. Assumption F3 ensures that all income realizations are bounded away from 0 and $+\infty$. Assumption D1 ensures that neither labor income nor investment income can generate unbounded expected discounted utility. Assumption D2 ensures that the current self either acts exponentially ($\beta = 1$) or underweights the future ($\beta \in [0,1)$).

5. FUNCTIONS OF LOCALLY BOUNDED VARIATION

In what follows, we shall make extensive use of the concept of a function of locally bounded variation.

DEFINITION 1: Let J be a subinterval of $\mathbb{R}$. Then the function $g: J \to \mathbb{R}$ is of *locally bounded variation* iff there exist increasing functions $g_+ : J \to \mathbb{R}$ and $g_- : J \to \mathbb{R}$ such that $g = g_+ - g_-$.

If $g: J \to \mathbb{R}$ is a function of locally bounded variation and J is open, then the function g_L defined by the formula $g_L(z) = g(z-)$ is the left-continuous regularization of g; the function g_R defined by the formula $g_R(z) = g(z+)$ is the right-continuous regularization of g; the function $\Delta g = g_R - g_L$ is the jump function of g; the measure g' with distribution function g_R is the first derivative of g; the measure $g'_d = \sum_{z \in \mathbb{R}} \Delta g(z) \delta_z$, where δ_z is the unit mass concentrated at z, is the discontinuous (or atomic) part of g'; and the measure $g'_c = g' - g'_d$ is the continuous (or non-atomic) part of g'.

REMARK 2: Lipschitz continuous functions, and hence continuously differentiable functions, have locally bounded variation.

6. CHARACTERIZATION OF EQUILIBRIUM

In this paper we focus on perfect equilibria in stationary Markov strategies that are bounded below in the sense that a self's continuation payoff is at least what she would get if all future selves were to consume all of their cash on hand.[14] With this in mind, we adopt the following convention.

CONVENTION 3: *We refer henceforth to perfect equilibria in stationary Markov strategies that are bounded below simply as equilibria*.

The equilibria of our model can be characterized in the usual way. Define $r:[0,+\infty)\to[0,+\infty)$ by the formula $r(x)=Rx+\bar{y}$, define $\underline{V}:[0,+\infty)\to[-\infty,+\infty)$ by the formula $\underline{V}(x)=U(x)+\sum_{t=1}^{\infty}\delta^t\int U(y)f(y)\,dy$, and define $\overline{V}:[0,+\infty)\to[-\infty,+\infty)$ by the formula $\overline{V}(x)=U(x)+\sum_{t=1}^{\infty}\delta^t U(r^t(x))$. Then we have the following theorem.

THEOREM 4 (Characterization): *A consumption function C is an equilibrium if and only if there exists a continuation-value function $V:[0,+\infty)\to[-\infty,+\infty)$ and a current-value function $W:[0,+\infty)\to[-\infty,+\infty)$ such that*

$$\underline{V}(x)\le V(x)\le \overline{V}(x),$$

$$V(x)=U(C(x))+\delta\int V(R(x-C(x))+y)f(y)\,dy,$$

$$W(x)=U(C(x))+\beta\delta\int V(R(x-C(x))+y)f(y)\,dy,$$

and

$$C(x)\in \operatorname*{argmax}_{c\in[0,x]} U(c)+\beta\delta\int V(R(x-c)+y)f(y)\,dy$$

for all $x\in[0,+\infty)$.

7. LOCAL REGULARITY OF THE CURRENT-VALUE FUNCTION

The Characterization Theorem (Theorem 4) shows that any equilibrium continuation-value function lies in the interval $[\underline{V},\overline{V}]$. In this section we fix an arbitrary Borel measurable $V\in[\underline{V},\overline{V}]$, and we establish the local regularity properties of the associated function $\mathfrak{W}V:[0,+\infty)\to[-\infty,+\infty)$ given by the formula

$$(\mathfrak{W}V)(x)=\max_{\gamma\in[0,1]}\left\{U(\gamma x)+\beta\delta\int V(R(1-\gamma)x+y)f(y)\,dy\right\}.$$

The section can be omitted on a first reading.

[14] Equilibria that are not bounded below do occur for some parameter values. For example, if $\underline{\rho}\ge 1$, then there exists a stationary equilibrium in which $C\equiv 0$ and $V\equiv -\infty$. For other parameter values, they can be ruled out. For example, if $\bar{\rho}<1$, then all equilibria are bounded below. We view equilibria that are not bounded below as economically pathological.

We begin with four lemmas. Define the function $G:[0,+\infty)\to\mathbb{R}$ by the formula $G(s)=\int V(Rs+y)f(y)\,dy$. Then we have the following lemma:

LEMMA 5: *G is twice continuously differentiable.*

PROOF: Changing variables according to the formula $z=Rs+y$, we obtain $G(s)=\int V(z)f(z-Rs)\,dz$, $G'(s)=-R\int V(z)f'(z-Rs)\,dz$, and $G''(s)=R^2\int V(z)f''(z-Rs)\,dz$. Since $\underline{V}\le V\le\overline{V}$ and $\operatorname{supp} f\subset[\underline{y},\bar{y}]$, we may apply the bounded convergence theorem to conclude that G, G', and G'' are all continuous. *Q.E.D.*

Define the function $w:[0,1]\times[0,+\infty)\to[-\infty,+\infty)$ by the formula $w(\gamma,x)=U(\gamma x)+\beta\delta G((1-\gamma)x)$. Then we have Lemma 6.

