Inverse Kinematics for the Rover's Arm

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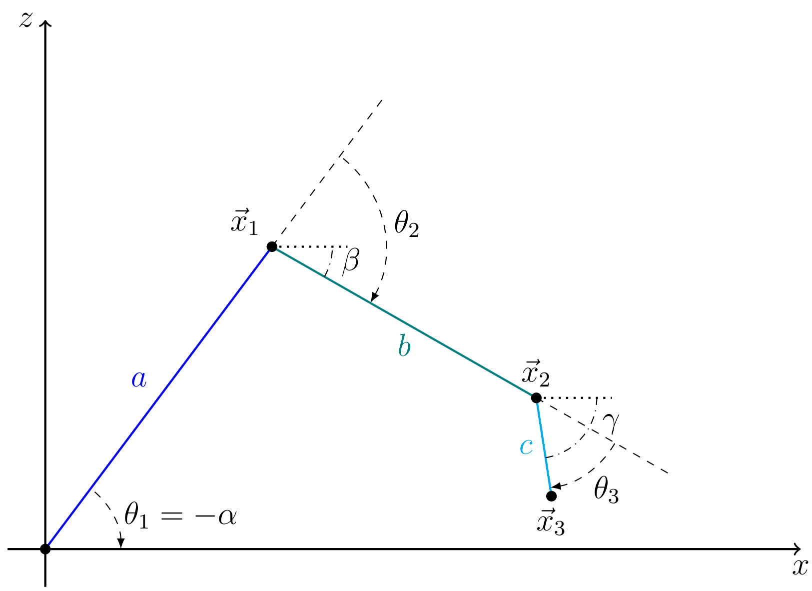


1 Background

On the rover's arm, each revolute joint's axis is along the positive y-axis, which is to the left from the point of view of the rover (the positive x-axis is forward and the positive z-axis is upward). Thus a positive rotation is clockwise as viewed from the right, like in this diagram. This results in unintuitive negations of angles sometimes—beware.

Note that we are using the angles α, β, and γ as relative to the positive x-axis, whereas the angles θ_1, θ_2, and θ_3 are relative to the previous arm link.

Goals:

1. Given a final position for the end effector ($\vec{x}_3$) and target pitch (γ), compute the angles $\theta_1, \theta_2, \theta_3$ required to reach the desired position with the correct orientation.

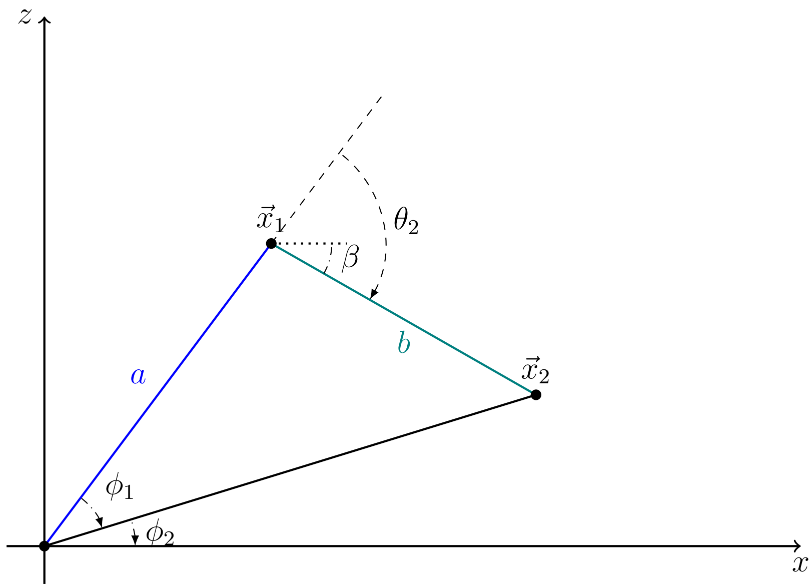
2. Given a velocity vector for the end effector ($\vec{x}_3' = \vec{v}$) and a target pitch angular velocity (γ'), compute the required joint velocities $\theta_1', \theta_2', \theta_3'$ required to move the arm as desired.

2 The Math

First we begin by performing the positional inverse kinematics. Begin by noting that we are given the position $\vec{x}_3 = \begin{bmatrix} x_3 \\ z_3 \end{bmatrix}$ and the pitch γ of the end effector. We can solve for $\vec{x}_2$ as follows (note the negation of γ):

$$\vec{x}_2 = \begin{bmatrix} x_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_3 - c\cos(-\gamma) \\ z_3 - c\sin(-\gamma) \end{bmatrix} = \begin{bmatrix} x_3 - c\cos(\gamma) \\ z_3 + c\sin(\gamma) \end{bmatrix}$$

Now we have reduced the problem by one joint. To solve the remaining problem, consider the following drawing:



Using the law of cosines, we have

$$x_2^2 + z_2^2 = a^2 + b^2 - 2ab\cos(\pi - \theta_2)$$

Note that $\cos(\pi - \theta_2) = -\cos(\theta_2)$, so we have

$$\cos(\theta_2) = \frac{x_2^2 + z_2^2 - a^2 - b^2}{2ab}$$

Thus

$$\boxed{\theta_2 = \cos^{-1}(\frac{x_2^2 + z_2^2 - a^2 - b^2}{2ab})}$$

Now we wish to find the angle $\theta_1 = \phi_1 + \phi_2$. Note that

$$\tan(-\phi_2) = \frac{z_2}{x_2}$$

so

$$\phi_2 = -\tan^{-1}(\frac{z_2}{x_2})$$

Again using the law of cosines, we have that

$$b^2 = a^2 + x_2^2 + z_2^2 - 2a\sqrt{x_2^2 + z_2^2}\cos(\phi_1) \implies$$

$$\cos(\phi_1) = \frac{a^2 + x_2^2 + z_2^2 - b^2}{2a\sqrt{x_2^2 + z_2^2}} \implies$$

$$\phi_1 = -\cos^{-1}(\frac{a^2 + x_2^2 + z_2^2 - b^2}{2a\sqrt{x_2^2 + z_2^2}})$$

signs for ϕ_1, ϕ_2 kinda cooked ngl

Thus we have

$$\boxed{\theta_1 = \phi_1 + \phi_2 = -\cos^{-1}(\frac{a^2 + x_2^2 + z_2^2 - b^2}{2a\sqrt{x_2^2 + z_2^2}}) - \tan^{-1}(\frac{z_2}{x_2})}$$

Now, we calculate θ_3. Note that

$$-\theta_1 - \theta_2 - \theta_3 = -\gamma$$

so then

$$\boxed{\theta_3 = \gamma - \theta_1 - \theta_2}$$

To make our formulas a little nicer, let $c^2 = x_2^2 + z_2^2$. Then we have

$$\boxed{\begin{aligned} &\theta_1 = -\cos^{-1}(\frac{c^2 + a^2 - b^2}{2ac}) - \tan^{-1}(\frac{z_2}{x_2}) \\ &\theta_2 = \cos^{-1}(\frac{c^2 - a^2 - b^2}{2ab}) \\ &\theta_3 = \gamma - \theta_1 - \theta_2 \end{aligned}}$$

For the second part of our calculations, we simply take the time derivative of each angle to find the corresponding angular velocity.

$$\frac{d}{dt}\theta_2 = \frac{d}{dt}(\cos^{-1}(\frac{c^2 - a^2 - b^2}{2ab})) = -[1 - (\frac{c^2 - a^2 - b^2}{2ab})^2]^{-\frac{1}{2}} \cdot \frac{d}{dt}(\frac{c^2 - a^2 - b^2}{2ab})$$

$$= -[1 - (\frac{c^2 - a^2 - b^2}{2ab})^2]^{-\frac{1}{2}} \cdot \frac{1}{2ab}(2c\frac{d}{dt}c)$$

Thus we have

$$\theta_2' = -[1 - (\frac{c^2 - a^2 - b^2}{2ab})^2]^{-\frac{1}{2}} \cdot \frac{1}{2ab}(2cc')$$

Similarly, for θ_1, we have

$$\frac{d}{dt}\theta_1 = \frac{d}{dt}(-\cos^{-1}(\frac{c^2 + a^2 - b^2}{2ac}) - \tan^{-1}(\frac{z_2}{x_2}))$$

$$= [1 - (\frac{c^2 + a^2 - b^2}{2ac})^2]^{-\frac{1}{2}} \cdot \frac{d}{dt}(\frac{c^2 + a^2 - b^2}{2ac}) - (1 + (\frac{z_2}{x_2})^2)^{-1} \cdot \frac{d}{dt}(\frac{z_2}{x_2})$$

$$= [1 - (\frac{c^2 + a^2 - b^2}{2ac})^2]^{-\frac{1}{2}} \cdot \frac{2cc' \cdot 2ac - 2ac'(c^2 + a^2 - b^2)}{4a^2c^2} - (1 + (\frac{z_2}{x_2})^2)^{-1} \frac{z_2' x_2 - z_2 x_2'}{x_2^2}$$

$$\theta_1' = [1 - (\frac{c^2 + a^2 - b^2}{2ac})^2]^{-\frac{1}{2}} \cdot \frac{c^2 c' - c'(a^2 - b^2)}{2ac^2} - (1 + (\frac{z_2}{x_2})^2)^{-1} \frac{z_2' x_2 - z_2 x_2'}{x_2^2}$$

We can then use θ_1' and θ_2' to compute θ_3':

$$\frac{d}{dt}\theta_3 = \frac{d}{dt}(-\gamma - \theta_1 - \theta_2) = -\gamma' - \theta_1' - \theta_2'$$

So, overall we have

$$\boxed{\begin{aligned}
&\theta_1' = [1 - (\frac{c^2 + a^2 - b^2}{2ac})^2]^{-\frac{1}{2}} \cdot \frac{c^2 c' - c'(a^2 - b^2)}{2ac^2} - (1 + (\frac{z_2}{x_2})^2)^{-1} \frac{z_2' x_2 - z_2 x_2'}{x_2^2} \\
&\theta_2' = -[1 - (\frac{c^2 - a^2 - b^2}{2ab})^2]^{-\frac{1}{2}} \cdot \frac{1}{2ab}(2cc') \\
&\theta_3' = \gamma' - \theta_1' - \theta_2'
\end{aligned}}$$