1. Eigenvalue problem

For an $n \times n$ square matrix $\underline{\underline{A}}$, is there a scalar λ and vector $\underline{x}$ such that:

$$\underline{\underline{A}}\,\underline{x} = \underset{\text{Eigenvalue}}{\lambda}\ \underset{\text{Eigenvector}}{\underline{x}}$$

We want nontrivial solutions ($\underline{x} \neq \underline{0}$). To find them:

$$\underline{\underline{A}}\,\underline{x} - \lambda \underline{x} = \underline{0}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda \underline{\underline{I}}\,\underline{x} = \underline{0}$$

$$\boxed{(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{x} = \underline{0}}$$

If $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ is invertible, we have only $\underline{x} = \underline{0}$ as a solution. Hence, we want this to be singular! This occurs when the determinant is zero:

$$\boxed{|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0}$$

This determinant creates a characteristic polynomial for λ that can be solved. Then, the $\underline{x}$ that corresponds to a given λ can be determined.

Ex: $\underline{\underline{A}} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = (-5-\lambda)(-2-\lambda) - 2\cdot 2 = (\lambda+5)(\lambda+2) - 4$$

$$= \lambda^2 + 7\lambda + 6 = (\lambda+1)(\lambda+6) = 0 \rightarrow \underline{\lambda = -1, -6}$$

$\lambda = -1$: $\begin{bmatrix} -5-(-1) & 2 \\ 2 & -2-(-1) \end{bmatrix} \underline{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \underline{x} = \underline{0}$

$$\left[\begin{array}{cc:c} -4 & 2 & 0 \\ 2 & -1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \quad x_1 - \frac{x_2}{2} = 0$$

can ignore since it's just zeros

x_2 free

This solution is not unique, which makes sense from the equation. We may then choose x_1 or x_2 so that the eigenvector has a nice norm or values. e.g.,

$$x_1 = 1, \; x_2 = 2 \longrightarrow \underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Repeat for $\lambda = -6$:

$$\begin{bmatrix} -5-(-6) & 2 \\ 2 & -2-(-6) \end{bmatrix} \underline{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \underline{x} = \underline{0}$$

Row 2 is 2 x Row 1, so we can just use Row 1! (This will be a pattern for 2x2 matrices).

$$x_1 + 2x_2 = 0 \rightarrow \underline{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Hence, the eigenvalues and eigenvectors are:

$$\boxed{\lambda_1 = -1, \; \underline{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \lambda_2 = -6, \; \underline{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$