LEMMA 6: *w is continuous.*

PROOF: This follows at once from the continuity of U and G. *Q.E.D.*

Define the optimal policy correspondence $\Gamma:[0,+\infty)\to[0,1]$ by the formula $\Gamma(x)=\operatorname{argmax}_{\gamma\in[0,1]}w(\gamma,x)$. Then we have the following lemma.

LEMMA 7: *Γ is nonempty valued, compact valued, and upper semicontinuous.*

PROOF: This follows from the continuity of w and the compactness of $[0,1]$. *Q.E.D.*

LEMMA 8: *For all $x>0$, $\Gamma(x)\subset(0,1]$.*

PROOF: This follows at once from the fact that $\partial w(\gamma,x)/\partial\gamma\to+\infty$ as $\gamma\to0+$. *Q.E.D.*

We are now in a position to prove the main result of this section. Put $W=\mathfrak{W}V$ and, for each $\gamma\in[0,1]$, define a function w_γ by the formula $w\gamma(x)=w(\gamma,x)$. Then we have the following theorem.

THEOREM 9 (Local Regularity): *Both W and W' are functions of locally bounded variation on $(0,+\infty)$. Moreover, for all $x_0>0$ and all $\xi\in(0,x_0)$ there exists $K>0$ such that $|W|\le K$, $|W'|\le K$, and $W''\ge-K$ on $[x_0-\xi,x_0+\xi]$.*

The inequality $|W'|\le K$ means, more explicitly, that the measure W' is nonsingular with respect to Lebesgue measure, and that the absolute value of the density of W' with respect to Lebesgue measure is bounded by K. The inequality $W''\ge-K$ means, more explicitly, that the negative part of the measure W'' is nonsingular with respect to Lebesgue measure, and that its density with respect to Lebesgue measure is bounded by K.

PROOF: Put $J = [x_0 - \xi, x_0 + \xi]$. Then $W|_J = \sup_{\gamma \in \Gamma(J)} w_\gamma|_J$. That is, the restriction of W to the interval J is the upper envelope of the restrictions of the functions $\{\omega_\gamma | \gamma \in \Gamma(\mathrm{J})\}$ to the interval J. Now, using the formulae for G, G', and G'' given in the proof of Lemma 5, it is easy to show that there exists $K > 0$ such that $|w_\gamma(x)|, |w'_\gamma(x)|, |w''_\gamma(x)| \leq K$ for all $\gamma \in \Gamma(J)$ and all $x \in J$. The inequality $|w_\gamma(x)| \leq K$ for all $\gamma \in \Gamma(J)$ implies at once that $|W| \leq K$ on J. Next, if $x_1, x_2 \in J$ and $\gamma_i \in \Gamma(x_i)$ for $i \in \{1,2\}$, then we have $W(x_2) = w_{\gamma 2}(x_2) \geq w_{\gamma 1}(x_2)$ and $W(x_1) = w_{\gamma 1}(x_1) \geq w_{\gamma 2}(x_1)$. Hence

$$W(x_2) - W(x_1) \leq w_{\gamma 2}(x_2) - w_{\gamma 2}(x_1) \leq K(x_2 - x_1)$$

and

$$W(x_2) - W(x_1) \geq w_{\gamma 1}(x_2) - w_{\gamma 1}(x_1) \geq K(x_2 - x_1).$$

That is, W is Lipschitz continuous with coefficient K on J. Hence W is of locally bounded variation on J, and $|W'| \leq K$ on J. Finally, define $q : \mathbb{R} \to \mathbb{R}$ by the formula $q(x) = (K/2)x^2$. Then $(w_\gamma + q)|_J$ is convex for all $\gamma \in \Gamma(J)$. Hence $(W + q)|_J = \sup_{\gamma \in \Gamma(J)} (w_\gamma + q)|_J$ is convex. Hence $(W + q)'$ is of locally bounded variation on J, and $W'' = (W + q)'' - q'' \geq 0 - K = -K$. *Q.E.D.*

Theorem 9 has the following corollaries, the proofs of which follow standard lines and are omitted.

COROLLARY 10: *The right and left derivatives $D_R W$ and $D_L W$ of W are well defined on* $(0, +\infty)$, *and we have $D_R W \geq D_L W$.*

In particular, W can have upward (or convex) kinks but it cannot have downward (or concave) kinks.

COROLLARY 11 (Envelope Principle): *For all $x > 0$:*
(i) *for all $\gamma \in \Gamma(x)$, $w'_\gamma(x) \in [D_L W(x), D_R W(x)]$;*
(ii) *there exists $\gamma_L \in \Gamma(x)$ such that $w'_{\gamma_L}(x) = D_L W(x)$;*
(iii) *there exists $\gamma_R \in \Gamma(x)$ such that $w'_{\gamma_R}(x) = D_R W(x)$.*

COROLLARY 12 (Shadow-Price Principle): *For all $x > 0$:*
(i) *for all $\gamma \in \Gamma(x)$, $U'(\gamma x) \in [D_L W(x), D_R W(x)]$;*
(ii) *there exists $\gamma_L \in \Gamma(x)$ such that $U'(\gamma_L x) = D_L W(x)$;*
(iii) *there exists $\gamma_R \in \Gamma(x)$ such that $U'(\gamma_R x) = D_R W(x)$.*

8. THE WEAK HYPERBOLIC EULER RELATION

We are now in a position to establish the WHER. The proof will be deferred to the end of the section.

