Notes on Baroclinic Sea Level Calculation

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1 Problem Formulation

Let p be the hydrostatic pressure, defined by

$$p(z) = g \int_z^\eta \rho(\zeta) d\zeta, \tag{1}$$

where η is the free surface height, and ρ is the density. The baroclinic pressure is defined by (Zaron and Ray, 2023)

$$p^*(z) = p(z) - \frac{1}{H} \int_{-H}^0 p(z) dz = p(z) - \frac{g}{H} \int_{-H}^0 \int_z^\eta \rho(\zeta) d\zeta dz, \tag{2}$$

where H is the depth of the seafloor. Since

$$p(0) = \frac{1}{H} \int_{-H}^0 p(0) dz, \tag{3}$$

the baroclinic pressure at $z = 0$ can be expressed in terms of

$$p^*(0) = \frac{1}{H} \int_{-H}^0 [p(0) - p(z)] dz = -\frac{g}{H} \int_{-H}^0 \int_z^0 \rho(\zeta) d\zeta dz. \tag{4}$$

This expression can be further simplified through integration by parts, in a process yielding

$$p^*(0) = -\frac{g}{H} \int_{-H}^0 (H + z) \rho(z) dz. \tag{5}$$

The baroclinic sea level is defined by

$$\eta^* = \frac{p^*(0)}{\rho_s g} = -\frac{1}{\rho_s H} \int_{-H}^0 (H + z) \rho(z) dz, \tag{6}$$

where ρ_s is the density at the surface.

If the full density field is used in the calculation, then the vertical average of the pressure is close to the pressure at the mid-depth of the water column, such that the resulting baroclinic sea level has an order of magnitude similar to the half-depth of the water column. The physical interpretation of this quantity is meaningless, unless the mean value is removed, since the baroclinic sea level is essentially a perturbation from the background state, whose precise definition depends on the context being examined. For internal tide propagation, the time average over one or several tidal periods should be considered as the background state.

Let $p'(z)$ be the pressure anomaly, defined by

$$p'(z) = g \int_z^\eta \rho'(\zeta) d\zeta, \tag{7}$$

where ρ' is the density anomaly with respect to the constant reference density ρ_0,

$$\rho'(z) = \rho(z) - \rho_0. \tag{8}$$

The difference between p and p' is given by

$$p_0(z) = p(z) - p'(z) = \rho_0 g(\eta - z). \tag{9}$$

This implies that

$$\begin{aligned} p^*(0) &= p(0) - \frac{g}{H} \int_{-H}^0 \left[\rho_0(\eta - z) + \int_z^\eta \rho'(\zeta) d\zeta \right] dz \\ &= p(0) - \rho_0 g \left(\eta + \frac{H}{2} \right) - \frac{g}{H} \int_{-H}^0 \int_z^\eta \rho'(\zeta) d\zeta dz. \end{aligned} \tag{10}$$

Without loss of generality, we can simply let ρ_s be the reference density. Assuming there exists a mixed layer with constant density at the surface, such that $p(0) = \rho_s g \eta$, this implies that

$$p^*(0) = -\rho_s g \frac{H}{2} - \frac{1}{H} \int_{-H}^0 p'(z) dz, \tag{11}$$

such that

$$\eta^* = -\frac{H}{2} - \frac{1}{\rho_s g H} \int_{-H}^0 p'(z) dz. \tag{12a}$$

Equivalently,

$$\eta^* = -\frac{H}{2} - \frac{1}{\rho_s H} \int_{-H}^0 (H + z) \rho'(z) dz. \tag{12b}$$

Although the baroclinic pressure is defined in terms of the full pressure field, the interpretation of the baroclinic sea level is not affected by a constant. Hence, we can conveniently redefine η^* as

$$\eta^* = -\frac{1}{\rho_s g H} \int_{-H}^0 p'(z) dz, \tag{13a}$$

or, equivalently,

$$\eta^* = -\frac{1}{\rho_s H} \int_{-H}^0 (H + z) \rho'(z) dz. \tag{13b}$$

In ocean models, the integration to zero is not always attainable. Nevertheless, if ρ_s is used as the reference density, then ρ' and hence p' are identically zero at the surface, so that replacing the upper limit of integration by η does not change the results,

$$\eta^* = -\frac{1}{\rho_s g H} \int_{-H}^\eta p'(z) dz = -\frac{1}{\rho_s H} \int_{-H}^\eta (H + z) \rho'(z) dz. \tag{14}$$

