A brief on Ceres Jacobians and Lie Jacobians

Joan Solà

Abstract

The aim of this document is to prove that the approaches to Jacobians in Ceres and Manif are mathematically equivalent.

1 Ceres Jacobians

If we use right-plus[1] for the manifold increment $\boldsymbol{\omega}$, the error function and its Jacobian can be written as

$$\mathbf{e} = e(\mathbf{x} \oplus \boldsymbol{\omega}) \tag{1}$$

$$\mathbf{J}^{\mathbf{e}}_{\boldsymbol{\omega}} = \mathbf{J}^{\mathbf{e}}_{\mathbf{x}\oplus\boldsymbol{\omega}} \cdot \mathbf{J}^{\mathbf{x}\oplus\boldsymbol{\omega}}_{\boldsymbol{\omega}} \tag{2}$$

$$\mathbf{J}^{\mathbf{e}}_{\mathbf{x}\oplus\boldsymbol{\omega}} = \left.\frac{\partial e(\mathbf{x} \oplus \boldsymbol{\omega})}{\partial(\mathbf{x} \oplus \boldsymbol{\omega})}\right|_{\boldsymbol{\omega}=0} = \frac{\partial e(\mathbf{x})}{\partial \mathbf{x}} \tag{3}$$

$$\mathbf{J}^{\mathbf{x}\oplus\boldsymbol{\omega}}_{\boldsymbol{\omega}} = \left.\frac{\partial(\mathbf{x} \oplus \boldsymbol{\omega})}{\partial \boldsymbol{\omega}}\right|_{\boldsymbol{\omega}=0} \tag{4}$$

(1) is the error function

(2) is the Jacobian computed using the chain rule.

(3) is the Jacobian computed in the constraint, as required by Ceres. For convenience, one uses the class "Jet" to compute it automatically.

(4) is the Jacobian computed in the local parametrization, as required by Ceres. The class storing the local parametrization in Ceres is now called "Manifold".

1.1 Implementation

Ceres uses regular differentation in vector spaces:

[1] Ceres actually uses left-plus, so this all needs to be re-done accordingly. See the Comments section at the end. The conclusions of equivalence do not vary, but the precise algebra will be slightly different.

$$\mathbf{J}^{\mathbf{e}}_{\mathbf{x}\oplus\boldsymbol{\omega}} \triangleq \lim_{\delta\mathbf{x}\to 0} \frac{e(\mathbf{x}+\delta\mathbf{x}) - e(\mathbf{x})}{\delta\mathbf{x}} \tag{5}$$

$$\mathbf{J}^{\mathbf{x}\oplus\boldsymbol{\omega}}_{\boldsymbol{\omega}} \triangleq \lim_{\delta\boldsymbol{\omega}\to 0} \left.\frac{\mathbf{x}\oplus(\boldsymbol{\omega}+\delta\boldsymbol{\omega}) - \mathbf{x}\oplus\boldsymbol{\omega}}{\delta\boldsymbol{\omega}}\right|_{\boldsymbol{\omega}=0} = \lim_{\delta\boldsymbol{\omega}\to 0} \frac{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}}{\delta\boldsymbol{\omega}} \tag{6}$$

Here, (5) is the vectorial derivative of our error function, and (6) is the derivative of the local parametrization of the state. In Ceres, (5) is computed automatically with the use of the "Jet" class, see (3), while (6) is computed in the local parametrization object, deriving from class "Manifold", see (4). The product (5) times (6) is performed by Ceres during the solving process, see (2), but the user has to provide separate expressions of (5) and (6).