THEOREM 13 (WHER): *Let C be an equilibrium. Then C is of locally bounded variation on* $(0, +\infty)$, *and we have the WHER*:

$$\begin{aligned}(U' \circ C)(x) \geq R\delta \int (U' \circ C)(R(x - C(x)) + y) f(y)\, dy \\ - R\delta(1-\beta) \int (U' \circ C)(R(x - C(x)) + y) \\ \times f(y) C'_c(R(x - C(x)) + dy) \\ - R\delta(1-\beta) \int \frac{\Delta(U \circ C)}{\Delta C}(R(x - C(x)) + y) \\ \times f(y) C'_d(R(x - C(x)) + dy),\end{aligned}$$

with equality if $C(x) < x$.

The right-hand side of the WHER involves three terms. We refer to these terms as the exponential term, the continuous hyperbolic term, and the discontinuous hyperbolic term. We also refer to the sum of the continuous and discontinuous hyperbolic terms as the hyperbolic term.

In order to build up some intuition for the WHER, we consider three corollaries.

COROLLARY 14 (EER): *If* $\beta = 1$, *then we have the EER*:

$$(U' \circ C)(x) \geq R\delta \int (U' \circ C)(R(x - C(x)) + y) f(y)\, dy,$$

with equality if $C(x) < x$.

In this relation, $(U' \circ C)(x)$ is the marginal utility of consumption today, and $(U' \circ C)(R(x - C(x)) + y)$ is the marginal utility of consumption tomorrow. The relation therefore states that the marginal utility of consumption today is at least $R\delta$ times the expectation of the marginal utility of consumption tomorrow.

COROLLARY 15 (SHER): *Suppose that there exists* $\overline{X} \in (0, +\infty)$ *such that*:
(i) $R(x - C(x)) + y \in [\underline{y}, \overline{X}]$ *for all* $x \in [\underline{y}, \overline{X}]$ *and all* $y \in \operatorname{supp} f$; *and*
(ii) *C is Lipschitz continuous on* $[\underline{y}, \overline{X}]$.
Then, *for all* $x \in [\underline{y}, \overline{X}]$, *we have the SHER*:

$$\begin{aligned}(U' \circ C)(x) \geq R\delta \int ((1 - (1-\beta) C')(U' \circ C)) \\ \times (R(x - C(x)) + y) f(y)\, dy,\end{aligned}$$

with equality if $C(x) < x$.

In other words, when $\beta < 1$, the discount factor δ in the EER must be replaced by the effective discount factor

$$\delta(1-(1-\beta)C'(R(x-C(x))+y)),$$

where $C'(R(x-C(x))+y)$ is the marginal propensity to consume tomorrow.

REMARK 16: The mathematical significance of the first hypothesis of Corollary 15 is that $[\underline{y}, \overline{X}]$ is an absorbing interval: once cash-on-hand falls in this interval, it remains there forever. The mathematical significance of the second hypothesis is that there exists $L > 0$ such that $|C(x_2) - C(x_1)| \leq L|x_2 - x_1|$ for all $x_1, x_2 \in [\underline{y}, \overline{X}]$. Equivalently: (i) C is of locally bounded variation on a neighborhood of $[\underline{y}, \overline{X}]$; (ii) the measure C' is nonsingular with respect to Lebesgue measure; and (iii) the absolute value of the density of C' with respect to Lebesgue measure is bounded by L.

REMARK 17: The economic significance of the first hypothesis of Corollary 15 is that we only expect to observe levels of cash-on-hand that lie in $[\underline{y}, \overline{X}]$. The economic significance of the second hypothesis is that C' can be interpreted in the usual way: the only complication is that it will in general be discontinuous.

REMARK 18: Theorem 10 below shows that the hypotheses of Corollary 15 are satisfied when β is close to 1.

Finally, let I be the identity mapping on $[0, +\infty)$, and let $M: [0, +\infty) \to [0, +\infty]$ be the generalized marginal utility of consumption given by the formula

$$M(x) = \left\{ \begin{array}{ll} U'(C(x)) & \text{if } x \text{ is a point of continuity of } C \\ \left(\dfrac{\Delta(U \circ C)}{\Delta C} \right)(x) & \text{if } x \text{ is a point of discontinuity of } C \end{array} \right\}.$$

Corollary 19 follows.

COROLLARY 19 (MHER): *We have the MHER*:

$$M(x) \geq R\delta \int M(R(x - C_L(x)) + y)$$
$$\times f(y)(I - (1-\beta)C)'(R(x - C_L(x)) + dy),$$

with equality if x is a point of continuity of C and $C(x) < x$.

Corollary 19 makes three points. First, in the hyperbolic model, it is more natural to formulate the Euler Relation in terms of the generalized marginal utility of consumption than in terms of the marginal utility of consumption. This is because, in the hyperbolic model, there may be jumps in the consumption function, and the current self must take these into account when evaluating the marginal utility of future consumption. Secondly, in order to formulate the Euler Relation in terms of the generalized marginal utility of consumption, it is necessary to work with the left-continuous version of C. In other words, it is necessary to assume that the consumer chooses the highest possible consumption level in cases where she is indifferent among more than one consumption choice. Thirdly, given that the Euler Relation is formulated in terms of the generalized marginal utility of consumption, the only difference between the SHER and the MHER is that the expression

$$(1-(1-\beta)C')(R(x-C(x))+y)\,dy$$

in the SHER is replaced by the expression

$$(I-(1-\beta)C)'(R(x-C_L(x))+dy)$$

in the MHER because C is not necessarily absolutely continuous.