1.1 Reference Density

More generally, if an arbitrary reference density ρ_0 is used, then

$$p^*(0) = \rho_s' g\eta - \frac{1}{2}\rho_0 g H - \frac{g}{H}\int_{-H}^{0}\int_{z}^{\eta} \rho'(\zeta) d\zeta dz. \tag{15}$$

where $\rho_s' = \rho_s - \rho_0$. This implies that

$$\eta^* = \left(\frac{\rho_s'}{\rho_s}\right)\eta - \left(\frac{\rho_0}{\rho_s}\right)\frac{H}{2} - \frac{1}{\rho_s g H}\int_{-H}^{0} p'(z) dz. \tag{16}$$

As before, the constant term may be dropped, such that η^* can be redefined as

$$\eta^* = \left(\frac{\rho_s'}{\rho_s}\right)\eta - \frac{1}{\rho_s g H}\int_{-H}^{0} p'(z) dz. \tag{17}$$

Equivalently, η^* may be defined in terms of the density anomaly,

$$\eta^* = -\frac{1}{\rho_s H}\int_{-H}^{0} (H+z)\rho'(z) dz. \tag{18}$$

In these cases, replacing the upper limit of integration by η leads to the differences

$$\frac{1}{\rho_s g H}\int_{0}^{\eta} p'(z) dz = \frac{\rho_s'}{\rho_s}\left(\frac{\eta^2}{2H}\right), \tag{19}$$

and

$$\frac{1}{\rho_s H}\int_{0}^{\eta} (H+z)\rho'(z) dz = \frac{\rho_s'}{\rho_s}\left(\eta + \frac{\eta^2}{2H}\right). \tag{20}$$

Since $\rho_s'/\rho_0 \ll 1$, the term $\eta^2/(2H)$ is generally insignificant in the deep ocean. Hence, we may approximate η^* with the following equivalent expressions,

$$\eta^* \approx \left(\frac{\rho_s'}{\rho_0}\right)\eta - \frac{1}{\rho_0 g H}\int_{-H}^{\eta} p'(z) dz, \tag{21a}$$

$$\eta^* \approx \left(\frac{\rho_s'}{\rho_0}\right)\eta - \frac{1}{\rho_0 H}\int_{-H}^{\eta} (H+z)\rho'(z) dz. \tag{21b}$$

2 Numerical Implementation

2.1 Integration of the Pressure Field

In MOM6, if `USE_EOS = True`, the pressure difference across a layer can be determined using the subroutine `int_density_dz`. In this case, the depth integral can be calculated layer by layer,

$$\int_{-H}^{\eta} p'(z) dz = \sum_{k=1}^{N}\int_{z_{k+1}}^{z_k} p'(z) dz, \tag{22}$$

where z_k represents the top of the k-th layer and z_{k+1} represents the bottom of the k-th layer. Within the k-th layer, where $z \in [z_{k+1}, z_k]$,

$$p'(z) = p'(z_k) + g \int_{z}^{z_k} \rho'(\zeta) d\zeta. \tag{23}$$

This implies that

$$\int_{z_{k+1}}^{z_k} p'(z) dz = h_k p'(z_k) + \int_{z_{k+1}}^{z_k} [p'(z) - p'(z_k)] dz, \tag{24}$$

where $h_k = z_k - z_{k+1}$ is the thickness of the k-th layer, and the integrated pressure difference on the right hand side is an optional output of the subroutine `int_density_dz`.

2.2 Integration of the Density Field: Method 1

Assuming, in an isopycnal or hybrid-isopycnal coordinate system, that density is constant within a layer, the pressure anomaly at the top of the k-th layer may be estimated numerically by

$$p'(z_k) = \sum_{j=1}^{k-1} g \rho'_j h_j = \sum_{j=1}^{k-1} g \rho'_j (z_j - z_{j+1}), \tag{25}$$

where ρ'_j is the density anomaly of the j-th layer. Note that $p'(z_1) = 0$, where $z_1 = \eta$ is the free surface. Suppose $\rho'_0 = 0$, the definition of $p'(z_k)$ is consistent for all $k \geq 1$, which may be represented by

$$p'(z_k) = \sum_{j=0}^{k-1} g(\rho'_{j+1} - \rho'_j) z_{j+1} - g \rho'_k z_k. \tag{26}$$

Within the k-th layer,

$$p'(z) = p'(z_k) + g \rho'_k (z_k - z), \tag{27}$$

which implies that

$$\int_{z_{k+1}}^{z_k} p'(z) dz = [p'(z_k) + g \rho'_k z_k](z_k - z_{k+1}) - \frac{1}{2} g \rho'_k (z_k^2 - z_{k+1}^2). \tag{28}$$

