原问题: $\arg\min_{g}\left\{\frac{fidelity}{2}\|f-g\|_2^2+\|\partial_{xx}g\|_1+\|\partial_{yy}g\|_1+\|\partial_{xy}g\|_1+\|\partial_{yx}g\|_1+paral1\cdot\|g\|_1\right\}$

重写为: $\begin{cases}\arg\min_{g}\left\{\frac{fidelity}{2}\|f-g\|_2^2+\|d_{xx}\|_1+\|d_{yy}\|_1+\|d_{xy}\|_1+\|d_{yx}\|_1+\|d_{l1}\|_1\right\}\\ s.t.\quad d_{xx}-\partial_{xx}g=0;\ d_{yy}-\partial_{yy}g=0;\ d_{xy}-\partial_{xy}g=0;\ d_{yx}-\partial_{yx}g=0;\ d_{l1}-paral1\cdot g=0\end{cases}$

增广拉式函数:

$$\begin{aligned}L_{mu}(g,d,w)=&\frac{fidelity}{2}\|f-g\|_2^2+\|d_{xx}\|_1+\|d_{yy}\|_1+\|d_{xy}\|_1+\|d_{yx}\|_1+\|d_{l1}\|_1\\&+W_{xx}^T(\partial_{xx}g-d_{xx})+W_{yy}^T(\partial_{yy}g-d_{yy})+\underbrace{2W_{xy}^T(\partial_{xy}g-d_{xy})}_{\text{因}\partial_{xy}g=\partial_{yx}g\text{对称性合并}}+W_{l1}^T\cdot(paral1\cdot g-d_{l1})\\&+\frac{mu}{2}\|\partial_{xx}g-d_{xx}\|_2^2+\frac{mu}{2}\|\partial_{yy}g-d_{yy}\|_2^2+2\cdot\frac{mu}{2}\|\partial_{xy}g-d_{xy}\|_2^2+\frac{mu}{2}\|paral1\cdot g-d_{l1}\|_2^2\end{aligned}$$

其中w为拉格朗日乘子、mu为惩罚参数，记 $b=\frac{w}{mu}$。

$L_{mu}(g,d,w)$对g求偏导:

$$\begin{aligned}\frac{\partial(L_{mu}(g,d,w))}{\partial g}=&fidelity(g-f)+\partial_{xx}W_{xx}+\partial_{yy}W_{yy}+2\partial_{xy}W_{xy}+paral1\cdot W_{l1}\\&+mu\left[(\partial_{xxxx}g-\partial_{xx}d_{xx})+(\partial_{yyyy}g-\partial_{yy}d_{yy})+2(\partial_{xyxy}g-\partial_{xy}d_{xy})+paral1\cdot(paral1\cdot g-d_{l1})\right]\\=&\,0\text{ 时}\end{aligned}$$

可整理得:

$$\begin{aligned}\left[fidelity+mu\cdot paral1^2+mu\cdot(\partial_{xxxx}+\partial_{yyyy}+2\partial_{xyxy})\right]g=&fidelity\cdot f+paral1\cdot(mu\cdot d_{l1}-W_{l1})\\&+\partial_{xx}(mu\cdot d_{xx}-W_{xx})+\partial_{yy}(mu\cdot d_{yy}-W_{yy})+2\partial_{xy}(mu\cdot d_{xy}-W_{xy})\end{aligned}$$

因记 $b=\frac{w}{mu}$，故等式两边同除mu后得:

$$\begin{aligned}\left[\frac{fidelity}{mu}+paral1^2+(\partial_{xxxx}+\partial_{yyyy}+2\partial_{xyxy})\right]g=&\frac{fidelity}{mu}\cdot f+paral1(d_{l1}-b_{l1})\\&+\partial_{xx}(d_{xx}-b_{xx})+\partial_{yy}(d_{yy}-b_{yy})+2\partial_{xy}(d_{xy}-b_{xy})\end{aligned}$$

到这一步只是想确认您的关于 d_{l1} 的等式约束确实是 $d_{l1}-paral1\cdot g=0$ (☆)

更新:

① 关于g的更新…

② 关于对偶变量d的更新:

(i) 关于 d_{l1} 的更新:

$$\begin{aligned}\arg\min_{d_{l1}}L_{mu}(g,d,w)&=\arg\min_{d_{l1}}\left\{\|d_{l1}\|_1+\underbrace{W_{l1}^T}_{b_{l1}=\frac{W_{l1}}{mu}}(paral1\cdot g-d_{l1})+\frac{mu}{2}\|paral1\cdot g-d_{l1}\|_2^2\right\}\times\frac{2}{mu}\\&=\arg\min_{d_{l1}}\left\{\frac{2}{mu}\|d_{l1}\|_1+2\cdot b_{l1}^T(paral1\cdot g-d_{l1})+\|paral1\,g-d_{l1}\|_2^2\right\}\\&=\arg\min_{d_{l1}}\left\{\frac{2}{mu}\|d_{l1}\|_1+\|(b_{l1}+paral1\cdot g)-d_{l1}\|_2^2-\underbrace{\|b_{l1}\|_2^2}_{d_{l1}\text{无关}}\right\}\end{aligned}$$

因为软阈值算子 $S_{\frac{\lambda}{\rho}}(\frac{u}{\rho}+\beta)=\arg\min_{\alpha}\left\{\frac{2\lambda}{\rho}\|\alpha\|_1+\|(\frac{u}{\rho}+\beta)-\alpha\|_2^2\right\}$

所以 $\arg\min_{d_{l1}}L_{mu}(g,d,w)=S_{\frac{1}{mu}}(b_{l1}+paral1\cdot g)$ (☆)

即 $S_{\frac{1}{mu}}(b_{l1}+paral1\cdot g)=sign(b_{l1}+paral1\cdot g)\cdot\max(0,\underline{abs[b_{l1}+paral1\cdot g]}-\frac{1}{mu})$

⇓

您的代码中 $\begin{cases}gsparse=g;\\ bsparse=bl1;\\ signd=abs(gsparse+bsparse)-\frac{1}{mu};\end{cases}$ ⇒ 疑似bug，是否少了paral1。

⇓

位于 SHIter/iter_sparse.m/第10行