REMARK 20: The consumption function C has at most a countable number of discontinuities. Hence, if the initial value x_0 of cash-on-hand is a point of continuity of C, then the probability that the equilibrium path hits a point of discontinuity of C is zero. Moreover, irrespective of whether x_0 is a point of continuity of C or not, the probability that the equilibrium path hits a point of discontinuity of C at any t greater than zero is zero. The fact that the condition for the MHER to hold as an equality is more restrictive than the condition for the WHER to hold as an equality therefore of very little practical importance.

We turn now to the proof of Theorem 13. This can be omitted at a first reading. We need three lemmas.

LEMMA 21: *Suppose that* $z_1 \in (0,+\infty)$, *and let g and h be two right-continuous functions of locally bounded variation on* $(0,+\infty)$ *such that* $g=h=0$ *on* $(0,z_1)$. *Then*

$$g(z_2)h(z_2)=\int_{[z_1,z_2]} g(z)h'(dz)+\int_{[z_1,z_2]} h_L(z)g'(dz)$$

for all $z_2 \in (z_1,+\infty)$.

PROOF: This result is a simple adaptation of Theorem VI.90 of Dellacherie and Meyer (1982). *Q.E.D.*

LEMMA 22: *Suppose that* $F:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable, and let g be a function of locally bounded variation on* $(0,+\infty)$. *Then* $F\circ g$ *is a function of locally bounded variation on* $(0,+\infty)$ *and, for all* $0<z_1\le z_2<+\infty$, *we have*

$$(F\circ g)(z_2+)-(F\circ g)(z_1-)$$
$$=\int_{[z_1,z_2]}(F'\circ g)(z)g_c'(dz)+\int_{[z_1,z_2]}\left(\frac{\Delta(F\circ g)}{\Delta g}\right)(z)g_c'(dz).$$

In particular, the measure $(F \circ g)'$ *is absolutely continuous with respect to the measure* g', *and we have*

$$(F \circ g)'(dz) = (F' \circ g)(z) g_c'(dz) + \left(\frac{\Delta(F \circ g)}{\Delta g} \right)(z) g_d'(dz).$$

PROOF: This result is a simple adaptation of formula VI.92.1 of Dellacherie and Meyer (1982). Cf. also formulae VI.93.1 and VI.93.2 of the same work. *Q.E.D.*

LEMMA 23: *Suppose that* C *is an equilibrium. Then* C *is of locally bounded variation on* $(0, +\infty)$.

PROOF: Corollary 12 implies that $D_L W \leq U' \circ C \leq D_R W$ in $(0, +\infty)$. Theorem 9 implies that $D_R W$ is of locally bounded variation in $(0, +\infty)$, and that $D_L W = (D_R W)_L$. So $U' \circ C$ is of locally bounded variation in $(0, +\infty)$. Lemma 22 therefore implies that $C = (U')^{-1} \circ (U' \circ C)$ is of locally bounded variation in $(0, +\infty)$. *Q.E.D.*

PROOF OF THEOREM 13: Fix $x > 0$ and put $\gamma(x) = C(x)/x$. Then

$$\begin{aligned} \frac{\partial w(\gamma(x), x)}{\partial \gamma} &= xU'(\gamma(x)x) + xR\beta\delta \int V(R(1-\gamma(x))x + y) f'(y)\, dy \\ &\geq 0, \end{aligned}$$

with equality if $\gamma(x) < 1$ (by the first-order condition for the maximization of w with respect to γ). So

$$U'(C(x)) \geq -R\beta\delta \int V(R(x - C(x)) + y) f'(y)\, dy,$$

with equality if $C(x) < x$. Now

$$\begin{aligned} &\beta \int V(R(x - C(x)) + y) f'(y)\, dy \\ &\quad = \int (W - (1-\beta) U \circ C)(R(x - C(x)) + y) f'(y)\, dy \end{aligned}$$

(because $V = W - (1-\beta) U \circ C$). Moreover,

$$\begin{aligned} &\int W(R(x - C(x)) + y) f'(y)\, dy \\ &\quad = -\int W'(R(x - C(x)) + y) f(y)\, dy \end{aligned}$$

(using Lemma 21 to integrate by parts)

$$= -\int (U' \circ C)(R(x - C(x)) + y) f(y)\, dy$$

(because $W' = U' \circ C$ except at a countable number of points); and

$$\int (U \circ C)(R(x - C(x)) + y) f'(y)\, dy$$

$$= \int (U \circ C_R)(R(x - C(x)) + y) f'(y)\, dy$$

(because $C = C_R$ except at a countable number of points)

$$= -\int f(y)(U \circ C_R)'(R(x - C(x)) + dy)$$

$$= -\int f(y)(U \circ C)'(R(x - C(x)) + dy)$$

(using Lemma 21 to integrate by parts and because $(U \circ C_R)' = (U \circ C)'$)

$$= -\int f(y)(U' \circ C)(R(x - C(x)) + y) C_c'(R(x - C(x)) + dy)$$

$$-\int f(y)\left(\frac{\Delta(U \circ C)}{\Delta C}\right)(R(x - C(x)) + y) C_d'(R(x - C(x)) + dy)$$

(using Lemma 22). This completes the proof of the WHER in the case $x > 0$. Finally, the WHER is trivial when $x = 0$, for in that case $U'(C(x)) = U'(0) = +\infty$. *Q.E.D.*

9. GLOBAL REGULARITY

In this section we prove a sharper version of the Local-Regularity Theorem (Theorem 9). This result is needed for the proof of the SHER. The section can be omitted on a first reading.

