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Computing Real Logarithm of a Real Matrix

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Abstract

In this paper we will be interested in characterizing and computing for a nonsingular real matrix $A \in R^{n\times n}$ a real matrix $X \in R^{n\times n}$ that satisfies $e^X = A$, that is, a logarithm of A. Firstly, we investigate the conditions under which such logarithm exists, unique, polynomial in A, or belongs to a particular class of matrices. Secondly, real Schur decomposition will be used to compute X.

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1 Introduction

Logarithms of matrices arise in various contexts. For example [1, 3, 8], for a physical system governed by a linear differential equation of the form

$$\frac{dy}{dt} = Xy,$$

where X is n-by-n unknown matrix. From observations of the state vector $y(t)$, if $y(0) = y_\circ$ then we know that

$$y(t) = e^{tX} y_\circ.$$

By taking n observations at $t = 1$ for n initial states consisting of the columns of the identity matrix, we obtain the matrix $A = e^X$. Under certain conditions on A, we can then solve for X, that is $X = \log A$. This raises the question of how to compute a logarithm of a matrix. We show that $S(A)$, the solution set of this matrix equation, is nonempty if and only if A is nonsingular.

In this paper we concerned with the real solvability of the matrix equation $e^X = A$ in case of real matrix A. Not every nonsingular real matrix have a real logarithm as the following example illustrates.

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Example 1 *Let* $A = \mathrm{diag}(1, -1)$*. Then any logarithm of* A *is given by*

$$\begin{aligned} X &= U\mathrm{diag}(\log(1), \log(-1))U^{-1} \\ &= U\mathrm{diag}(2\pi i j_1, i\pi(2j_2+1))U^{-1} \end{aligned}$$

where $j_1, j_2 \in Z$ *and* U *is any nonsingular matrix commuting with* A*. All these logarithms are matrices with noncomplex conjugate eigenvalues. Hence it can not be similar to a real matrix and no real logarithm of* A *can be obtained.*

The existence of a real logarithm of a real matrix is discussed in Section 2. In Section 2, we also characterize such logarithm, that is, set the conditions for which X is polynomial in A, symmetric, positive definite, or orthogonal.

The computation of a real logarithm X of a real matrix A arises in many system identification, one of which is the mathematical modeling of dynamic systems [4]. In Section 3, we propose a technique to compute such X based on the real Schur decomposition.

2 Characterization of a real logarithm

In the following theorem we give a set of conditions on the matrix A that guarantees the existence of a real logarithm of A. We start by a lemma which constructs a real logarithm of a particular 2-by-2 matrices.

Lemma 2 *The 2-by-2 real matrices of the form*

$$A_1 = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \quad \lambda > 0 \text{ and } A_2 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad b \neq 0$$

have real logarithms given by

$$X_1 = \begin{bmatrix} Log\lambda & \pi \\ -\pi & Log\lambda \end{bmatrix} \text{ and } X_2 = \begin{bmatrix} \theta & \mu \\ -\mu & \theta \end{bmatrix},$$

respectively, where $e^{\theta \pm i\mu} = a \pm ib$*.*

Proof. To prove this lemma, it is enough to show that $e^{X_1} = A_1$ and $e^{X_2} = A_2$. For the matrix X_1, there exists a nonsingular matrix V such that

$$\begin{aligned} e^{X_1} &= e^{V\mathrm{diag}(Log\lambda + i\pi, Log\lambda - i\pi)V^{-1}} \\ &= V(-\lambda I)V^{-1} = -\lambda I = A_1. \end{aligned}$$

For X_2, it is clear that X_2 is a normal matrix, and since A_2 and X_2 are diagonalizable and commuting, they are simultaneously diagonalizable [7]. Then

$$\begin{aligned} e^{X_2} &= e^{U\mathrm{diag}(\theta + i\mu, \theta - i\mu)U^{-1}} \\ &= U\mathrm{diag}(a+ib, a-ib)U^{-1} \\ &= A_2. \end{aligned}$$

■

Next we set up a sufficient and necessary condition for the existence of a real logarithm X of a real matrix $A \in R^{n\times n}$. A proof of the following theorem can be found in Culver [5], and Ulig [6], however the proof we provide is different, it is based on the usage of real Jordan canonical form and the previous lemma.