2 Manif Jacobians

We have the error function and its Jacobian,

$$\mathbf{e} = e(\mathbf{x}) \tag{7}$$

$$\mathcal{J}^{\mathbf{e}}_{\mathbf{x}} = \frac{\mathcal{D}e(\mathbf{x})}{\mathcal{D}\mathbf{x}} \tag{8}$$

2.1 Implementation

We use differentiation on the manifold using right plus and minus:

$$\mathcal{J}^{\mathbf{e}}_{\mathbf{x}} = \frac{\mathcal{D}e(\mathbf{x})}{\mathcal{D}\mathbf{x}} \triangleq \lim_{\delta\boldsymbol{\omega}\to 0} \frac{e(\mathbf{x}\oplus\delta\boldsymbol{\omega}) \ominus e(\mathbf{x})}{\delta\boldsymbol{\omega}} \tag{9}$$

In manif, the Jacobian is computed in one go, without resorting to the chain rule as in Ceres.

3 Comparative

Let us assume $\mathbf{e}$ is always a vector. Then, $\ominus$ is the regular subtraction '$-$', and (9) becomes

$$\mathcal{J}^{\mathbf{e}}_{\mathbf{x}} = \lim_{\delta\boldsymbol{\omega}\to 0} \frac{e(\mathbf{x}\oplus\delta\boldsymbol{\omega}) - e(\mathbf{x})}{\delta\boldsymbol{\omega}} \tag{10}$$

We need to see whether (5) times (6) is equal to (10),

$$\mathbf{J}^{\mathbf{e}}_{\boldsymbol{\omega}} = \mathbf{J}^{\mathbf{e}}_{\mathbf{x}\oplus\boldsymbol{\omega}} \cdot \mathbf{J}^{\mathbf{x}\oplus\boldsymbol{\omega}}_{\boldsymbol{\omega}} \tag{11}$$

$$= \lim_{\delta\mathbf{x}\to 0} \frac{e(\mathbf{x}+\delta\mathbf{x}) - e(\mathbf{x})}{\delta\mathbf{x}} \cdot \lim_{\boldsymbol{\omega}\to 0} \frac{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}}{\delta\boldsymbol{\omega}} \tag{12}$$

Let us call $\delta\mathbf{x} \triangleq \mathbf{x} \oplus \delta\boldsymbol{\omega} - \mathbf{x}$. We have that $\delta\mathbf{x} \to 0$ iff $\delta\boldsymbol{\omega} \to 0$, so we can develop,

$$\begin{aligned}
\mathbf{J}^{\mathbf{e}}_{\boldsymbol{\omega}} &= \lim_{\delta\mathbf{x}\to 0} \frac{e(\mathbf{x}+\delta\mathbf{x}) - e(\mathbf{x})}{\delta\mathbf{x}} \cdot \lim_{\delta\boldsymbol{\omega}\to 0} \frac{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}}{\delta\boldsymbol{\omega}} && (13)\\
&= \lim_{\delta\boldsymbol{\omega}\to 0} \frac{e(\mathbf{x}+\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}) - e(\mathbf{x})}{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}} \cdot \lim_{\delta\boldsymbol{\omega}\to 0} \frac{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}}{\delta\boldsymbol{\omega}} && (14)\\
&= \lim_{\delta\boldsymbol{\omega}\to 0} \frac{e(\mathbf{x}+\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}) - e(\mathbf{x})}{\cancel{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}}} \cdot \frac{\cancel{\mathbf{x}\oplus\delta\boldsymbol{\omega} - \mathbf{x}}}{\delta\boldsymbol{\omega}} && (15)\\
&= \lim_{\delta\boldsymbol{\omega}\to 0} \frac{e(\mathbf{x}\oplus\delta\boldsymbol{\omega}) - e(\mathbf{x})}{\delta\boldsymbol{\omega}} && (16)
\end{aligned}$$

which is exactly equal to (10).

4 Comments

1. We observe that the right plus appears in both definitions (1) and (9). So, should Ceres use left-plus and manif right-plus, the expressions would not match.
2. In effect, ceres uses left-plus in its shipped local-parametrizations. To use right-plus and thus be compatible with manif, one needs to define new local-parametrizations using right-plus.
3. In wolf, we use right plus for the local parametrizations, so we should be fine with manif.