Put $\underline{V}^- = -(\underline{V} \wedge 0)$ and $\overline{V}^+ = +(\overline{V} \vee 0)$, define $N_1 : [0, +\infty) \to [0, +\infty)$ by the formula $N_1(x) = \underline{V}^-(\underline{y}) \vee \overline{V}^+(Rx - \bar{y})$, and define $N_2 : [0, +\infty) \to [0, +\infty)$ by the formula $N_2(x) = (U'(x)/x) \vee N_1(x)$. Then we have the following theorem.

THEOREM 24 (Global Regularity): *There exist $K > 0$ such that, for all $V \in [\underline{V}, \overline{V}]$,*
(i) $(1 - \beta)U + \beta \underline{V} \le \mathfrak{W}V \le (1 - \beta)U + \beta \overline{V}$,
(ii) $U' \le (\mathfrak{W}V)' \le U' \vee (KN_1)$, *and*
(iii) $(\mathfrak{W}V)'' \ge -KN_2$ *on* $(0, +\infty)$.

PROOF: Put $W = \mathfrak{W}V$. We have

$$W(x) \leq \max_{\gamma \in [0,1]} \left\{ U(\gamma x) + \beta\delta \int \overline{V}(R(1-\gamma)x + y) f(y)\, dy \right\}$$
$$\leq (1-\beta) \max_{\gamma \in [0,1]} \{U(\gamma x)\}$$
$$+ \beta \max_{\gamma \in [0,1]} \left\{ U(\gamma x) + \delta \int \overline{V}(R(1-\gamma)x + y) f(y)\, dy \right\}$$
$$\leq (1-\beta)U(x) + \beta \overline{V}(x)$$

and

$$W(x) \geq \max_{\gamma \in [0,1]} \left\{ U(\gamma x) + \beta\delta \int \underline{V}(R(1-\gamma)x + y) f(y)\, dy \right\}$$
$$\geq U(x) + \beta\delta \int \underline{V}(y) f(y)\, dy$$
$$= (1-\beta)U(x) + \beta \left(U(x) + \delta \int \underline{V}(y) f(y)\, dy \right)$$
$$\geq (1-\beta)U(x) + \beta \underline{V}(x).$$

This completes the proof of part 1.

Next, Corollary 12 tells us that, for all $x > 0$, there exist $\gamma_L, \gamma_R \in \Gamma(x)$ such that

$$D_H W(x) = w'_{\gamma H}(x) = \gamma_H U'(\gamma_H x) + (1-\gamma_H)\beta\delta G'((1-\gamma_H)x)$$

for $h \in \{L, R\}$. Moreover the first-order condition tells us that

$$\left\{ \begin{matrix} xU'(\gamma_H x) - x\beta\delta G'((1-\gamma_H)x) \geq 0 \\ \gamma_H \leq 1 \end{matrix} \right\}$$

with at least one equality. Hence $D_H W(x) = U'(x) \vee \beta\delta G'((1-\gamma_H)x)$. Now

$$|G'((1-\gamma_H)x)| = \left| -R \int V(R(1-\gamma_H)x - y) f'(y)\, dy \right|$$
$$\leq R \int |V|(R(1-\gamma_H)x + y) |f'|(y)\, dy$$
$$\leq R \int N_1(x) |f'|(y)\, dy \leq R \|f'\|_1 N_1(x).$$

Hence there exists $K_1 > 0$ such that

$$U'(x) \vee (-K_1 N_1(x)) \leq D_L W(x) \leq D_R W(x) \leq U'(x) \vee (+K_1 N_1(x)).$$

Noting that $U'(x) \vee (-K_1 N_1(x)) = U'(x)$, we obtain part 2.

Finally, for all $x > 0$ and all $\gamma \in \Gamma(x)$, we have

$$w''_{\gamma}(x) = \gamma^2 U''(\gamma x) + (1-\gamma)^2 \beta\,\delta G''((1-\gamma)x).$$

Moreover

$$\begin{aligned}\gamma^2 U''(\gamma x) &\geq -\frac{\gamma\bar{\rho}}{x} U'(\gamma x) \geq -\frac{\gamma\bar{\rho}}{x} D_R W(x) \\ &\geq -\frac{\gamma\bar{\rho}}{x}(U'(x) \vee (+K_1 N_1(x)))\end{aligned}$$

(by U4, Corollary 12 and part 2) and

$$\begin{aligned}|G''((1-\gamma)x)| &= \left| -R^2 \int V(R(1-\gamma_H)x + y) f''(y)\, dy \right| \\ &\leq R^2 \|f''\|_1 N_1(x).\end{aligned}$$