Theorem 3 *Let $A \in R^{n\times n}$ be a real matrix. Then there exists a real logarithm X of A if and only if A is nonsingular and each Jordan block of A belonging to negative eigenvalue occurs an even number of times.*

Proof. Let X be a real logarithm of A, that is, $e^X = A$, By using [7, Th 3.4.5] each complex Jordan block (if exists) of any size occur in the Jordan canonical form of a real matrix in conjugate pairs. Hence we may suppose that the Jordan canonical form of X is

$$J_X = \operatorname{diag}(J_{m_1}(x_1), \dots, J_{m_r}(x_r), B_{2m_{r+1}}, \dots, B_{2m_p}), \tag{1}$$

where $x_1, \dots, x_r$ are real, $x_{r+1}, \dots, x_p$ are complex, and $x_1, x_2, \dots, x_p$ are not necessarily distinct and $B_{2m_s} = \operatorname{diag}(J_{m_s}(x_s), J_{m_s}(\overline{x_s}))$. The Jordan canonical form J_A of $A = e^X$ has the form

$$J_A = \operatorname{diag}(J_{m_1}(e^{x_1}), \dots, J_{m_r}(e^{x_r}), B'_{2m_{r+1}}, \dots, B'_{2m_p}), \tag{2}$$

where $B'_{2m_s} = \operatorname{diag}(J_{m_s}(e^{x_s}), J_{m_s}(e^{\overline{x_s}}))$, and $\overline{x}$ denotes the complex conjugate of x. Clearly, $e^{x_k} \neq 0$ for any $x_k \in C$, then A must be nonsingular. Moreover, $e^{x_k} < 0$ only if $\operatorname{Im}(x_k) \neq 0$, in which case $e^{x_k} = e^{\overline{x_k}}$. Thus negative eigenvalues of A must be associated with Jordan blocks which occur in pairs.

Conversely, let $A \in R^{n\times n}$ satisfy the conditions in the theorem. From the real Jordan canonical form we have

$$A = S\operatorname{diag}(J_{m_1}(\lambda_1), \dots, J_{m_q}(\lambda_q), J_{2m_{q+1}}(\lambda_{q+1}), \dots, J_{2m_p}(\lambda_p))S^{-1}, \tag{3}$$

where S is a real n-by-n nonsingular matrix, $\lambda_1, \dots, \lambda_q$ are positive and $\lambda_{q+1}, \dots, \lambda_p$ are either negative or complex eigenvalues of A that are not necessarily distinct. It is easy to check that the Jordan canonical form of $\log^{(j)} J_{m_k}(\lambda_k)$ is given by

$$\log^{(j)} J_{m_k}(\lambda_k) = \begin{bmatrix} \log^{(j)} \lambda_k & 1/\lambda_k & -1/2\lambda_k^2 & \dots & \frac{(-1)^{m_k-2}}{(m_k-1)\lambda_k^{m_k-1}} \\ 0 & \log^{(j)} \lambda_k & 1/\lambda_k & \dots & \frac{(-1)^{m_k-3}}{(m_k-2)\lambda_k^{m_k-2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/\lambda_k \\ 0 & 0 & 0 & \dots & \log^{(j)} \lambda_k \end{bmatrix} \tag{4}$$

where $\log^{(j)} z$ is a branch of $\log z$ defined by

$$\log^{(j)} z = Logz + 2\pi i j, \quad j = 0, \pm 1, \ldots .$$

As for $k = q+1, \ldots, p$, $J_{2m_k}(\lambda_k)$ has the form

$$J_{2m_k} = \begin{bmatrix} L_k & I & 0 & \ldots & 0 \\ 0 & L_k & I & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I \\ 0 & 0 & 0 & \ldots & L_k \end{bmatrix},$$

where $L_k = \begin{bmatrix} a_k & b_k \\ -b_k & a_k \end{bmatrix}$ corresponding to either complex conjugate eigenvalues $\lambda = a_k + ib_k$ and $\overline{\lambda} = a_k - ib_k$, $b_k \neq 0$, each with multiplicity m_k, or to a pair of negative eigenvalues, that is, $a_k < 0$ and $b_k = 0$. We can use the integration definition to find $\log^{(j)} J_{2m_k}(\lambda_k)$ for a certain branch of $\log z$. Namely,