Let B_k be defined, for $k \geq 1$, by

$$B_k = p'(z_k) + g \rho'_k z_k = \sum_{j=0}^{k-1} g(\rho'_{j+1} - \rho'_j) z_{j+1}, \tag{29}$$

and let $B_0 = 0$. The depth integral of p' may be represented in terms of

$$\begin{aligned} \int_{-H}^{\eta} p'(z) dz &= \sum_{k=1}^{N} \int_{z_{k+1}}^{z_k} p'(z) dz \\ &= \sum_{k=0}^{N} \left[B_k (z_k - z_{k+1}) - \frac{1}{2} g \rho'_k (z_k^2 - z_{k+1}^2) \right]. \end{aligned} \tag{30}$$

Note that

$$\begin{aligned}\sum_{k=0}^{N} B_k(z_k - z_{k+1}) &= \sum_{k=0}^{N-1}(B_{k+1} - B_k)z_{k+1} + B_N H \\ &= \sum_{k=0}^{N-1} g(\rho'_{k+1} - \rho'_k)(z_{k+1}^2 + z_{k+1}H).\end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned}\sum_{k=0}^{N} \rho'_k(z_k^2 - z_{k+1}^2) &= \sum_{k=0}^{N-1}(\rho'_{k+1} - \rho'_k)z_{k+1}^2 - \rho_N H^2 \\ &= \sum_{k=0}^{N-1}(\rho'_{k+1} - \rho'_k)(z_{k+1}^2 - H^2).\end{aligned} \tag{32}$$

Combining these results together yields

$$\int_{-H}^{\eta} p'(z)dz = \frac{1}{2}\sum_{k=0}^{N-1} g(\rho'_{k+1} - \rho'_k)(z_{k+1} + H)^2. \tag{33}$$

Note that this expression remains unchanged if calculated using the full density field, in which case ρ_0 would be the reference density since $\rho'_0 = 0$.

2.3 Integration of the Density Field: Method 2

Alternatively, we may evaluate the following depth integral,

$$\begin{aligned}\int_{-H}^{\eta}(H + z)\rho'(z)dz &= \sum_{k=1}^{N}\int_{z_{k+1}}^{z_k}(H + z)\rho'(z)dz \\ &= \sum_{k=0}^{N}\rho'_k\left[H(z_k - z_{k+1}) + \frac{1}{2}(z_k^2 - z_{k+1}^2)\right].\end{aligned} \tag{34}$$

Note that

$$\sum_{k=0}^{N}\rho'_k(z_k - z_{k+1}) = \sum_{k=0}^{N-1}(\rho'_{k+1} - \rho'_k)(z_{k+1} + H), \tag{35}$$

and that

$$\sum_{k=0}^{N}\rho'_k(z_k^2 - z_{k+1}^2) = \sum_{k=0}^{N-1}(\rho'_{k+1} - \rho'_k)(z_{k+1}^2 - H^2). \tag{36}$$

Hence, the algorithm is identical to that derived previously,

$$\int_{-H}^{\eta}(H + z)\rho'(z)dz = \frac{1}{2}\sum_{k=0}^{N-1}(\rho'_{k+1} - \rho'_k)(z_{k+1} + H)^2. \tag{37}$$

2.4 Integration of the Density Field: Summary

If we let g'_k be the reduced gravity across the interface located at $z = z_k$, defined by

$$g'_k = g\left(\frac{\rho_k - \rho_{k-1}}{\rho_0}\right), \quad k \geq 2, \tag{38}$$

and let $g'_1 = g\rho_1/\rho_0$, then the following expression is also equivalent,

$$\frac{1}{\rho_0}\int_{-H}^{\eta} p'(z)dz = \frac{1}{2}\left[\sum_{k=1}^{N} g'_k(z_k + H)^2 - g(\eta + H)^2\right]. \tag{39}$$

To summarize, the full algorithm for calculating η^* is given by

$$\eta^* \approx \left(\frac{g'_1}{g} - 1\right)\eta - \frac{1}{2H}\left[\sum_{k=1}^{N}\left(\frac{g'_k}{g}\right)(z_k + H)^2 - (\eta + H)^2\right]. \tag{40}$$

References

E. D. Zaron and R. D. Ray. Clarifying the distinction between steric and baroclinic sea surface height. *Journal of Physical Oceanography*, 53:2591–2596, 2023. https://doi.org/10.1175/JPO-D-23-0073.1.