Hence

$$w''_{\gamma}(x) \geq -\frac{\bar{\rho}}{x}(U'(x) \vee (+K_1 N_1(x))) - R^2 \|f''\|_1 N_1(x).$$

Now $N_1(x)$ dominates $U'(x)/x$ and $N_1(x)/x$ for large x, and $U'(x)/x$ dominates $N_1(x)$ and $N_1(x)/x$ for small x. Hence there exists $K_2 > 0$ such that $w''_{\gamma}(x) \geq -K_2 N_2(x)$. Noting that $N_2(x)$ is continuous in x and independent of γ, we obtain part 3. *Q.E.D.*

Theorem 24 has three corollaries. For all Borel measurable $V \in [\underline{V}, \overline{V}]$, put $\mathfrak{C}V = (U')^{-1} \circ D_R \mathfrak{W}V$ and $\mathfrak{V}V = \mathfrak{W}V - (1-\beta)U \circ \mathfrak{C}V$. Denote the bounds on W' and W'' given in Theorem 24 by $\underline{W}_1$, $\overline{W}_1$, and $\underline{W}_2$. Let I denote the identity mapping on $(0, +\infty)$. Then the first corollary is as follows.

COROLLARY 25: *There is a continuous function* $\overline{C}_1 : (0, +\infty) \to (0, +\infty)$ *such that, for all Borel measurable* $V \in [\underline{V}, \overline{V}]$, *we have* $0 \leq \mathfrak{C}V \leq I$ *and* $(\mathfrak{C}V)' \leq \overline{C}_1$.

PROOF: The inequality $0 \leq \mathfrak{C}V \leq I$ is immediate from the fact that $+\infty > (\mathfrak{W}V)' \geq U'$. As for the inequality $(\mathfrak{C}V)' \leq \overline{C}_1$, put $A = (U')^{-1}$. Let $W_1 : (0, +\infty) \to (0, +\infty)$ be any function such that: (i) $W_1(x) = (\mathfrak{W}V)'(x)$ if x is a point of continuity of $(\mathfrak{W}V)'$; and (ii) $W_1(x) \in [(D_L \mathfrak{W}V)(x), (D_R \mathfrak{W}V)(x)]$ and

$$A'(W_1(x)) = \left(\frac{\Delta(A \circ (\mathfrak{W}V)')}{\Delta((\mathfrak{W}V)')} \right)(x) \qquad \text{otherwise}.$$

Put $\underline{A} = \min\{A'(\lambda) \mid \lambda \in [\underline{W}_1, \overline{W}_1]\}$. Then

$$\begin{aligned}(\mathfrak{C}V)' &= (A' \circ (\mathfrak{W}V)')(\mathfrak{W}V)''_c + \frac{\Delta(A \circ (\mathfrak{W}V)')}{\Delta((\mathfrak{W}V)')}(\mathfrak{W}V)''_d \\ &= (A' \circ W_1)(\mathfrak{W}V)'' \leq (A' \circ W_1)\underline{W}_2 \leq \underline{A}\,\underline{W}_2\end{aligned}$$

(by Lemma 22, by definition of W_1, because $A' < 0$ and by definition of $\underline{A}$ respectively). *Q.E.D.*

The second corollary is as follows.

COROLLARY 26: *There is a continuous function* $\underline{V}_1 : (0, +\infty) \to \mathbb{R}$ *such that, for all Borel measurable* $V \in [\underline{V}, \overline{V}]$, *we have* $\underline{V} \leq \mathfrak{V}V \leq \overline{V}$ *and* $(\mathfrak{V}V)' \geq \underline{V}_1$.

PROOF: We have

$$\begin{aligned}(\mathfrak{V}V)(x) &= (\mathfrak{W}V)(x) + (1-\beta)\delta \int V(R(x - (\mathfrak{C}V)(x)) + y) f(y)\, dy \\ &\geq (1-\beta)U(x) + \beta \underline{V}(x) + (1-\beta)\delta \\ &\quad \times \int \underline{V}(R(x - (\mathfrak{C}V)(x)) + y) f(y)\, dy = \underline{V}(x).\end{aligned}$$

Similarly, $(\mathfrak{V}V)(x) \leq \overline{V}(x)$. As for the inequality $(\mathfrak{V}V)' \geq \underline{V}_1$, let $C_0 : (0, +\infty) \to (0, +\infty)$ be any function such that: (i) $C_0(x) = (\mathfrak{C}V)(x)$ if x is a point of continuity of $\mathfrak{C}V$; and (ii) $C_0(x) \in [C_R(x), C_L(x)]$ and

$$U'(C_0(x)) = \left(\frac{\Delta(U \circ \mathfrak{C}V)}{\Delta(\mathfrak{C}V)}\right)(x) \qquad \text{otherwise}.$$

Then

$$\begin{aligned}(\mathfrak{V}V)' &= (\mathfrak{W}V)' - (1-\beta)\left((U' \circ \mathfrak{C}V)(\mathfrak{C}V)'_c + \frac{\Delta(U \circ \mathfrak{C}V)}{\Delta(\mathfrak{C}V)}(\mathfrak{C}V)'_d\right) \\ &= (\mathfrak{W}V)' - (1-\beta)(U' \circ C_0)(\mathfrak{C}V)' \\ &\geq \underline{W}_1 - (1-\beta)(U' \circ C_0)\overline{C}_1 \\ &\geq \underline{W}_1 - (1-\beta)\overline{W}_1\overline{C}_1\end{aligned}$$

(by Lemma 22, by definition of C_0, because $U' > 0$ and because $U' \circ C_0 \leq D_R \mathfrak{W}V$ respectively). *Q.E.D.*

The third corollary is the following.