$$\log^{(j)} J_{2m_k}(\lambda_k) = \frac{1}{2\pi i} \int_{\Gamma} (\log^{(j)} z)(zI - J_{2m_k}(\lambda_k))^{-1} dz, \tag{5}$$

where Γ encloses the eigenvalues $\lambda_k, \overline{\lambda_k}$ of $J_{2m_k}(\lambda_k)$. The inverse $(zI - J_{2m_k}(\lambda_k))^{-1}$ can be shown to take the general form

$$\begin{bmatrix} (zI-L_k)^{-1} & (zI-L_k)^{-2} & (zI-L_k)^{-3} & \ldots & (zI-L_k)^{-m_k} \\ 0 & (zI-L_k)^{-1} & (zI-L_k)^{-2} & \ldots & (zI-L_k)^{-m_k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (zI-L_k)^{-1} \end{bmatrix}. \tag{6}$$

Substituting (6) in (5) and integrating along Γ, we have

$$\log^{(j)} J_{2m_k}(\lambda_k) = \begin{bmatrix} \log^{(j)} L_k & L_k^{-1} & -\frac{1}{2}(L_k^{-1})^2 & \ldots & \frac{(-1)^{m_k-2}(L_k^{-1})^{m_k-1}}{m_k-1} \\ 0 & \log^{(j)} L_k & L_k^{-1} & \ldots & \frac{(-1)^{m_k-3}(L_k^{-1})^{m_k-2}}{m_k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & L_k^{-1} \\ 0 & 0 & 0 & \ldots & \log^{(j)} L_k \end{bmatrix}. \tag{7}$$

where by using the previous lemma, L_k has a real logarithm of the form

$$\log^{(j)} L_k = \begin{bmatrix} \theta_k & \mu_k \\ -\mu_k & \theta_k \end{bmatrix}, \quad \log^{(j)}(a_k \pm ib_k) = \theta_k \pm i\mu_k.$$

Now set

$$\begin{aligned} X &= S\text{diag}(\log^{(j_1)}(J_{m_1}(\lambda_1)), \dots, \log^{(j_q)}(J_{m_q}(\lambda_q)), \log^{(j_{q+1})}(J_{2m_{q+1}}(\lambda_{q+1})), \dots \\ &\quad \dots, \log^{(j_p)}(J_{2m_p}(\lambda_p)))S^{-1} \end{aligned}$$

where for $k = 1, 2, \dots, q$, each $\log^{(j_k)}(J_{m_k}(\lambda_k))$ is defined by equation (4) and $\log^{(j_k)}(J_{2m_k}(\lambda_k))$ is given by (7) for all $k = q + 1, \dots, p$. Clearly if we take the logarithms of the Jordan blocks in these forms with particular choice of j in (4) we can get a real logarithm of A. ∎

Also, a characterization of the uniqueness of the real logarithm in term of the spectrum of A , $\sigma(A)$, is given in the next theorem; Culver [5].

Theorem 4 *Let $A \in R^{n\times n}$. Then there exists a unique real logarithm X of A if and only if A is nonderogatory and all the eigenvalues of A are positive real , that is, if all the eigenvalues of A are positive and no Jordan blocks of A belonging to the same eigenvalue appear more than once.*

Now we deal with the real polynomial solvability of $e^X = A$, that is, the existence of polynomial $p(z)$ such that $X = p(A)$ and $e^X = A$. The following theorem establishes the conditions for real logarithm X of A to be polynomial in A.

Theorem 5 *Let $A \in R^{n\times n}$, be nonsingular matrix with Jordan canonical form*

$$A = S\text{diag}(J_{m_1}(\lambda_1), J_{m_2}(\lambda_2), \dots, J_{m_p}(\lambda_p))S^{-1}$$

then $X \in S(A)$ is polynomial in A if and only if the same value of the scalar logarithm is used for the same eigenvalue of A, that is, if $e^{x_k} = \lambda_k$ for every $k = 1, 2, \dots, p$, then $\lambda_i = \lambda_j$ implies that $x_i = x_j$ for all $1 \le i, j \le p$.

From the previous theorem we conclude that if $A \in R^{n\times n}$ has any negative eigenvalues, no real solution of $e^X = A$ can be polynomial in A.

Suppose that A is a real n-by-n matrix, next we give the additional conditions on A for which its real logarithm is real normal, symmetric, skew symmetric, positive (semi) positive or orthogonal logarithm. We start by the following lemma on which our results are based.

