1. Nonlinear first-order ODEs

These ODEs can be very hard to work with, but we are often interested in behavior near a steady state where $\underline{y}' = \underline{0}$. We can linearize these around the steady state!

For the system of ODEs

$$\underline{y}' = \underline{f}(\underline{y})$$

We solve for the steady state $\underline{y}' = \underline{0} \rightarrow f(\underline{y}_0) = \underline{0}$

Then, take the Taylor series about $\underline{y}_0$:

$$\underline{f}(\underline{y}) \approx \underbrace{\underline{f}(\underline{y}_0)}_{\underline{0}} + \underline{\underline{J}}(\underline{y}_0)(\underline{y} - \underline{y}_0) \qquad \underline{\underline{J}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \text{ Jacobian matrix}$$

If we define $\Delta \underline{y} = \underline{y} - \underline{y}_0$, then $\Delta \underline{y}' = \underline{y}'$ and

$$\boxed{\Delta \underline{y}' = \underline{\underline{J}}(\underline{y}_0)\Delta \underline{y}}$$

This is a linear system with constant coefficients! We can use it to understand the behavior of the system near y_0, but it will break down the farther away we get.

2. Ex: Lotka-Volterra (predator-prey) model

$$N_1' = aN_1 - bN_1N_2 \qquad a, b, k, \ell > 0$$

$$N_2' = kN_1N_2 - \ell N_2$$

Steady state: $N_1' = N_1(a - bN_2) = 0$, $N_2' = N_2(kN_1 - \ell) = 0$ $\longrightarrow$ $\underset{\substack{\uparrow \\ \text{boring, all dead !!}}}{(0,0)}$ or $\underbrace{\left(\frac{\ell}{k}, \frac{a}{b}\right)}_{\underline{N}_0}$

Linearize: $f_1 = aN_1 - bN_1N_2$, $f_2 = kN_1N_2 - \ell N_2$ $\longrightarrow$ $\underline{\underline{J}} = \begin{bmatrix} \partial f_1/\partial N_1 & \partial f_1/\partial N_2 \\ \partial f_2/\partial N_1 & \partial f_2/\partial N_2 \end{bmatrix} = \begin{bmatrix} a - bN_2 & -bN_1 \\ kN_2 & kN_1 - \ell \end{bmatrix}$

$$\underline{\underline{J}}(\underline{N}_0) = \begin{bmatrix} 0 & -b\ell/k \\ \frac{ka}{b} & 0 \end{bmatrix}$$

$$\Delta \underline{N}' = \underline{\underline{J}}(N_0)\Delta N \longrightarrow \begin{vmatrix} \Delta N_1' = (-b\ell/k)\Delta N_2 \\ \Delta N_2' = (ka/b)\Delta N_1 \end{vmatrix}$$

$$\Delta \underline{N}' = \underbrace{\begin{bmatrix} 0 & -b\ell/k \\ \frac{ka}{b} & 0 \end{bmatrix}}_{\underline{\underline{A}}} \Delta \underline{N}$$

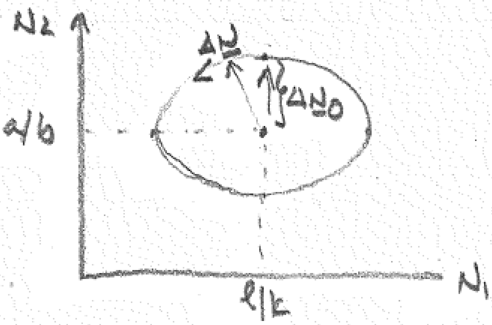
$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = \begin{vmatrix} -\lambda & -bl/k \\ \frac{ka}{b} & -\lambda \end{vmatrix} = \lambda^2 + \left(\frac{ka}{b}\right)\left(\frac{bl}{k}\right) = \lambda^2 + al = 0 \rightarrow \lambda_{1,2} = \pm i\sqrt{al}$$

$\lambda_{1,2}$ is pure imaginary, so this is a center. It rotates counter clockwise because $-\frac{bl}{k} - \frac{ka}{b} < 0$.

We can solve this as complex eigenvectors, then we would convert to sine and cosine using:

$$e^{ix} = \cos x + i \sin x$$

The result is an ellipse! (A center)



Increased predators (2) → decreased prey (1) because of predators → decreased predators from lack of prey → growth of prey then predators!