COROLLARY 27: *There exists* $\ell_0 \in (0, +\infty)$ *such that, for all Borel measurable* $V \in [\underline{V}, \overline{V}]$, *we have* $\mathfrak{C}V = I$ *on* $[0, \ell_0]$.

PROOF: Put $\ell_0 = \min\{x \mid KN_1(x) \geq U'(x)\}$. *Q.E.D.*

REMARK 28: Using a different representation of $\mathfrak{W}V$ as the upper envelope of smooth functions, it can be shown that $(\mathfrak{W}V)'' \geq U'' \circ \mathfrak{C}V$. This estimate does not provide an a priori lower bound for $(\mathfrak{W}V)''$ because it involves the endogenous quantity $\mathfrak{C}V$. It does however yield sharper versions of Corollaries 25 and 26, namely $(\mathfrak{C}V)' \leq 1$ and $(\mathfrak{V}V)' \geq (\mathfrak{W}V)'$. In particular, for all V there exists $l \in (0, +\infty)$ such that $\mathfrak{C}V = I$ on $[0, \ell)$ and $\mathfrak{C}V < I$ on $(\ell, +\infty)$.

We are now in a position to establish existence and continuous dependence on the parameter β. Define two functions of locally bounded variation on $(0, +\infty)$ to be equivalent if and only if they are equal at all points of continuity, and let $\mathscr{BV}^0_{\text{loc}}$ denote the set of equivalence classes of functions of locally bounded variation on $(0, +\infty)$. Let $\mathscr{BV}^1_{\text{loc}}$ denote the set of functions that, together with their first derivative, are in $\mathscr{BV}^0_{\text{loc}}$. Let S_V be the set of $V \in \mathscr{BV}^0_{\text{loc}}$ such that $\underline{V} \leq V \leq \overline{V}$ and $V' \geq \underline{V}_1$. View $\mathfrak{W}$ as a mapping from S_V to $\mathscr{BV}^1_{\text{loc}}$, and view $\mathfrak{C}$ and $\mathfrak{V}$ as mappings from S_V to $\mathscr{BV}^0_{\text{loc}}$. Let $\mathscr{BV}^0_{\text{loc}}$ be endowed with the topology generated by the sets of the form

$$\left\{ g \left| \int_{x_1}^{x_2} |g - g_0|(x)\, dx < \varepsilon, \left| \int_0^{+\infty} h(x)(g - g_0)'(dx) \right| < \varepsilon \right. \right\}$$

for some $g_0 \in \mathscr{BV}^0_{\text{loc}}$, $0 < x_1 < x_2 < +\infty$, $\varepsilon > 0$ and continuous $h : (0, +\infty) \to \mathbb{R}$ of compact support. Let $\mathscr{BV}^1_{\text{loc}}$ be endowed with the topology generated by the sets of the form

$$\{ g \,|\, g \in N(g_0),\, g' \in N(g_0') \}$$

for some $g_0 \in \mathscr{BV}^1_{\text{loc}}$, some open neighborhood $N(g_0)$ of g_0 in $\mathscr{BV}^0_{\text{loc}}$, and some open neighborhood $N(g_0')$ of g_0' in $\mathscr{BV}^0_{\text{loc}}$. Then we have the following theorem.

THEOREM 29: *For all $\beta \in [0,1]$, let $\mathscr{E}(\beta)$ denote the set of triples $(W, V, C) \in \mathscr{BV}^1_{\text{loc}} \times \mathscr{BV}^0_{\text{loc}} \times \mathscr{BV}^0_{\text{loc}}$ such that W is the current-value function, V is the continuation-value function, and C is the consumption function of an equilibrium. Then $\mathscr{E}$ is nonempty valued, compact valued, and upper semicontinuous.*

PROOF: The set S_V is a nonempty compact convex subset of $\mathscr{BV}^0_{\text{loc}}$. Moreover the a priori estimates contained in Theorem 9 and Corollaries 25 and 26 imply that $\mathfrak{W}$, $\mathfrak{C}$, and $\mathfrak{V}$ are continuous, and that $\mathfrak{V}$ is a self-map of S_V. We may therefore apply Tychonov's fixed-point theorem to conclude that $\mathscr{E}$ is nonempty valued, and standard arguments to conclude that $\mathscr{E}$ is compact valued and upper semicontinuous. *Q.E.D.*

10. THE STRONG HYPERBOLIC EULER RELATION REVISITED

In this section we show that, if $\delta R < 1$ and β is sufficiently close to 1, then there is an absorbing interval on which all equilibria C are Lipschitz continuous. In other words, the hypotheses of Corollary 15 are satisfied.

THEOREM 30: *Suppose that* $\delta R < 1$. *Then there exists* $\overline{\beta} \in [0,1)$ *and* $\overline{X} \in (0,+\infty)$ *such that, for all* $\beta \in [\overline{\beta}, 1]$ *and all equilibria* C:
(i) $R(x - C(x)) + y \in [\underline{y}, \overline{X}]$ *for all* $x \in [\underline{y}, \overline{X}]$ *and all* $y \in \operatorname{supp} f$; *and*
(ii) C *is Lipschitz continuous on* $[\underline{y}, \overline{X}]$.