Lemma 6 *Let $A \in R^{n\times n}$ be a nonsingular matrix, and the negative eigenvalues of A, if exist, occur an even number of times, then A has a real normal logarithm if and only if A is normal.*

Proof. Suppose that there exists a real normal logarithm X of a real matrix A. Then there exists a real orthogonal matrix $Q \in R^{n\times n}$ such that

$$X = Q\text{diag}(D_1, \dots, D_s, D_{s+1}, \dots, D_p)Q^T$$

where D_j is 1-by-1 real matrix for all $j = 1, 2, \ldots, s$ and D_j is 2-by-2 real matrix for all $j = s+1, \ldots, p$, each of them have the form

$$D_j = \begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix}$$

corresponding to the complex conjugate eigenvalues $x_j = a_j + ib_j$ and $\overline{x}_j = a_j - ib_j$. Since $e^X = A$, then

$$A = Q\mathrm{diag}(e^{D_1}, \ldots, e^{D_s}, e^{D_{s+1}}, \ldots, e^{D_p})Q^T$$

Clearly, D_j and D_j^T are commuting for all $j = 1, 2, \ldots, p$. It follows that

$$e^{D_j} e^{D_j^T} = e^{D_j + D_j^T}, \quad \text{for all} j = 1, 2, \ldots, p$$

then $AA^T = A^T A$, that is, A is normal.

Conversely, consider that A is normal matrix. Then there exists a real orthogonal matrix $Q \in R^{n\times n}$ such that

$$A = Q\mathrm{diag}(D_1, \ldots, D_s, D_{s+1}, \ldots, D_p)Q^T$$

where D_j is a positive number for $j = 1, \ldots, s$, and for $j = s+1, \ldots p$, D_j is 2-by-2 matrix of the form

$$\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix} \quad or \quad \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

corresponding to a complex conjugate eigenvalues $a_j + ib_j$ and $a_j - ib_j$, or to a pair of negative eigenvalues, that is, $\alpha < 0$. Then by using Lemma 1 we can construct a real logarithm X of A. ■

In the previous lemma, the nonsingular real normal matrix A has a real normal logarithm X with prescribed spectrum $\sigma(X) \subset K$, $K \subset C$ if and only if the scalar equation $e^x = \lambda$ has a solution in K for every $\lambda \in \sigma(A)$.

Theorem 7 *Let $A \in R^{n\times n}$, be nonsingular real matrix, that has a real logarithm, that is, each Jordan block of A belonging to a negative eigenvalue occurs an even number of times. Then*

(*a*) *There exists a symmetric logarithm of A if and only if A is positive definite. This logarithm is unique.*

(*b*) *There exists a skew-symmetric logarithm of A if and only if A is orthogonal.*

(*c*) *There exists a real positive definite logarithm (positive semidefinite) of A if and only if $A - I > 0$ $(A - I \geq 0)$.*

(*d*) *There exists an orthogonal logarithm of* A *if and only if* A *is normal and for every* $\lambda \in \sigma(A)$, *we have* $|\log \lambda| = 1$, *that is, for every* $\lambda = a + ib \in \sigma(A)$, *we have*

$$(\log \sqrt{a^2 + b^2})^2 + (\tan^{-1} \frac{b}{a})^2 = 1$$

Proof. (a) The proof of this assertion follows due to the obvious fact that the scalar equation $e^x = \lambda$ has a unique real solution if and only if $\lambda > 0$. Hence the matrix equation $e^X = A$ has a symmetric solution if and only if A is positive definite.
(b) Similarly, the scalar equation $e^x = \lambda$ has a solution belongs to iR if and only if $|\lambda| = 1$. Then the matrix equation $e^X = A$ has a skew symmetric solution if and only if A is orthogonal.
(c) The logarithm X of A is positive definite (semidefinite) if and only if all the solutions of the scalar equation $e^x = \lambda$ are positive (nonnegative). Namely X is positive definite (semidefinite) if and only if A is a real normal and $\lambda > 1 (\lambda \geq 1)$ for all $\lambda \in \sigma(A)$.
(d) Since all the solutions of the scalar equation $e^x = \lambda$ lie on the unit circle if and only if $|\log \lambda| = 1$, therefor the matrix equation $e^X = A$ has an orthogonal solution if and only if A is real normal and $|\log \lambda| = 1$ for all $\lambda \in \sigma(A)$. ■

3 Computation of a real logarithm X

In this section we study the problem of computing a real logarithm of a real matrix. Our main tool for such computation is the real Schur decomposition of the real matrix A.

Let $A \in R^{n \times n}$ be a nonsingular real matrix with no negative eigenvalues, then there exists an orthogonal matrix $Q \in R^{n \times n}$, such that

$$A = QTQ^T = Q \begin{bmatrix} T_{11} & T_{12} & T_{13} & \dots & T_{1m} \\ 0 & T_{22} & T_{23} & \dots & T_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & T_{mm} \end{bmatrix} Q^T. \tag{8}$$

Here each block T_{ii} is either 1-by-1 or 2-by-2 with complex conjugate eigenvalues λ_i and $\overline{\lambda_i}$, $\lambda_i \neq \overline{\lambda}_i$. Since A and T are similar, we have

$$\log A = Q \log TQ^T,$$

so that $\log A$ is real if and only if $\log T$ is real. Hence we need an algorithm for computing a real logarithm of the upper triangular block matrix T. If we suppose that $F = \log T = (F_{ij})$, we look for those F which are functions of T, and hence F will inherit the upper triangular block structure from T. First we compute

$$F_{ii} = LogT_{ii}, \quad \text{for all } i = 1, 2, \ldots, m.$$

Once the diagonal blocks of F are known, the blocks in the strict upper triangular of F can be derived from the commutativity result $FT = TF$. Indeed by computing (i, j) entries in this equation, we get

$$\sum_{k=i}^{j} F_{ik}T_{kj} = \sum_{k=i}^{j} T_{ik}F_{kj}, \quad j > i$$

and thus, if $\sigma(T_{ii}) \cap \sigma(T_{jj}) = \phi$, $i \neq j$, we obtain an equation with unique solution [2], namely

$$F_{ij}T_{jj} - T_{ii}F_{ij} = T_{ij}F_{jj} - F_{ii}T_{ij} + \sum_{k=i+1}^{j-1} (T_{ik}F_{kj} - F_{ik}T_{kj}), \tag{9}$$

where F_{ij} are computed one superdiagonal at a time. This Sylvester equation results in a linear system of order 1,2 or 4 that can be solved using standard methods.

From this algorithm for constructing F from its diagonal blocks we conclude that F is real, and consequently $\log A$ is real if and only if each of the blocks F_{ii} is real. Next we discuss the real logarithms $\log T_{ii}$ of 2-by-2 a real matrix with complex conjugate eigenvalues.

Lemma 8 *Let $B = (b_{ij}) \in R^{2\times 2}$ with complex conjugate eigenvalues $\lambda = a+ib$ and $\overline{\lambda} = a - ib$, $b \neq 0$. Then B has a countable real logarithms.*

Proof. Since B has complex conjugate eigenvalues λ and $\overline{\lambda}$, $\lambda \neq \overline{\lambda}$, and $\lambda = a + ib$, then there exists a nonsingular matrix $V \in C^{2\times 2}$, such that

$$B = V\text{diag}(\lambda, \overline{\lambda})V^{-1}.$$

Then B can be written in the form $B = aI + ibVKV^{-1} = aI + bW$, where $W = iVKV^{-1}$ and $K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Clearly W is a real matrix. Thus any logarithm X of B is given by

$$\begin{aligned} X &= V\text{diag}(\log^{(j_1)} \lambda, \log^{(j_2)} \overline{\lambda})V^{-1} \\ &= V\text{diag}(\theta + i\mu + i2\pi j_1, \theta - i\mu + i2\pi j_2)V^{-1} \end{aligned}$$

where $\theta = Log\,|\lambda|$, $\mu = Arg\lambda$ and $Arg\overline{\lambda} = -Arg\lambda$. Then the set of all logarithms of a 2-by-2 real matrix (with complex conjugate eigenvalues) is a countable set, each logarithm is given by

$$X = \theta I + \mu W + VEV^{-1} \tag{10}$$

where $E = i2\pi \text{diag}(j_1, j_2)$. In fact equation (10) gives all the possible solutions of $e^X = B$. The logarithm in equation (10) is real if and only if VEV^{-1} is a real matrix, that is, if and only if $j_1 = -j_2$. In this case $VEV^{-1} = i2\pi j_1 VKV^{-1} = 2\pi j_1 W$. Then any real logarithm X of B has the form

$$X = \theta I + (\mu + 2\pi j_1)W,$$

where $W = \frac{1}{b}(B - aI)$. And indeed once θ and μ are known we have a countable set of real logarithms. ■

The set of real logarithms of a real 2-by-2 matrix (with complex conjugate eigenvalues) can also obtained in an alternative approach by using Lagrange interpolation as follows.