PROOF: Note first that, if $\beta = 1$, then there is a unique equilibrium (W_1, V_1, C_1). It is well known that $W_1'' \leq 0$ and $C_1' \in [0,1]$. Moreover, the methods of Schechtman and Escudero (1977) show that there exist $\lambda \in (0,1)$ and $\bar{x} \in (0,+\infty)$ such that $R(x - C_1(x)) + \bar{y} \leq \lambda x$ for all $x \geq \overline{X}$.

Next, since $W_1'' \leq 0$ and $C_1' \in [0,1]$, W_1 and C_1 are both Lipschitz continuous on $[\underline{y}, \overline{X}]$. Hence, since $\mathscr{E}$ is compact valued and upper semicontinuous,

$$\sup\{\|W - W_1\|_\infty \mid (W,V,C) \in \mathscr{E}(\beta)\} \qquad \text{and}$$
$$\sup\{\|C - C_1\|_\infty \mid (W,V,C) \in \mathscr{E}(\beta)\}$$

both converge to 0 as $\beta \to 1$, where $\|\cdot\|_\infty$ denotes the supremum norm on $[\underline{y}, \overline{X}]$. The existence of the absorbing interval then follows from the uniform convergence of C to C_1 on $[\underline{y}, \overline{X}]$.

Finally, for all $x \in [\underline{y}, \overline{X}]$ and all $\gamma \in [0,1]$, we have

$$\begin{aligned}
&\int (W - (1-\beta) U \circ C)(R(1-\gamma)x + y) f''(y)\, dy \\
&\quad = \int (W_1 + (W - W_1) - (1-\beta) U \circ C)(R(1-\gamma)x + y) f''(y)\, dy \\
&\quad = \int W_1''(R(1-\gamma)x + y) f(y)\, dy \\
&\qquad + \int ((W - W_1) - (1-\beta)(U \circ C))(R(1-\gamma)x + y) f''(y)\, dy.
\end{aligned}$$

Hence, since $W_1'' \leq 0$ and since W and C converge uniformly to W_1 and C_1 on $[\underline{y}, \overline{X}]$,

$$\limsup_{\beta \to 1} \sup_{x \in [\underline{y}, \overline{X}],\, \gamma \in [0,1]} \int (W - (1-\beta) U \circ C)(R(1-\gamma)x + y) f''(y)\, dy \leq 0.$$

We conclude that there exists $\overline{\beta}_2 \in [0,1)$ such that, for all $\beta \in [\overline{\beta}_2, 1]$ and all $(W,V,C) \in \mathscr{E}(\beta)$,

$$\sup_{x \in [\underline{y}, \overline{X}],\, \gamma \in [0,1]} U''(\gamma x) + R^2 \delta \int (W - (1-\beta) U \circ C)(R(1-\gamma)x + y) f''(y)\, dy < 0.$$

In other words, the second derivative of the current self's objective function is bounded away from 0. It follows at once that: the current self possesses a unique best response for all $x \in [\underline{y}, \overline{X}]$; this best response can be characterized by the first-order condition; and, applying the Lipschitz Implicit Function Theorem[15] to the first-order condition, the consumption function is Lipschitz continuous on $[\underline{y}, \overline{X}]$. *Q.E.D.*

REMARK 31: Theorem 30 has been proven despite the fact that the liquidity constraint binds over some range of X.

REMARK 32: There are other cases in which the conclusions of Theorem 30 hold. For example, they hold if $\beta\delta$ is close to 0.

REMARK 33: It does not appear to be possible to strengthen the second conclusion of Theorem 30. Numerical simulations suggest that C is rarely, if ever, Lipschitz continuous on the whole of $[0, +\infty)$.

REMARK 34: The method of continuity used in the proof of Theorem 30 can be applied to obtain additional properties of C. For example: if U and f are both three times continuously differentiable, and if β is sufficiently close to 1, then $C' > 0$ on $[\underline{y}, \overline{X}]$; and if U and f are both four times continuously differentiable, if there exists $\theta > 1$ such that $U'U'''/U''U'' \equiv \theta$, and if β is sufficiently close to 1, then $C'' \leq 0$ on $[\underline{y}, \overline{X}]$.[16]

11. CONCLUSION

This paper analyzes the intrapersonal game that arises when a consumer with a quasi-hyperbolic discount function faces liquidity constraints and uncertain income. When the consumption function is Lipschitz continuous—a property that holds in a neighborhood of $\beta = 1$—the Hyperbolic Euler Relation reduces to

$$U'(C(x_t)) \geq E_t R[C'(x_{i+1})\beta\delta + (1 - C'(x_{t+1}))\delta]U'(C(x_{t+1})).$$

In general, the hyperbolic Euler Relation is characterized by an endogenous effective discount factor, which varies with the marginal propensity to consume.

The results in this paper generalize. First, our results apply to the finite-horizon case. More interestingly, our arguments apply when there is more than one state

[15] Cf., Clarke et al. (1998, Theorem 3.1.9, p. 108).

[16] The consumption function C will in general have a downward kink at the point ℓ at which the liquidity constraint ceases to bind. So C'' is in general a measure.

variable: e.g., multiple assets including illiquid assets and/or state variables that capture endogenous preferences like habit formation.

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