Let $B = (b_{ij}) \in R^{2\times 2}$ with complex conjugate eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$, then there exists a polynomial $r(z)$ of the first degree given by

$$r(z) = (\log^{(j_1)} \lambda)\frac{(z - \overline{\lambda})}{(\lambda - \overline{\lambda})} + (\log^{(j_2)} \overline{\lambda})\frac{(z - \lambda)}{(\overline{\lambda} - \lambda)}$$

where $\log^{(j)} z$ is a branch of $\log z$. Hence we can define a logarithm X of B as

$$\begin{aligned} X &= r(B) = (\log^{(j_1)} \lambda)\frac{(B - \overline{\lambda}I)}{2ib} + (\log^{(j_2)} \overline{\lambda})\frac{(B - \lambda I)}{-2ib} \\ &= \frac{i}{2b}\left[(\log^{(j_2)} \overline{\lambda})(B - \lambda I) - (\log^{(j_1)} \lambda)(B - \overline{\lambda} I)\right] \\ &= \frac{i}{2b}\left[(Log\overline{\lambda} + i2\pi j_2)(B - \lambda I) - (Log\lambda + i2\pi j_1)(B - \overline{\lambda} I)\right] \\ &= \frac{i}{2b}\left[\overline{(Log\lambda - i2\pi j_2)(B - \overline{\lambda} I)} - (Log\lambda + i2\pi j_1)(B - \overline{\lambda} I)\right] \end{aligned}$$

where $j_1, j_2 \in Z$. This logarithm is real if the matrix in the bracket is pure imaginary, that is, if $j_1 = -j_2$. For example if we set $j_1 = j_2 = 0$, we have the principal logarithm $LogB$, namely

$$\begin{aligned} LogB &= \frac{i}{2b}\left[\overline{(Log\lambda)(B - \overline{\lambda} I)} - (Log\lambda)(B - \overline{\lambda} I)\right] \\ &= -\frac{i}{2b} 2i\text{Im}\left[(Log\lambda)(B - \overline{\lambda} I)\right] \\ &= \frac{1}{b}\text{Im}\left[(Log\lambda)(B - \overline{\lambda} I)\right]. \end{aligned}$$

Hence

$$LogB = \frac{1}{b}\begin{bmatrix} bLog\,|\lambda| + (b_{11} - a)Arg\lambda & b_{12}Arg\lambda \\ b_{21}Arg\lambda & bLog\,|\lambda| + (b_{22} - a)Arg\lambda \end{bmatrix}, \quad (11)$$

where $a = \frac{1}{2}(b_{11} + b_{22})$ and $b = \frac{1}{2}\sqrt{-(b_{11} - b_{22})^2 - 4b_{12}b_{21}}$.
We summarize the previous steps in the following algorithm.
Algorithm special-real-logarithm (B, j_1)
(This algorithm computes a real logarithm of a real 2-by-2 matrix B.)
Input: A, j_1
$a = (b_{11} + b_{22})/2;$
$b = sqrt(-(b_{11} - b_{22})^2 - 4b_{12}b_{21})/2;$
$\theta = \frac{1}{2}Log(a^2 + b^2);$
$\mu = \tan^{-1}(b/a);$
$X = \theta I + \frac{1}{b}(\mu + 2\pi j_1)(B - aI).$

Now we can give an algorithm to compute a real logarithm of a block upper triangular real matrix T. Assume that T defined by equation (8) such that $T_{11}, \dots, T_{rr}$ are 1-by-1 and $T_{r+1,r+1}, \dots, T_{mm}$ are 2-by-2 matrices with complex conjugate eigenvalues. The following algorithm compute a real logarithm F of T.

Algorithm general-real-logarithm
Input: T
for $i = 1$ **to** r
  $F_{ii} = \log(T_{ii})$
endfor
for $i = r + 1$ **to** m
  $F_{ii} =$ special-real-logarithm (B, j_1)
endfor
$SUM = 0;$
for $i = 1$ **to** m
  for $j = 2$ **to** m
    if $(j - 1 \geq i + 1)$ **then**
      for $k = i + 1$ **to** $j - 1$
        $SUM = SUM + T_{ik}F_{kj} - F_{ik}T_{kj}$
      endfor
    endif
  Solve $F_{ij}T_{jj} - T_{ii}F_{ij} = T_{ij}F_{jj} - F_{ii}T_{ij} + SUM.$
  (This system of equations can be solved by any standard method)
  endfor
endfor.

Note that, if A is a real normal matrix then the above algorithm computes the real logarithms even if A has negative or repeated eigenvalues provided that the negative eigenvalues occur in pairs.

3.1 Real logarithm of real normal matrix

If $A \in R^{n\times n}$ is a normal matrix and each of its negative eigenvalue occurs an even number of times, then Theorem 6 implies that there exists a real orthogonal matrix $Q \in R^{n\times n}$ such that

$$A = Q\text{diag}(D_1, \dots, D_s, D_{s+1}, \dots, D_p)Q^T,$$

where D_j is a positive number for $j = 1, \dots, s$, and for $j = s+1, \dots p$, D_j is a 2-by-2 matrix of the form

$$\begin{bmatrix} a_j & b_j \\ -b_j & a_j \end{bmatrix} \quad or \quad \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

corresponding to a pair of complex conjugate eigenvalues $a_j + ib_j$ and $a_j - ib_j$, or to a pair of negative eigenvalues, that is, $\alpha < 0$. Then by using Lemma 1 we can find a real matrix X_j such that $e^{X_j} = D_j$ for all $j = 1, 2, \dots, p$, and consequently

$$X = Q\text{diag}(X_1, X_2, \dots, X_j)Q^T$$

is a real logarithm of A. If A has negative eigenvalues then there is no real logarithm of A which is a polynomial in A.

Example 9 *Consider the normal matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

with eigenvalues 6, $-1.5 \pm 0.866i$. *The real Schur decomposition of* A *is given by* $A = QTQ^T = Q\text{diag}(D_1, D_2)Q^T$, *where*

$$Q = \begin{bmatrix} 0.5774 & 0.3004 & 0.7592 \\ 0.5774 & 0.5073 & -0.6397 \\ 0.5774 & -0.8077 & -0.1195 \end{bmatrix},$$

and

$$T = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -1.5 & 0.866 \\ 0 & -0.866 & -1.5 \end{bmatrix}.$$

Then by using Lemma 1, there exists a real logarithm X *of* A *of the form*

$$X = Q\text{diag}(X_1, X_2)Q^T,$$

where $X_1 = 1.792$ *and* $X_2 = \begin{bmatrix} 0.549 & 2.618 \\ -2.618 & 0.549 \end{bmatrix}$. *Consequently*

$$X = \begin{bmatrix} 0.9634 & -1.0969 & 1.9258 \\ 1.9259 & 0.9634 & -1.0969 \\ -1.0970 & 1.9258 & 0.9634 \end{bmatrix}.$$

References

[1] R. C. Allen and S. A. Pruess, An analysis of an inverse problem in ordinary differential equations, SIAM J. Sci. Statist. Comput., 2(1981), 176-185.

[2] A. Bjorck and S. Hammarling, A Schur method for the square root of a matrix, Linear Algebra Appl., 52(1983), 127-140.

[3] S. H. Cheng, N. J. Higham, C. S. Kenney, and A. J. Laub, Approximating the logarithm of a matrix to specified accuracy, SIAM J. Matrix Anal. Appl., 22(2000), 1112-1125.

[4] W. J. Culver, An analytic theory of modeling for a class of minimal energy control systems, SIAM journal control, 2(1964), 267-294.

[5] W. J. Culver, On the existence and uniqueness of the real logarithm of a matrix, Proc. Amer. Math., 17(1966), 1146-1151.

[6] J. C. Evard and F. Uhlig, On the Matrix Equation $f(X) = A$, Linear Algebra Appl., 162(1992), 447-519.

[7] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press., Cambridge, 1991.

[8] B. Singer and S. Spilerman, The representation of social processes by markov models, Amer. J. Sociology, 82(1976), 1-54.

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